بسم الله الرحمن الرحیم

فصل چهارم

خصوصیات زبان‌های منظم (۲)

Properties of Regular Languages (2)

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More Applications

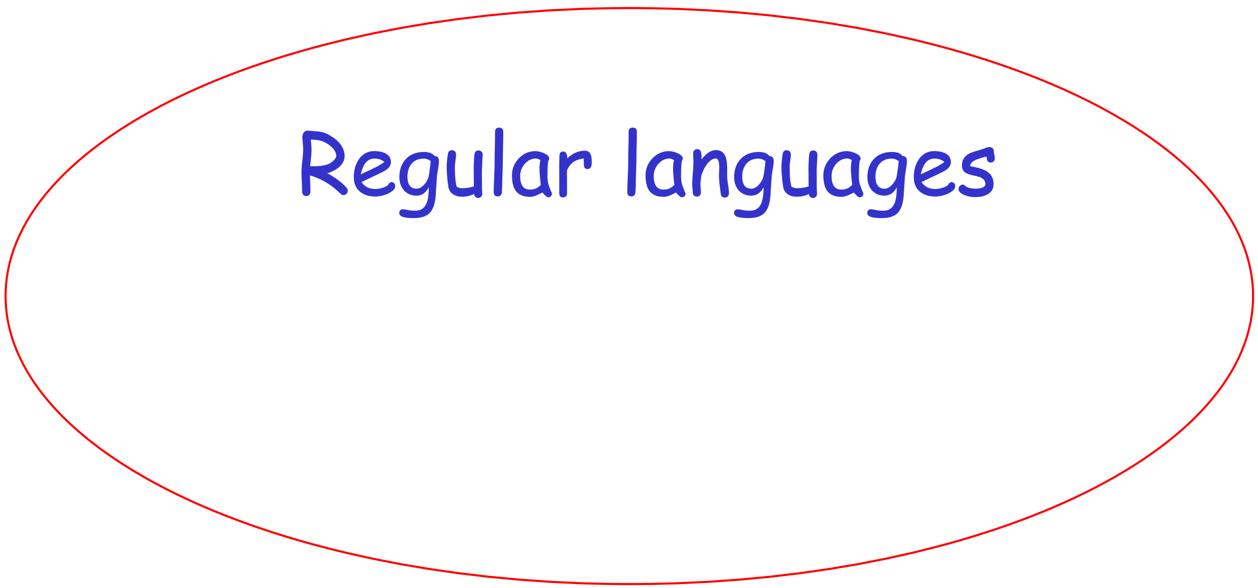
of

the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages $L = \{ww^R : w \in \Sigma^*\}$



Theorem: The language

$$L = \{ww^R : w \in \Sigma^*\} \quad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{ww^R : w \in \Sigma^*\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{ww^R : w \in \Sigma^*\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

pick $w = a^m b^m b^m a^m$

Write $a^m b^m b^m a^m = x\, y\, z$

From the Pumping Lemma
it must be that length $|x\, y| \leq m, \quad |y| \geq 1$

$$a^m b^m b^m a^m = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{...ab...bb...ba...a}_{z}$$

$$\overbrace{a...aa...a...a}^{m}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}$$

$y = a^k, \quad k \geq 1$

We have: $x\,y\,z = a^m b^m b^m a^m$

$$y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\,y^2\,z = x\,y\,y\,z = a^{m+k} b^m b^m a^m \in L$$

Therefore: $a^{m+k}b^m b^m a^m \in L$

BUT: $L = \{ww^R : w \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k}b^m b^m a^m \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

Regular languages

Theorem: The language

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

pick $w = a^m b^m c^{2m}$

Write $a^m b^m c^{2m} = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$a^m b^m c^{2m} = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...bc...cc...c}_{z}$$

$$a^m b^m c^{2m} = \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}$$

$y = a^k, \quad k \geq 1$

We have: $$x\,y\,z = a^m b^m c^{2m}$$

$$y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $$x\,y^i\,z \;\in\; L$$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^0\,z \;\in\; L$

$$x\,y^0\,z = x\,z = a^{m-k} b^m c^{2m} \quad \in\; L$$

Therefore: $a^{m-k}b^{m}c^{2m} \in L$

BUT: $L = \{a^{n}b^{l}c^{n+l} : \; n, l \geq 0\}$

$\Downarrow$

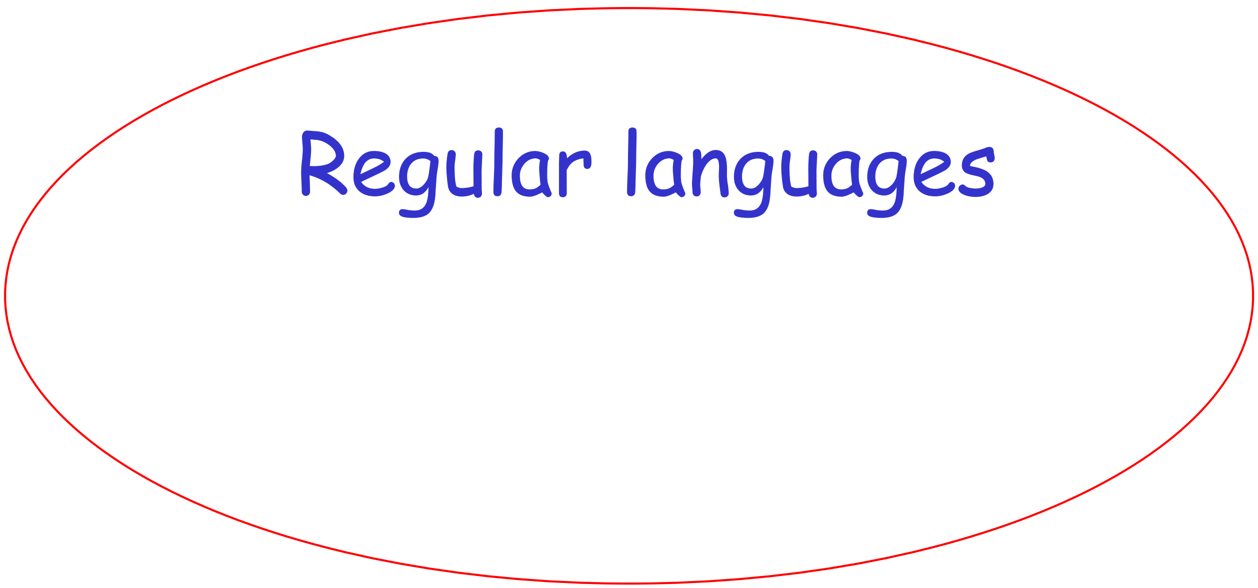
$a^{m-k}b^{m}c^{2m} \notin L$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages $L = \{a^{n!}:\ n \geq 0\}$



Theorem: The language $L = \{a^{n!} : n \geq 0\}$

is not regular

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{a^{n!} : \; n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

pick $w = a^{m!}$

Write $a^{m!} = x\,y\,z$

From the Pumping Lemma

it must be that length $|x\,y| \leq m, \quad |y| \geq 1$

$$a^{m!} = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}a...a}^{m}\overbrace{a...aa...a}^{m!-m}$$

$$z = a...aa...aa...a$$

$y = a^k, \quad 1 \leq k \leq m$

We have: $$x\,y\,z = a^{m!}$$

$$y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $$x\,y^i\,z \in L$$

$$i = 0, 1, 2, \ldots$$

Thus: $$x\,y^2\,z \in L$$

$$x\,y^2\,z = x\,y\,y\,z = a^{m!+k} \in L$$

Therefore: $a^{m!+k} \in L \qquad 1 \le k \le m$

And since: $L = \{a^{n!} : \ n \ge 0\}$

$$\Downarrow$$

There is p : $m!+k = p! \qquad 1 \le k \le m$

However: $m!+k \leq m!+m$ for $m>1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m+1)$$

$$= (m+1)!$$

$$\Downarrow$$

$$m!+k < (m+1)!$$

$$\Downarrow$$

$m!+k \neq p!$ for any p

Therefore: $a^{m!+k} \in L$

BUT: $L = \{a^{n!} : n \geq 0\}$ and $1 \leq k \leq m$

$\Downarrow$

$a^{m!+k} \notin L$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Closure under Other Operations

Homomorphism

Suppose Σ and Γ are alphabets.
Then a function

$$f : \Sigma \rightarrow \Gamma^*$$

is called a homomorphism.

Homomorphism

Extended definition (for strings):

$$f : \Sigma^* \to \Gamma^*$$

$$w = a_1 a_2 \ldots a_n$$

$$h(w) = h(a_1 a_2 \ldots a_n) = h(a_1) h(a_2) \ldots h(a_n)$$

Homomorphism

If L is a language on Σ, then its homomorphic image is defined as:

$$h(L) = \{h(w) : w \in L\}$$

Homomorphism

If L is a language on Σ, then its **homomorphic image** is defined as:

$$h(L) = \{ h(w) : w \in L \}$$

Example

$$\Sigma = \{a, b\} \qquad \Gamma = \{a, b, c\}$$

$$\begin{cases} h(a) = ab \\ h(b) = bbc \end{cases}$$

$$h(aba) = abbbcab$$

$$L = \{aa, aba\} \Rightarrow h(L) = \{abab, abbbcab\}$$

Homomorphism

If L is the language on Σ of a regular expression r,

then

the regular expression for $h(L)$ is obtained by applying the homomorphism to each Σ symbol of r.

Example

$$\Sigma = \{a, b\} \qquad \Gamma = \{b, c, d\}$$

$$\begin{cases} h(a) = dbcc \\ h(b) = bdc \end{cases}$$

$$r = (a + b^*)(aa)^*$$

$$L = L(r)$$

$$h(r) = (dbcc + (bdc)^*)(dbccdbcc)^*$$

Homomorphism

The family of regular languages is closed under homomorphism:

If L is regular, then so is $h(L)$.

Proof:

Let $L(r) = L$ for some regular expression r.

Prove: $h(L(r)) = L(h(r))$.

Right Quotient

Let L_1 and L_2 be languages on the same alphabet.

Then the right quotient of L_1 with L_2 is defined as:

$$L_1 / L_2 = \{x : xy \in L_1 \wedge y \in L_2\}$$

Right Quotient

$$L_1 / L_2 = \{x : xy \in L_1 \wedge y \in L_2\}$$

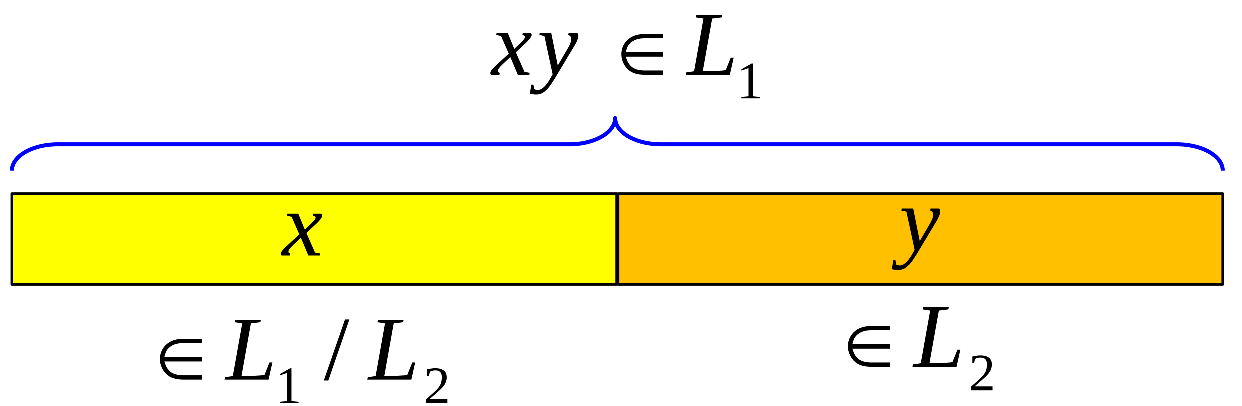
$$\overbrace{\underbrace{\boxed{\quad x \quad}}_{\in L_1 / L_2}\underbrace{\boxed{\quad y \quad}}_{\in L_2}}^{xy \in L_1}$$

Right Quotient

$$L_1 = \{a^n b^m \mid n \ge 1, m \ge 0\} \bigcup \{ba\}$$

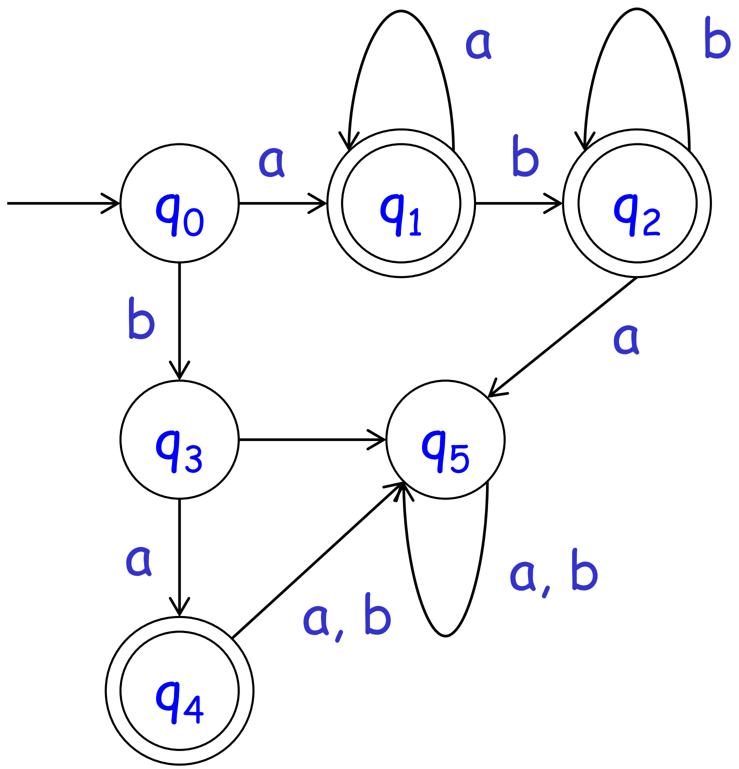
$$L_2 = \{b^m \mid m \ge 1\}$$

$$L_1 / L_2 = \{a^n b^m \mid n \ge 1,\ m \ge 0\}$$



Example

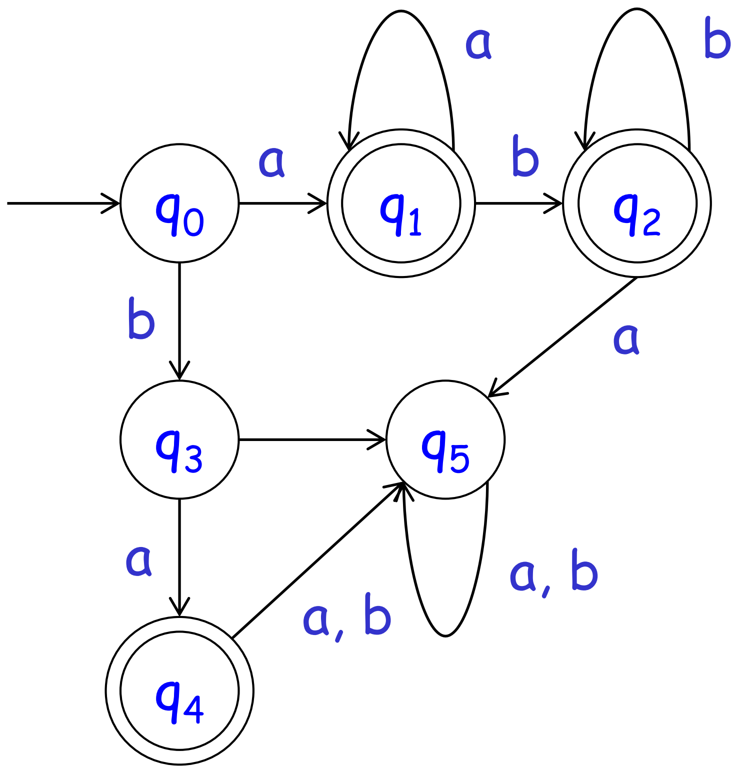
$$L_1 = \{a^n b^m \mid n \geq 1, m \geq 0\} \bigcup \{ba\}$$



Example

$$L_1 = \{a^n b^m \mid n \geq 1, m \geq 0\} \bigcup \{ba\}$$

$$L_2 = \{b^m \mid m \geq 1\}$$



$$\delta^*(q_0, x) = q_i$$

$$\delta^*(q_i, y) \in F \text{ and } y \in L_2$$

Example

$$L_1 = \{a^n b^m \mid n \ge 1, m \ge 0\} \cup \{ba\}$$

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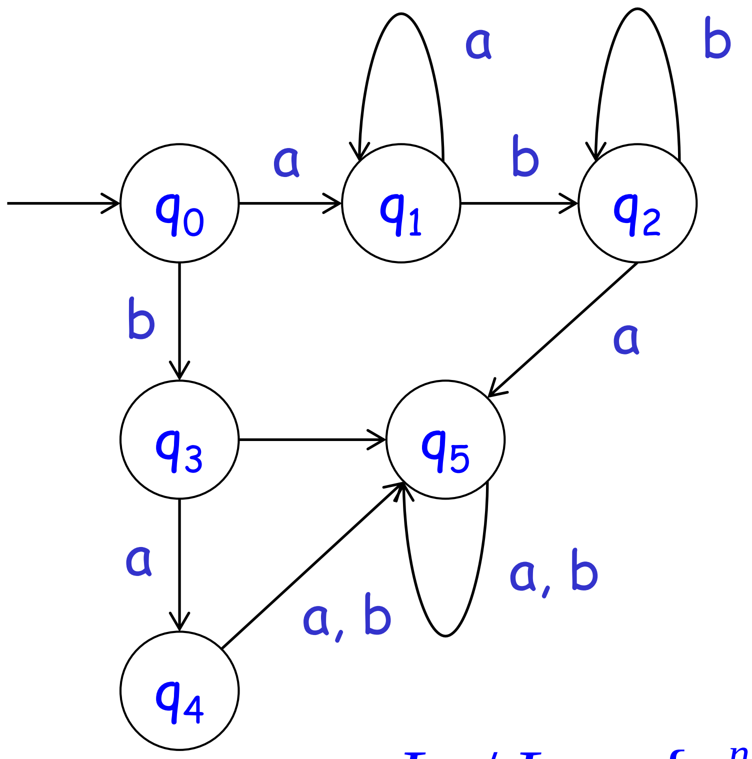


$$\delta^*(q_0, x) = q_i$$

$$\delta^*(q_i, y) \in F \text{ and } y \in L_2$$

$$L_1 / L_2 = \{a^n b^m \mid n \ge 1,\ m \ge 0\}$$

Right Quotient

The family of regular languages is closed under right quotient:

If L_1 and L_2 are regular, then so is L_1 / L_2.

Proof

If L_1 and L_2 are regular, we can find a dfa for L_1 / L_2:

$M = (Q, \Sigma, \delta, q_0, F)$ accepts L_1

$\hat{M} = (Q, \Sigma, \delta, q_0, \hat{F})$ accepts L_1 / L_2

If $y \in L_2$ and $\delta^*(q_i, y) \in F$

Then: add q_i to $\hat{F}$

$$M_i = (Q, \Sigma, \delta, q_i, F) \quad , L(M_i) \cap L_2 \neq \emptyset$$

Example

$L_1 = L(a^*baa^*)$

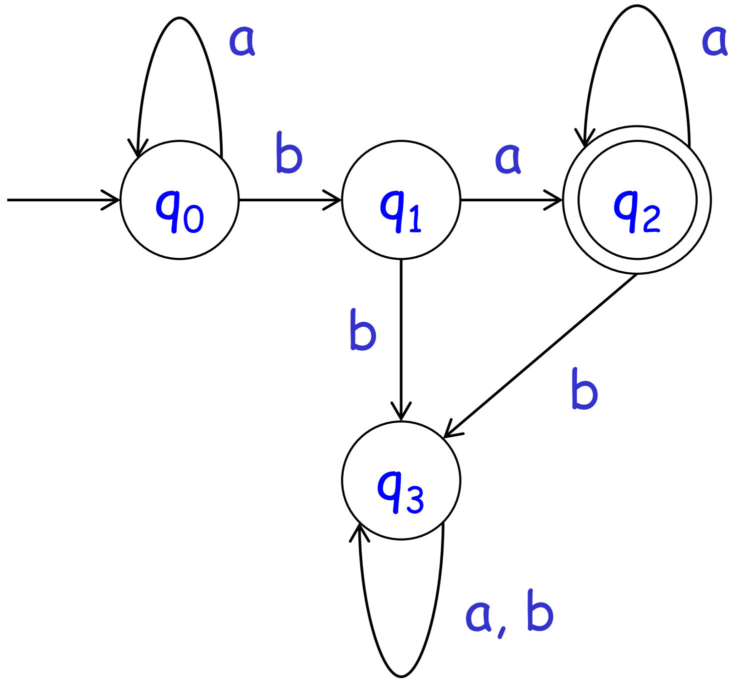
$L_2 = L(ab^*)$

$$L_1 / L_2 = ?$$

Example

$$L_1 = L(a^* baa^*)$$

$$L_2 = L(ab^*)$$



$L(M_0) \cap L_2 = \varnothing$

$L(M_1) \cap L_2 = \{a\}$

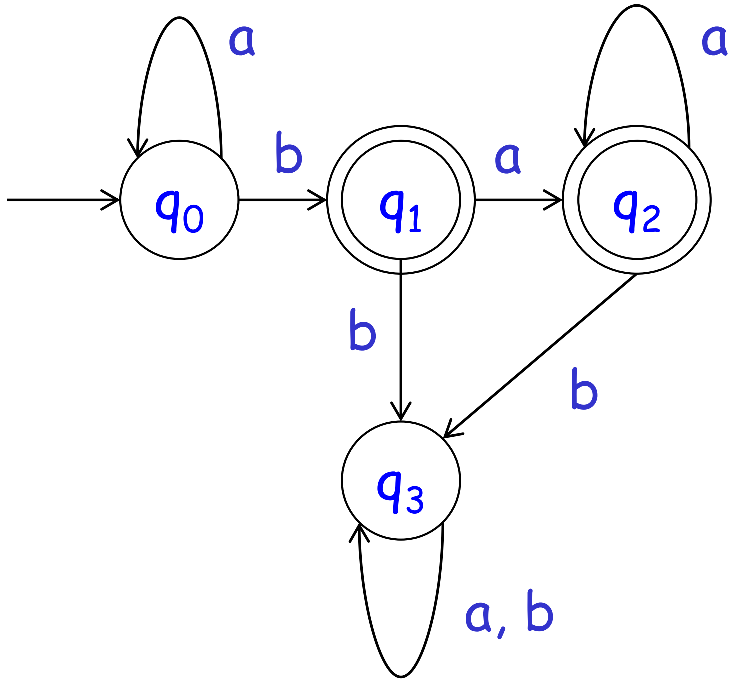
$L(M_2) \cap L_2 = \{a\}$

$L(M_3) \cap L_2 = \varnothing$

Example

$$L_1 = L(a^* baa^*)$$

$$L_2 = L(ab^*)$$



$L(M_0) \cap L_2 = \emptyset$

$L(M_1) \cap L_2 = \{a\}$

$L(M_2) \cap L_2 = \{a\}$

$L(M_3) \cap L_2 = \emptyset$