Problem 1

(a)

$$x(t) = e^{-t}u(-t) + 2e^{-2t}u(t)$$

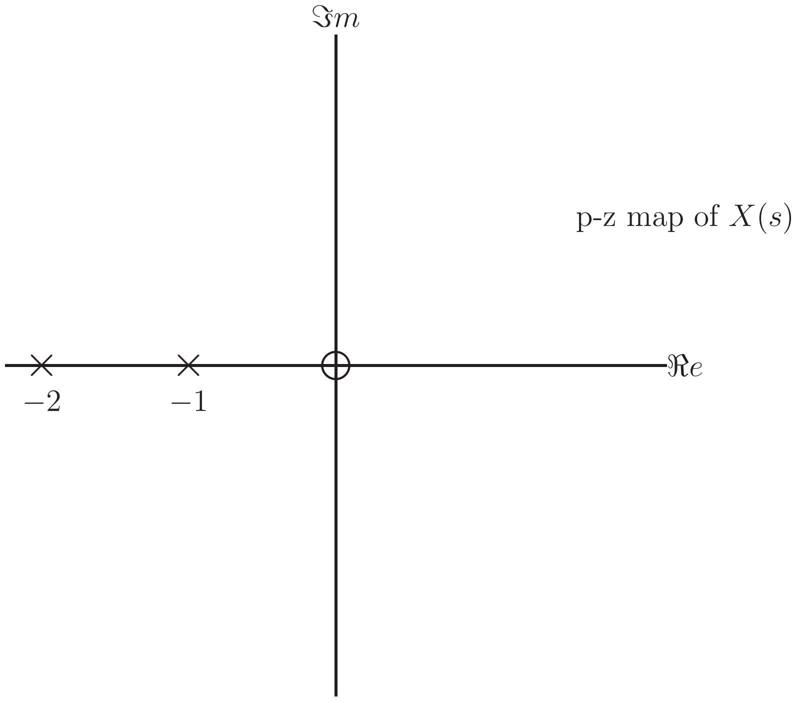
Using Laplace transforms of elementary functions (table 9.2), we find,

$$e^{-t}u(-t) \longleftrightarrow \frac{-1}{s+1}, \qquad R_1 = \mathcal{R}e\{s\} < -1$$
$$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}, \qquad R_2 = \mathcal{R}e\{s\} > -2$$

Using the linearity property,

$$\begin{aligned} X(s) &= \frac{-1}{s+1} + \frac{2}{s+2}, && \text{ROC containing } R_1 \bigcap R_2 \\ &= \frac{s}{(s+1)(s+2)} && \text{ROC} = -2 < \mathcal{R}e\{s\} < -1 \end{aligned}$$

The pole-zero plot is shown below:



(b)

$$\begin{aligned} x(t) &= (e^t \cos t)u(-t) + u(-t) \\ &= \left(e^t(\frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt})\right) u(-t) + u(-t) \\ &= \frac{1}{2}e^{(1+j)t}u(-t) + \frac{1}{2}e^{(1-j)t}u(-t) + u(-t) \end{aligned}$$

Using Laplace transforms of elementary functions (table 9.2), we find,

$$\begin{aligned} e^{(1+j)t}u(-t) &\longleftrightarrow \frac{-1}{s-(1+j)}, \qquad R_1 = \mathcal{R}e\{s\} < 1 \\ e^{(1-j)t}u(-t) &\longleftrightarrow \frac{-1}{s-(1-j)}, \qquad R_2 = \mathcal{R}e\{s\} < 1 \\ u(-t) &\longleftrightarrow \frac{-1}{s} \qquad R_3 = \mathcal{R}e\{s\} < 0 \end{aligned}$$

Using the linearity property,

$$\begin{aligned} X(s) &= \frac{-1/2}{s-(1+j)} + \frac{-1/2}{s-(1-j)} + \frac{-1}{s}, \qquad \text{ROC containing } R_1 \bigcap R_2 \bigcap R_3 \\ &= \frac{-(1/2)s^2 + (1/2)s(1-j) - (1/2)s^2 + (1/2)s(1+j) - s^2 + 2s - 2}{s(s^2 - 2s + 2)} \qquad \text{ROC=} \ \mathcal{R}e\{s\} < 0 \\ &= \frac{-1(2s^2 - 3s + 2)}{s(s^2 - 2s + 2)} \qquad \text{ROC=} \ \mathcal{R}e\{s\} < 0 \end{aligned}$$

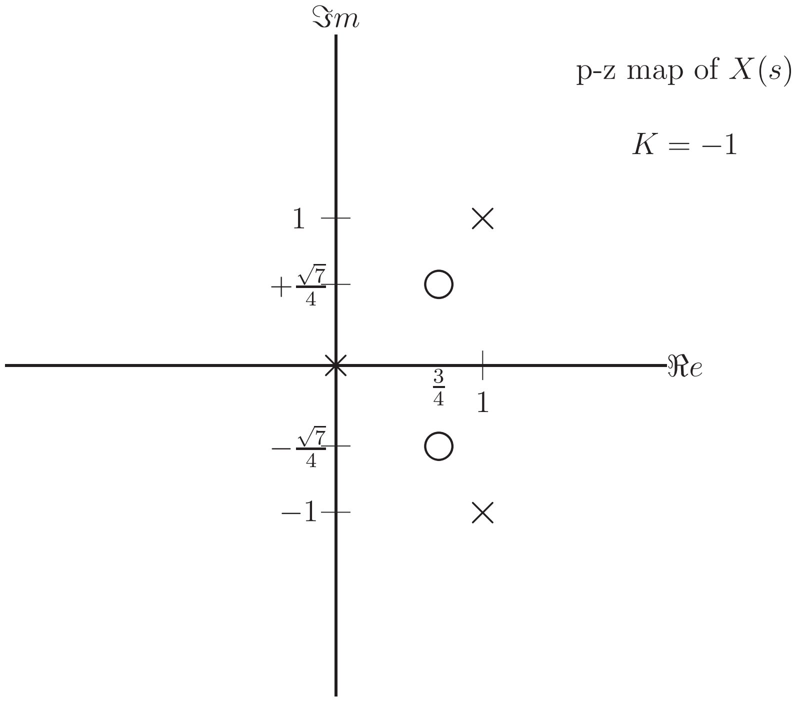
Solving the denominator, we see that poles are located at,

$$\begin{aligned} s &= 1 \pm j \\ s &= 0 \end{aligned}$$

Solving the numerator, we see that zeros are located at,

$$s = \frac{3}{4} \pm j\frac{\sqrt{7}}{4}$$

The pole-zero plot is shown below:



Problem 2

(a) We are given,

$$X(s) = \frac{s-25}{s^2 - s - 12} = \frac{s-25}{(s-4)(s+3)} \qquad -3 < \mathcal{R}e\{s\} < 4$$

Using partial fraction expansion:

$$\frac{s-25}{(s-4)(s+3)} = \frac{A}{s-4} + \frac{B}{s+3}$$

multiply both sides by $(s-4)$ and plug-in $s = 4$,

$$\begin{aligned} \frac{s-25}{s+3} &= A \\ A &= -3 \end{aligned}$$

multiply both sides by $(s+3)$ and plug-in $s = -3$,

$$\begin{aligned} \frac{s-25}{s-4} &= B \\ B &= 4 \end{aligned}$$

Therefore,

$$X(s) = \frac{-3}{s-4} + \frac{4}{s+3} \qquad -3 < \mathcal{R}e\{s\} < 4$$

Using the table of Laplace transforms for elementary functions (table 9.2) and given ROC, we find

$$x(t) = 3e^{4t}u(-t) + 4e^{-3t}u(t)$$

(b) We are given,

$$X(s) = \frac{2s^2+7s+9}{(s+2)^2} = \frac{2s^2+7s+9}{s^2+4s+4} \qquad \mathcal{R}e\{s\} > -2$$

As the degree of the numerator is equal to that of the denominator, we need to use long division to divide the numerator by the denominator before finding the partial fraction expansion:

$$X(s) = 2 - \frac{s-1}{(s+2)^2}$$

Now we can find the partial fraction expansion of the second term on the right hand side of the equality:

$$\frac{s-1}{(s+2)^2} = \frac{A}{(s+2)^2} + \frac{B}{s+2}$$

multiply both sides by $(s+2)^2$ and plug-in $s=-2$,

$$\begin{aligned} s-1 &= A + B(s+2) \\ A &= -3 \end{aligned}$$

multiply both sides by $(s+2)^2$, plug-in A and $s=1$,

$$\begin{aligned} s-1 &= A + B(s+2) \\ B &= 1 \end{aligned}$$

Therefore,

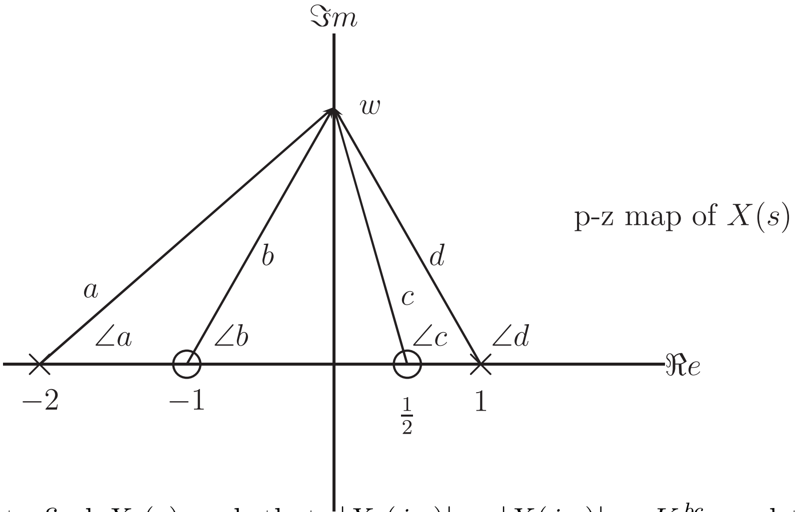
$$\begin{aligned} X(s) &= 2 - \left(\frac{-3}{(s+2)^2} + \frac{1}{s+2}\right) \qquad \mathcal{R}e\{s\} > -2 \\ &= 2 + \frac{3}{(s+2)^2} - \frac{1}{s+2} \qquad \mathcal{R}e\{s\} > -2 \end{aligned}$$

Using the table of Laplace transforms for elementary functions (table 9.2) and given ROC, we find

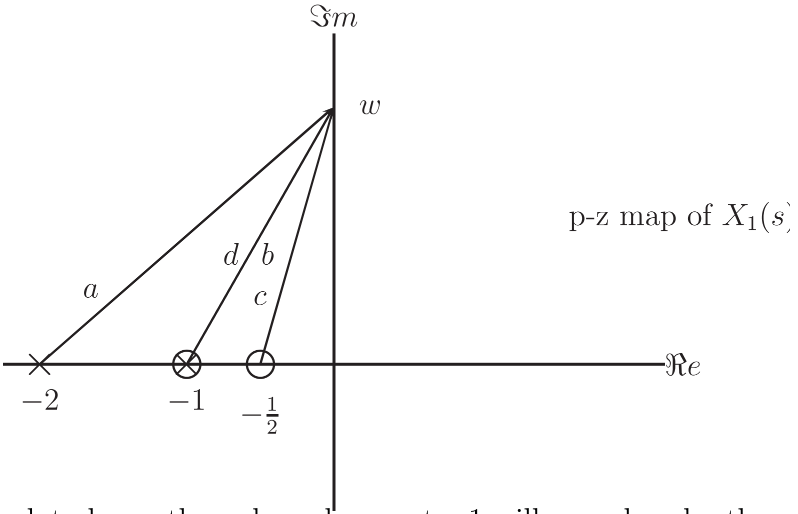
$$x(t) = 2\delta(t) + 3te^{-2t}u(t) - e^{-2t}u(t)$$

Problem 3 **(O & W 9.24 (f))**

We are given $X(s)$. $|X(jw)|$ for any w can be calculated as $K\frac{bc}{ad}$ where K is any real number and a, b, c, d are vector magnitudes as shown below in the pole-zero diagram of $X(s)$.



We need to find $X_1(s)$ such that, $|X_1(jw)| = |X(jw)| = K\frac{bc}{ad}$, and there are no poles and zeros in the right-half plane.
Reflecting the pole (at 1 on real axis) and zero (at $\frac{1}{2}$ on real axis) along jw-axis or imaginary axis, we can conserve the magnitude d and c from the pole and zero respectively. The resulting pole-zero diagram will be as follows:



From the plot above, the pole and zero at -1 will cancel each other. Therefore,

$$X_1(s) = \frac{K(s+\frac{1}{2})}{(s+2)}$$

It is important to note that, from p-z map of $X(s)$, as $b = d$,

$$|X(jw)| = K\frac{bc}{ad} = K\frac{c}{a} = |X_1(jw)|$$

Now, we need to find $X_2(jw)$ such that $\angle X_2(jw) = \angle X(jw)$ and there are no poles or zeros in the right-half plane of p-z map of $X_2(s)$. From the p-z map of $X(s)$ shown on the previous page, we can write the phase, $\angle X(jw)$, as

$$\angle X(jw) \;=\; \angle b + \angle c - \angle a - \angle d$$

If we reflect the pole at $s = 1$ along jw-axis, the contribution to overall phase from the reflected pole becomes $-(\pi - \angle d) = -\pi + \angle d$. Notice that the sign of contributed angle ($\angle d$) has flipped. Now, lets convert that pole to a zero. The contribution to overall phase from the resulting zero is $+(\pi - \angle d) = \pi - \angle d$.
The above two operations, reflecting a pole along the jw-axis and changing it to zero, just add $+\pi$ to overall phase. In order to keep the phase unchanged, we can multiply the resulting Laplace transform by -1 as $-1 = e^{-j\pi}$ will subtract π from the overall phase.

Similarly, if we reflect the zero at $s = \frac{1}{2}$ along jw-axis, change that zero to a pole, and multiply resulting the Laplace transform with -1, the phase will remain unchanged.

Therefore,

$$\begin{aligned} X_2(s) &= K(-1)(-1)\frac{(s+1)(s+1)}{(s+2)(s+\frac{1}{2})} \\ &= \frac{K(s+1)^2}{(s+2)(s+\frac{1}{2})} \end{aligned}$$

Problem 4 **(O & W 9.26)**

We need to find the Laplace transform of $y(t)$ using the properties of Laplace transform. The two properties, time shifting (table 9.1 of text book) and time scaling by -1 (page $686 - 687$ of text book), that we will need are:

$$\begin{aligned} x(t-t_0) &\longleftrightarrow e^{-st_0}X(s) \\ x(-t) &\longleftrightarrow X(-s) \end{aligned}$$

The ROC is unchanged for the time shifting property. For the time scaling by -1 property, ROC is reflected about the $j\omega$ or imaginary axis in s-plane.

$$\begin{aligned} y(t) &= x_1(t-2) * x_2(-t+3) \\ \text{where} \quad x_1(t) &= e^{-2t}u(t) \\ \text{and} \quad x_2(t) &= e^{-3t}u(t) \end{aligned}$$

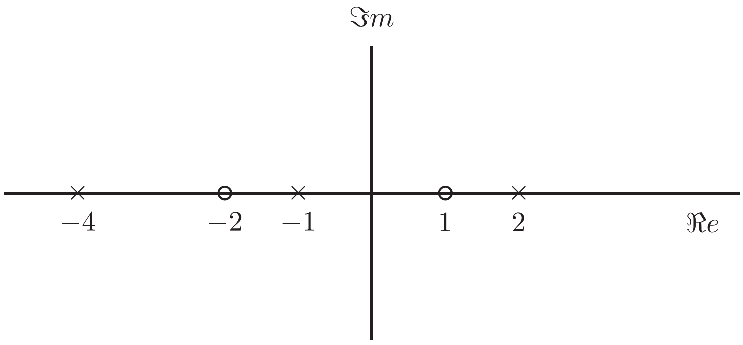
taking the Laplace transform

$$\begin{aligned} x_1(t) &\longleftrightarrow \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2 \\ x_1(t-2) &\longleftrightarrow e^{-2s}\frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2 \\ x_2(t) &\longleftrightarrow \frac{1}{s+3}, \qquad \mathcal{R}e\{s\} > -3 \\ x_2(-t) &\longleftrightarrow \frac{1}{-s+3}, \qquad \mathcal{R}e\{s\} < 3 \\ x_2(-(t-3)) &\longleftrightarrow e^{-3s}\frac{1}{-s+3}, \qquad \mathcal{R}e\{s\} < 3 \end{aligned}$$

Using the convolution property of Laplace transform (table 9.1),

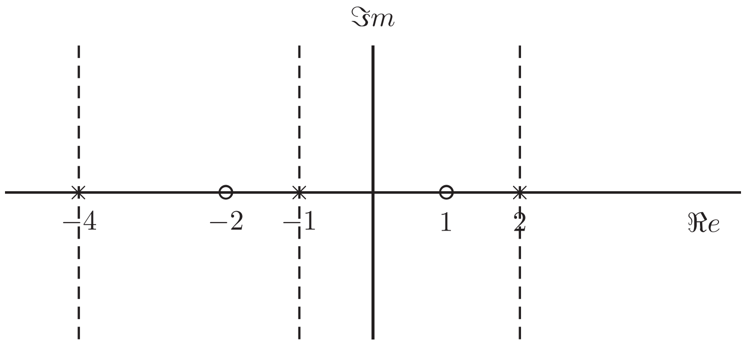
$$\begin{aligned} Y(s) &= X_1(s)X_2(s), \qquad \text{at least } R_1 \cap R_2 \\ &= \frac{e^{-2s}}{s+2} \times \frac{e^{-3s}}{-s+3} \\ Y(s) &= \frac{-e^{-5s}}{s^2-s-6}, \qquad -2 < \mathcal{R}e\{s\} < 3 \end{aligned}$$

Problem 5 Consider an LTI system for which the system function $H(s)$ is rational and has the pole-zero pattern shown below:



(a) Indicate all possible ROC's that can be associated with this pole-zero pattern.

The ROCs are bounded by vertical lines through the location of the poles as in the figure below:



Thus, the possible ROCs are:

$$\begin{aligned} & \Re e\{s\} < -4 \\ -4 < & \Re e\{s\} < -1 \\ -1 < & \Re e\{s\} < 2 \\ 2 < & \Re e\{s\} \end{aligned}$$

(b) For each ROC identified in Part (a), specify whether the associated system is stable and/or causal.

Causal systems have ROCs that are to the right of the right-most pole. Stable systems are systems whose ROCs include the $j\omega$-axis.

$\Re e\{s\} < -4$: Not Causal. Not Stable

$-4 < \Re e\{s\} < -1$: Not Causal. Not Stable

$-1 < \Re e\{s\} < 2$: Not Causal. Stable

$2 < \Re e\{s\}$: Causal. Not Stable

Problem 6 Draw a direct-form representation for the causal LTI system with system function

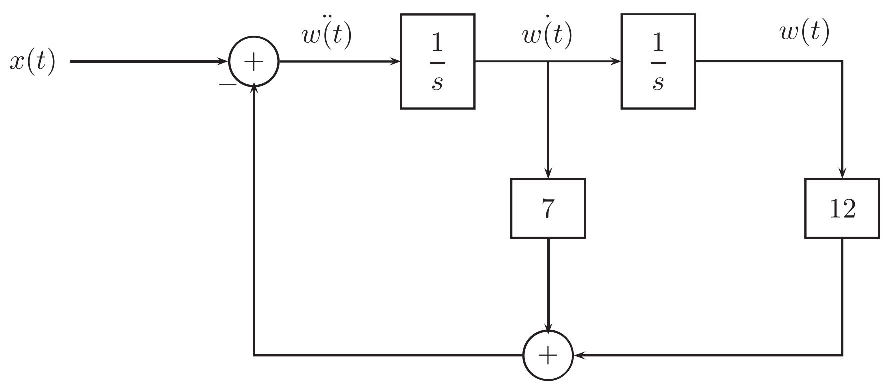
$$H(s) = \frac{s(s+1)}{(s+3)(s+4)}.$$

Note that the system function can be represented as follows:

$$\begin{aligned} H(s) &= \frac{s(s+1)}{(s+3)(s+4)} \\ &= \frac{s^2+s}{s^2+7s+12} \\ &= \frac{Y(s)}{W(s)}\frac{W(s)}{X(s)} \\ &= \underbrace{s^2+s}_{\frac{Y(s)}{W(s)}}\underbrace{\frac{1}{s^2+7s+12}}_{\frac{W(s)}{X(s)}}. \end{aligned}$$

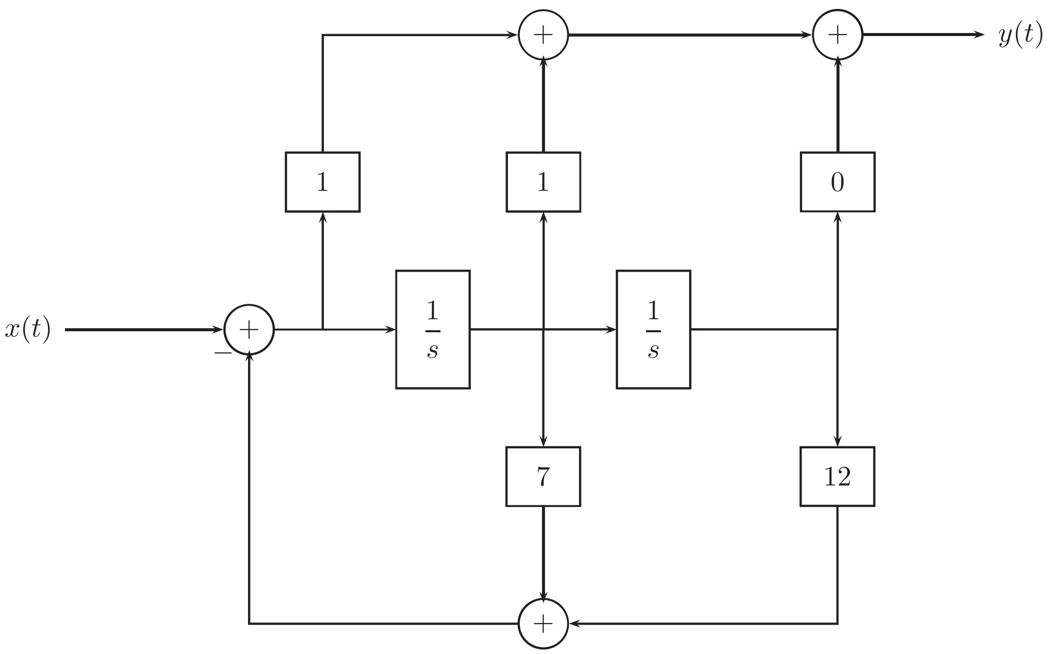
Thus, we can see the system $H(s)$ as a cascade of two systems, i.e., $Z(s) = \frac{Y(s)}{W(s)}$ which accounts for the zeros and $P(s) = \frac{W(s)}{X(s)}$ which accounts for the poles.

First, let's draw a block diagram representation of the system $P(s)$. Since the system is of second order, we would like to represent the system using only two integrators in cascade.



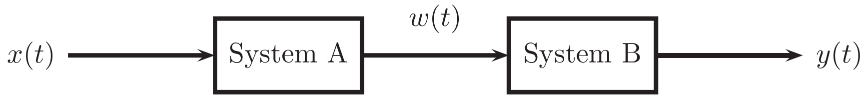
Here, note that $\ddot{w}(t) = \frac{d^2w(t)}{dt^2}$ and $\dot{w}(t) = \frac{dw(t)}{dt}$. It is easy to confirm that the above implementation of the system indeed represents the system $P(s)$ by recursively applying Black's formula.

Now, we would like to use the system above to implement the other system, $Z(s)$. Note that s in Laplace domain corresponds to differentiation in time domain. Thus, $y(t)$ is nothing but a linear combination of different orders of derivatives of $w(t)$; in our case, $y(t) = 1\times\ddot{w}(t)+1\times\dot{w}(t)+0\times w(t)$. Thus, a direct form representation of the overall system $H(s)$ is as shown below:



From this representation of the system, the numbers in the gain boxes are picked off simply from the coefficients of the numerator and denominator for a given rational system function. Often, when the gain terms are 1, they are omitted. When the gain terms are 0, the branch is often omitted.

Problem 7 Consider the cascade of two LTI systems as depicted below:



where we have the following:

- System A is causal with impulse response

$$h(t) = e^{-2t}u(t)$$

- System B is causal and is characterized by the following differential equation relating its input, $w(t)$, and output, $y(t)$:

$$\frac{dy(t)}{dt} + y(t) = \frac{dw(t)}{dt} + \alpha w(t)$$

- If the input $x(t) = e^{-3t}$, the output $y(t) = 0$.

1. Find the system function $H(s) = Y(s)/X(s)$, determine its ROC and sketch its pole-zero pattern. Note: Your answer should only have numbers in them (i.e., you have enough information to determine the value of α).

2. Determine the differential equation relating $y(t)$ and $x(t)$.

Solutions:

1. The overall system function $H(s)$ is $H_A(s) \times H_B(s)$ since the systems' A and B are cascaded together. From O&W Table 9.2 Laplace transforms of elementary functions,

$$H_A(s) = \frac{1}{s+2}, \Re e\{s\} > -2.$$

$H_B(s)$ is determined by letting $x(t) = e^{st}$ in the differential equation given for system B. Then $y(t) = H(s)e^{st}$ and we find

$$H_B(s) = \frac{s+\alpha}{s+1}, \Re e\{s\} > -1.$$

The ROC was known to be $\Re e\{s\} > -1$ and not $\Re e\{s\} < -1$ because we are told system B is causal. We need to solve for α. Along with the differential equation relating the input to the output for system B, we are told that if $x(t) = e^{-3t}$, an eigenfunction of an LTI system, then $y(t) = 0$.

Here, the concept we wished to test was the eigenfunction property of LTI systems. However, this problem was ill-posed. Namely, in order to apply the eigenfunction property, -3 from e^{-3t}, should have been in the ROC of $H(s)$ so that $H(-3)$ had existed. However, since the ROC of $H(s)$ is $\Re e\{s\} > -1$, $H(-3)$ does not exist. Therefore, you did not have enough information to determine $H(s)$ for this problem. The solution below is just to illustrate a possible method to obtain $H(s)$ assuming that $H(-3)$ existed.

Thus, $H_B(s)|_{s=-3} = 0$. This constraint allows us to solve for α. Specifically,

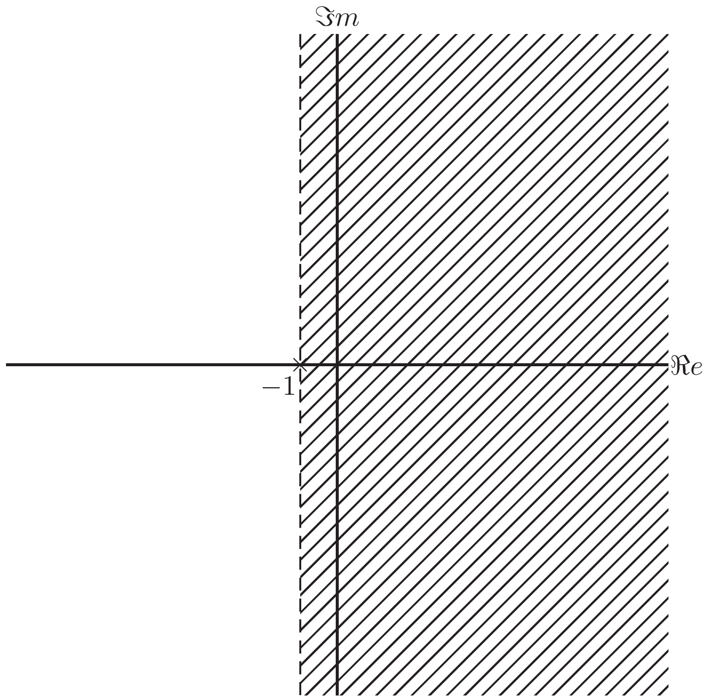
$$H_B(-3) = \frac{-3+\alpha}{-3+1} = 0 \longrightarrow \alpha = 3.$$

Therefore, the overall cascaded system function is

$$H(s) = \frac{s+3}{(s+1)(s+2)}, \Re e\{s\} > -1$$

The ROC must not have any poles in it and so the overall ROC must be to the right of the all poles in the system.

The pole-zero plot is shown below.



2. We can determine the differential equation relating $y(t)$ and $x(t)$ from the system function found in (a). Since $H(s) = \frac{Y(s)}{X(s)}$, we multiply the denominator of $H(s)$ by $Y(s)$ and we multiply the numerator of $H(s)$ by $X(s)$:

$$Y(s)(s^2 + 3s + 2) = X(s)(s + 3).$$

Distributing on both sides:

$$s^2Y(s) + 3sY(s) + 2Y(s) = sX(s) + 3X(s)$$

Because of linearity, we can take the inverse Laplace transform of each term above and get the differential equation:

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} + 3x(t).$$

Problem 8 Suppose we are given the following information about a causal and stable LTI system with impulse response $h(t)$ and a rational function $H(s)$:

- The steady state response to a unit step, i.e., $s(\infty) = \frac{1}{3}$.
- When the input is $e^t u(t)$, the output is absolutely integrable.

- The signal
$$\frac{d^2h(t)}{dt^2}+5\frac{dh(t)}{dt}+6h(t)$$
is of finite duration.

- $h(t)$ has exactly one zero at infinity.

Determine $H(s)$ and its ROC.

Solution: To determine $H(s)$ and its ROC, we need to analyze and combine all the information given.

The first piece of information we are given is that the system is causal and stable. Because the system is causal, we know that the ROC is right-sided. Because the system is stable, we know that the ROC includes the $j\omega$-axis.

The next piece of information, *The steady state response to a unit step, i.e.,* $s(\infty)=\frac{1}{3}$, gives us information about $H(s)|_{s=0}$. Note that the step response is $s(t)=\int_{-\infty}^{t}h(\tau)\mathrm{d}\tau$. Thus, for $t=\infty$, $s(\infty)=\int_{-\infty}^{\infty}h(\tau)\mathrm{d}\tau$. But this is the same Laplace transform equation that can be used to solve for $H(s)|_{s=0}$. Thus, $H(0)=\frac{1}{3}$.

The next piece of information, *When the input is* $e^{t}u(t)$*, the output is absolutely integrable,* gives us information about a zero of $H(s)$. We know that if $x(t)=e^{t}u(t)$ then $X(s)=\frac{1}{s-1}, \Re e\{s\}>1$ and $Y(s)=H(s)X(s)$. The ROC for $Y(s)$ will be *at least* the intersection of the ROC for $X(s)$ with the ROC for $H(s)$. If $y(t)$ is absolutely integrable, then we can take a Fourier transform of it, i.e., the ROC includes the $j\omega$-axis. Because of Property 2 in Chapter 9 of O&W, the ROC of any system, $Y(s)$ included, does not include any poles. Thus, the pole in $Y(s)$ at $s=1$ is eliminated by having a zero at $s=1$ in $H(s)$.

Jumping to the fourth bullet point, *h(t) has exactly one zero at infinity,* gives us information about the relative orders of the numerator and denominator for a rational tranform. Specifically, the order of the denominator is one greater than the order of the numerator. Thus, we know that the denominator has two poles.

The third bullet point gives us information about the poles of $H(s)$. By Property 3 in Chapter 9 of O&W, if a signal is of finite duration and is absolutely integrable then the ROC of the signal is the entire s-plane. We know that $\dfrac{d^2h(t)}{dt^2}+5\dfrac{dh(t)}{dt}+6h(t)$ is of finite duration. Is it also absolutely integrable? We can show that it is by looking individually at each of the terms in the function. We know that $h(t)$ is absolutely integrable because it is stable, i.e., its ROC includes the $j\omega$-axis. Multiplying $h(t)$ by 6 (a constant) will not change its absolute integrability. From Table 9.1, the ROC of the deriviative of a function includes the ROC of the original function. Thus, $5\frac{\mathrm{d}h(t)}{\mathrm{d}t}$ will include the $j\omega$-axis and be absolutely integrable. Likewise the second derivative, $\frac{\mathrm{d}^2h(t)}{dt^2}$ will include the ROC of the first derivative and thus, it is absolutely integrable also. From Table 9.2, the sum of 3 functions has at least the intersection of the ROC's of each of the three functions. Since the ROC of each of these functions includes the $j\omega$-axis, the sum will include the $j\omega$-axis and thus, the sum is absolutely integrable. Because the ROC of this signal, $\frac{d^2h(t)}{dt^2}+5\frac{dh(t)}{dt}+6h(t)$ is the entire s-plane, there can be no poles except at ∞. However, we know that $H(s)$ has at least two poles. In order for the signal to have no poles, we must make sure that the poles of $H(s)$ are cancelled by zeros of the signal. Taking the Laplace transform of the signal gives us

$$s^2H(s)+5sH(s)+6H(s)=(s+2)(s+3)H(s).$$

Thus, $H(s)$ can only have two poles, at $s=-2$ and $s=-3$.
Combining all the information about the poles and zeros gives us,

$$H(s)=K\frac{s-1}{(s+2)(s+3)}, \Re e\{s\}>-2.$$

Finally, we choose K such that $H(0)=\frac{1}{3}$. That is, $K=-2$ and

$$H(s)=-2\frac{s-1}{(s+2)(s+3)}, \Re e\{s\}>-2.$$

Problem 5 Consider the basic feedback system of Figure 11.3 (a) on p.819 of O&W. Determine the closed-loop system impulse response when

$$H(s)=\frac{1}{s+5}, \quad G(s)=\frac{2}{s+2}.$$

We can use Black's Formula to compute the system function for the entire system, $Q(s)$. This is given by:

$$\begin{aligned} Q(s) &= \frac{\frac{1}{s+5}}{1+\frac{2}{(s+5)(s+2)}} \\ &= \frac{(s+2)}{s^2+7s+12} \end{aligned}$$

Next, we can use partial fraction expansion to write $Q(s)$ as a sum of 1st order terms.

$$\begin{aligned} Q(s) &= \frac{s+2}{s^2+7s+12} \\ &= \frac{s+2}{(s+4)(s+3)} \\ &= \frac{2}{s+4}-\frac{1}{s+3} \end{aligned}$$

Since this is a feedback system, we know it is causal. Thus, we find the inverse Laplace transform for the system function using the causal part to obtain the inpulse response as follows:

$$q(t)=2e^{-4t}u(t)-e^{-3t}u(t)$$