Chapter 1

1) $\forall w \in \Sigma^* \implies (w^R)^R = w$

$w = a_1 a_2 \cdots a_n \quad n \geq 0$

$a_i \in \Sigma$

اثبات با استقرا روی طول رشته $|w|$

induction basis : $\underline{|w| = 0} \Rightarrow w = \lambda \Rightarrow w^R = \underbrace{\lambda^R}_{\lambda} \Rightarrow (w^R)^R = \lambda^R = \lambda$ ✓

induction hypothesis : $\underline{|w| = k} \Rightarrow (w^R)^R = w$

فرض

induction step : $\underline{|w| = k+1} \Rightarrow (w^R)^R = w$

باید اثبات شود

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proof :

برهان

$|w| = k+1 \quad , k \geq 0 \Rightarrow w = ua \ , |u| = k \ , a \in \Sigma$

$(w^R) = (ua)^R = au^R$.

$(u^R)^R = u$ فرض استقرا

$(w^R)^R = (au^R)^R = (u^R)^R a = ua = w$

و حکم اثبات می‌شود.

۲) $L = L^*$?

where $L = L^*$?

آ) $L_1 = \{a^n b^{n+1} : n \geq 0\}$

$L_2 = \{w : n_a(w) = n_b(w)\}$

ب)

ب) $\boxed{L = L^2 ,\ \lambda \in L \Rightarrow L = L^*}$

شرط لازم و کافی

$L^0 = \{\lambda\}$

$L^1 = L = L^2$

$L^2 = L$

$L^3 = L^2 = L$

$\vdots$

$L^i = L \quad i > 0$

$L^* = \bigcup_{i=0}^{\infty} L^i = \underbrace{\{\lambda\} \cup L}_{\lambda \in L} = L$

$L_1^2 = \{uv : u \in L_1, v \in L_1\}$

$= \{a^n b^{n+1} a^m b^{m+1} : n, m \geq 0\}$

$\neq L_1$

آ) $\lambda \in L_1$ ✗

$L_1 = L_1^2$?

$\begin{cases} \lambda \in L_1^* \\ \lambda \notin L_1 \end{cases} \Rightarrow L_1 \neq L_1^*$

$L_2 \qquad \lambda \in L_2$ ✓

$L_2 = L_2^2$?

$L_2^2 = \{wv : n_a(w) = n_b(w) \wedge n_a(v) = n_b(v)\}$

$= \{u : n_a(u) = n_b(u)\}$

$= L_2$ ✓

$\therefore L_2 = L_2^*$ ✓

۳) $\exists L \subseteq \Sigma^* (\overline{L^*} = \overline{L}^*)$ × False.

$$\begin{cases} \lambda \in L^* \Rightarrow \lambda \notin \overline{L^*} \\ \lambda \in \overline{L}^* \end{cases} \Rightarrow \overline{L}^* \neq \overline{L^*}$$

$$\overline{\underset{M}{L^*}} = \Sigma^* - \underset{M}{L^*}$$

f) $\forall L_1, L_2 \subseteq \Sigma^* \Rightarrow (L_1 L_2)^R = L_2^R L_1^R$

$$
\begin{aligned}
L_2^R L_1^R &= \{v^R : v \in L_2\}\{w^R : w \in L_1\} \\
&= \{v^R w^R : v \in L_2 \wedge w \in L_1\} \\
&= \{(wv)^R : v \in L_2 \wedge w \in L_1\} \\
&= \{(wv)^R : wv \in L_1 L_2\} \\
&= \{u^R : u \in L_1 L_2\} \\
&= (L_1 L_2)^R \quad \checkmark
\end{aligned}
$$

۵) $\Sigma = \{a, b\}$

$G = (V, T, S, P)$

آ) $L_1 = \{w \in \Sigma^* : n_a(w) = 1\}$

$S \rightarrow BaB$

تولید b ها → $B \rightarrow Bb \mid \lambda$

$G_1 = (\{S, B\}, \{a, b\}, S, P)$

ب) $L_2 = \{w \in \Sigma^* : n_a(w) \geq 1\}$

$S \rightarrow BaB$

هر ترکیبی از a و b → $B \rightarrow aB \mid bB \mid \lambda$

$G_2 = (\{S, B\}, \{a, b\}, S, P)$

ج) $L_3 = \{w \in \Sigma^* : 0 \leq n_a(w) \leq 3\}$

$S \rightarrow BaB \mid BaBaB \mid BaBaBaB \mid B$

$B \rightarrow Bb \mid \lambda$

$G_3 = (\{S, B\}, \{a, b\}, S, P)$

4) $L_i = L(G_i)$

آ) $L_1 = \{a^n b^m : n \geq 0, m > n\}$

$S_1 \longrightarrow aS_1B \mid B$

$B \longrightarrow bB \mid b$

ب) $L_2 = \{a^n \overbrace{b^{2n}}^{(bb)^n} : n \geq 0\}$

$S_2 \longrightarrow aS_2bb \mid \lambda$

پ) $L_3 = \{a^{n+2} b^m : n \geq 1\}$

$S_3 \longrightarrow aaaAB'$
$A \longrightarrow aA \mid \lambda$
$B' \longrightarrow bB' \mid \lambda$

ت) $L_4 = \{a^n b^{n-3} : n \geq 3\}$
$= \{a^{m+3} b^m : m \geq 0\}$

$S_4 \longrightarrow aaaA'$
$A' \longrightarrow aA'b \mid \lambda$

ث) $L_5 = L_1 L_2$ $S_5 \longrightarrow S_1 S_2$

ج) $L_6 = L_1 \cup L_2$

$S_6 \longrightarrow S_1 \mid S_2$

چ) $L_7 = L_1^3$

$S_7 \longrightarrow S_1 S_1 S_1$

ح) $L_8 = L_1^*$

$S_8 \longrightarrow S_8 S_1 \mid \lambda$

یا

$S_8 \longrightarrow S_1 S_8 \mid \lambda$

خ) $L_9 = L_1 - \overline{L_4}$
$= L_1 \cap L_4$
$= \emptyset$

$S \longrightarrow S$

v) $L = \{ww^R : w \in \{a,b\}^+\}$

$S \longrightarrow aSa \mid bSb \mid aa \mid bb$

$G = (V, T, S, P)$

$= (\{S\}, \{a,b\}, S, P)$

1)

$$S \longrightarrow Aa$$
$$A \longrightarrow B$$
$$B \longrightarrow Aa$$

$G = (V, T, S, P)$

$T = \{a\}$, $V = \{A, B, S\}$

$L(G) = \emptyset$