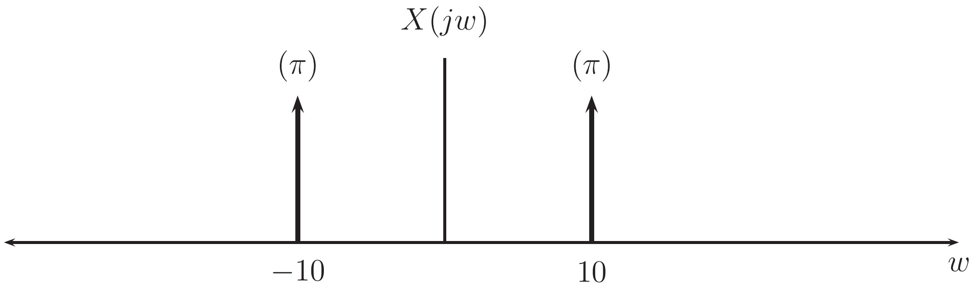
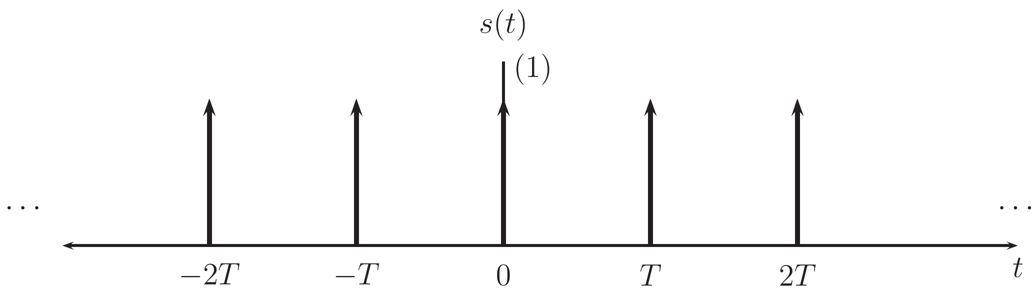
Problem 1

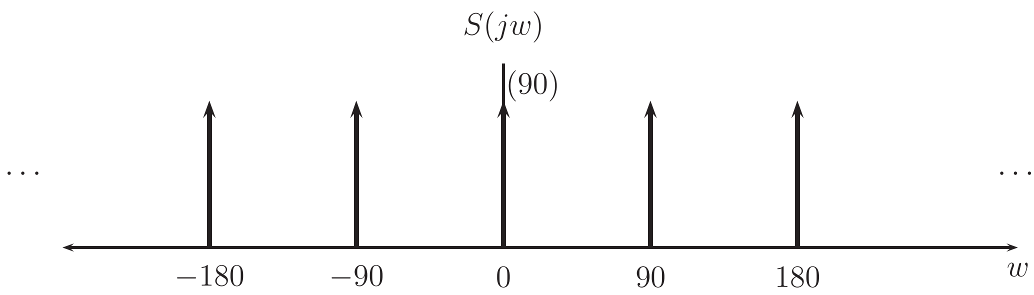
(a) We are given $x(t) = \cos(10t)$. Here, $\omega_o = 10$ rad/sec. Taking the Fourier transform of $x(t)$,



The sampling function, $s(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$, with $T = \frac{2\pi}{90}$.



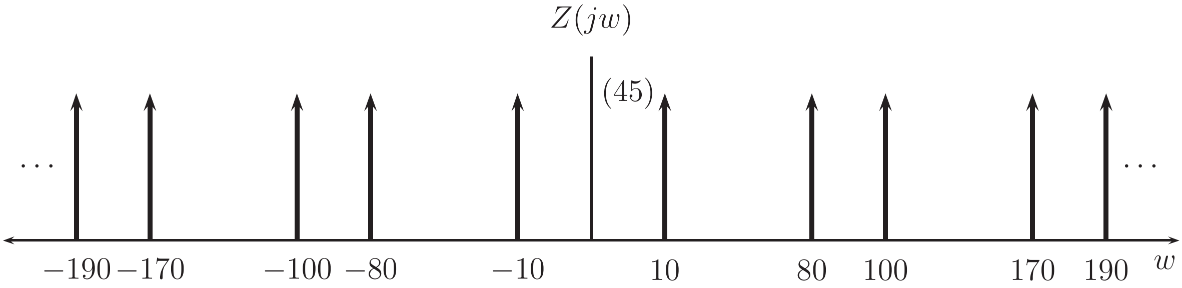
Taking the Fourier transform of $s(t)$ (note that $\omega_s = \frac{2\pi}{T} = 90$),



Using the multiplication property, $z(t) = x(t)s(t)$ in frequency domain is $Z(jw) = \frac{1}{2\pi}(X(jw) * S(jw))$, i.e. we need to convolve $X(jw)$ with the periodic impulse train in $S(jw)$ and scale the amplitude by $\frac{1}{2\pi}$ (see section 7.1.1 in O&W).

$$\begin{aligned} z(t) &= \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT) \\ Z(jw) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)S(j(w-\theta))d\theta \end{aligned}$$

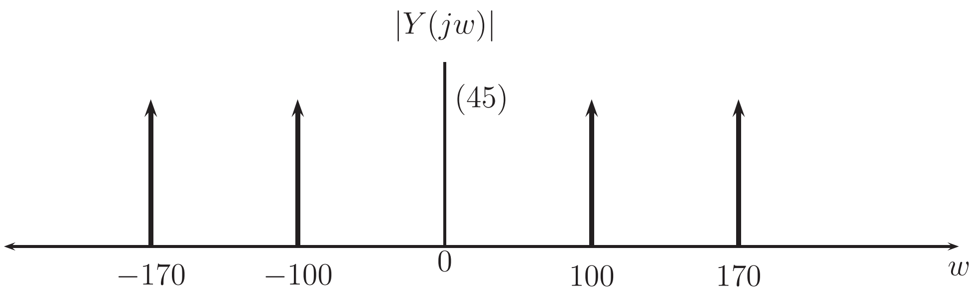
Therefore, $Z(jw)$ is as follows:



(b) $y(t)$ is the output from the band-pass filter, $H(jw)$, with input $z(t)$ as derived in part (a). We know,

$$Y(jw) = H(jw)Z(jw)$$

Let us consider $|Y(jw)|$ and $\angle Y(jw)$ separately. $|Y(jw)|$ is the band-pass filtered version of $|Z(jw)|$ with frequency components between 90 to 180 and -180 to -90 rad/sec.

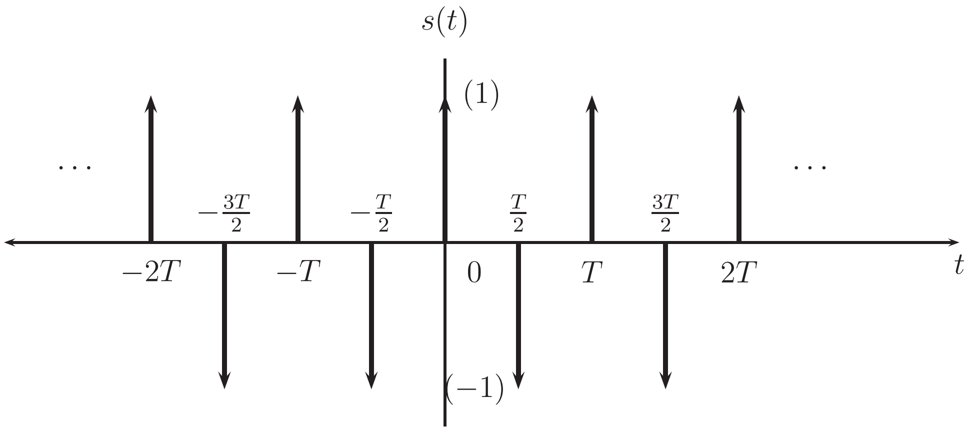


$$\begin{aligned}\angle Y(jw) &= \angle H(jw) + \angle Z(jw) \\ &= -\frac{\pi w}{200} + 0 = -\frac{\pi w}{200}\end{aligned}$$

Combining the magnitude and angle, $Y(jw) = |Y(jw)|e^{j\angle Y(jw)}$.
Consider $Y(jw)$ as the Fourier transform of the sum of two sinusoidal signals; one with $w_o = 100$ and another with $w_o = 170$. Using the time-shifting property of Fourier transform, $x(t - t_o) \overset{\mathcal{FT}}{\longleftrightarrow} e^{-jwt_o}X(jw)$,

$$\begin{aligned}y(t) &= \frac{45}{\pi}\cos(100(t - \frac{\pi}{200})) + \frac{45}{\pi}\cos(170(t - \frac{\pi}{200})) \\ &= \frac{45}{\pi}\cos(100t - \frac{\pi}{2}) + \frac{45}{\pi}\cos(170t - \frac{17\pi}{20})\end{aligned}$$

(c) Now the sampling function $s(t)$ is changed with $T = \frac{2\pi}{90}$,



$$s(t) \;=\; \sum_{k=-\infty}^{\infty} \delta(t-kT) - \sum_{k=-\infty}^{\infty} \delta(t-kT-\frac{T}{2})$$

Taking the Fourier transform,

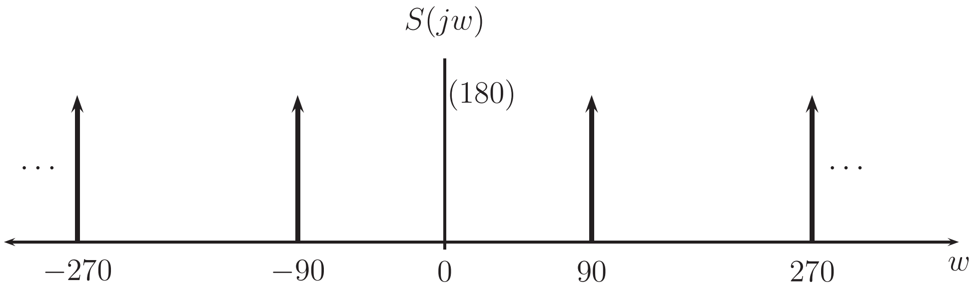
$$\begin{aligned}
S(jw) &= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(w-k\frac{2\pi}{T}) - \frac{2\pi}{T}e^{-jw\frac{T}{2}}\sum_{k=-\infty}^{\infty} \delta(w-k\frac{2\pi}{T}) \\
&= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(w-k\frac{2\pi}{T}) - \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} e^{-jk\frac{2\pi}{T}\frac{T}{2}}\delta(w-k\frac{2\pi}{T}) \\
&= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(w-k\frac{2\pi}{T}) - \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} (e^{-j\pi})^k\delta(w-k\frac{2\pi}{T}) \\
&= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(w-k\frac{2\pi}{T}) - \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} (-1)^k\delta(w-k\frac{2\pi}{T})
\end{aligned}$$

Seperating the odd and even terms of k,

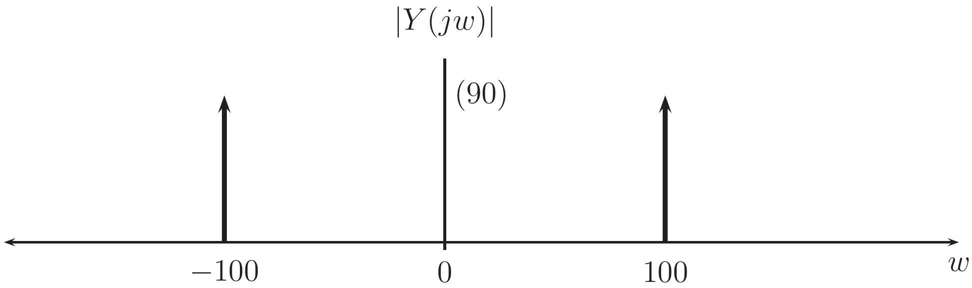
$$\begin{aligned}
S(jw) &= \frac{2\pi}{T}\sum_{k=even} \delta(w-k\frac{2\pi}{T}) - \frac{2\pi}{T}\sum_{k=even} \delta(w-k\frac{2\pi}{T}) \\
&\quad + \frac{2\pi}{T}\sum_{k=odd} \delta(w-k\frac{2\pi}{T}) + \frac{2\pi}{T}\sum_{k=odd} \delta(w-k\frac{2\pi}{T}) \\
&= \frac{4\pi}{T}\sum_{k=odd} \delta(w-k\frac{2\pi}{T})
\end{aligned}$$

$x(t) = \cos(10t)$ as before. To find $Z(jw)$, we need to convolve $X(jw)$ with the impulse train in $S(jw)$ and scale the result by $\frac{1}{2\pi}$.

$S(jw)$ is as sketched below,



The convolution will place two scaled impulses (from $X(jw)$) centered at each impulse in the impulse train of $S(jw)$. Finally, $H(jw)$ will only pass impulses that exist between 90 to 180 and -180 to -90 radians. We plot $|Y(jw)|$ (output from $H(jw)$) as follows:



As derived in part (b), $\angle Y(jw) = \angle H(jw) = -\frac{\pi w}{200}$.
From the plot of $|Y(jw)|$ and the $\angle Y(jw)$, we can view $y(t)$ as a time-shifted cos functions. Therefore,

$$\begin{aligned} y(t) &= \frac{90}{\pi}\cos(100(t-\frac{\pi}{200})) \\ &= \frac{90}{\pi}\cos(100t-\frac{\pi}{2}) \end{aligned}$$

Problem 2 (O&W 7.30 except let $x_c(t) = \delta(t - \frac{T}{2})$)

(a) We are given $x_c(t)$

$$\begin{aligned} x_c(t) &= \delta(t - \frac{T}{2}) \\ X_c(jw) &= \mathrm{e}^{-jw\frac{T}{2}} \end{aligned}$$

We take the Fourier transform of the system's differential equation and find the frequency response, $H(jw)$, of the system.

$$\begin{aligned} \frac{dy_c(t)}{dt} + y_c(t) &= x_c(t) \\ jwY_c(jw) + Y_c(jw) &= X_c(jw) \\ H(jw) = \frac{Y_c(jw)}{X_c(jw)} &= \frac{1}{1 + jw} \end{aligned}$$

Now, we can write,

$$\begin{aligned} Y_c(jw) &= X_c(jw)H(jw) = \mathrm{e}^{-jw\frac{T}{2}}\frac{1}{1+jw} \\ y_c(t) &= e^{-(t-\frac{T}{2})}u(t - \frac{T}{2}) \end{aligned}$$

(b) $y[n] = y_c(nT)$ where $y_c(t)$ is as defined in part (a). Therefore, $y_c(nT)$ will pick-up values from $y_c(t)$ at nT time values with $n = 0, 1, 2, \ldots$

$$\begin{aligned} y[n] = y_c(nT) &= e^{-nT+\frac{T}{2}}u[n-1] \\ &= (e^{\frac{T}{2}})(e^{-T})(e^{-T})^{n-1}u[n-1] \end{aligned}$$

Using the time-shifting property of DTFT and basic DTFT table,

$$Y(e^{jw}) = e^{-\frac{T}{2}}e^{-jw}\frac{1}{1 - e^{-T}e^{-jw}}$$

Now we choose $H(e^{jw})$ such that:

$$\begin{aligned} y[n] * h[n] &= w[n] = \delta[n] \\ Y(e^{jw})H(e^{jw}) &= 1 \\ H(e^{jw}) &= \frac{1}{e^{-\frac{T}{2}}e^{-jw}}(1 - e^{-T}e^{-jw}) \\ H(e^{jw}) &= e^{\frac{T}{2}}e^{jw} - e^{-\frac{T}{2}} \end{aligned}$$

Taking the inverse FT,

$$h[n] = e^{\frac{T}{2}}\delta[n+1] - e^{-\frac{T}{2}}\delta[n]$$

Problem 3

First, we need to find frequency response of the DT filter, $y[n] = \frac{3}{4}y[n-2] + x[n] + \frac{1}{4}x[n-1]$ When $x[n] = \delta[n]$, $y[n] = h[n]$. Therefore,

$$\begin{aligned} h[n] &= \frac{3}{4}h[n-2] + \delta[n] + \frac{1}{4}\delta[n-1] \\ H(e^{j\Omega}) &= \frac{3}{4}e^{-j2\Omega}H(e^{j\Omega}) + 1 + \frac{1}{4}e^{-j\Omega} \\ H(e^{j\Omega}) &= \frac{1+\frac{1}{4}e^{-j\Omega}}{1-\frac{3}{4}e^{-j2\Omega}}, \qquad |\Omega| < \pi \end{aligned}$$

It is given that $X(j\omega) = 0$ for $|\omega| \geq \frac{\pi}{T}$ and we have a sampling frequency, $\omega_s = \frac{2\pi}{T}$. So there will be no aliasing.

Therefore, the effective frequency response of the entire CT system, $H_c(jw)$, is related to the frequency response of the DT system, $H(e^{j\Omega})$, by (assume $\Omega = wT$ and find appropriate range of w):

$$H_c(j\omega) = \begin{cases} H(e^{j\omega T}), & |w| \leq \frac{\pi}{T} = \frac{\omega_s}{2} \\ 0, & |w| > \frac{\pi}{T} \end{cases}$$

$$H_c(j\omega) = \begin{cases} \frac{1+\frac{1}{4}e^{-j\omega T}}{1-\frac{3}{4}e^{-j2\omega T}}, & |w| \leq \frac{\pi}{T} = \frac{\omega_s}{2} \\ 0, & |w| > \frac{\pi}{T} \end{cases}$$

Problem 4 O&W 7.22

Solution:
In this problem we need to figure out a range of values for the sampling period, T, to recover y(t) completely from $y_p(t)$. To do this we need to determine the bandwidth of the original $Y(j\omega)$ and use the sampling theorem. By the convolution property, $Y(j\omega) = X_1(j\omega)X_2(j\omega)$. The bandwidth of $Y(j\omega)$ then will be the bandwidth of the smaller of the two bandwidths, $X_1(j\omega)$ or $X_2(j\omega)$. Hence, $Y(j\omega) = 0$ for $|\omega| > 1000\pi$. Then, using the sampling theorem,

$$\omega_s = \frac{2\pi}{T} > 2\omega_m = 2(1000\pi).$$

This gives the range of T as $0 < T < 0.001$ seconds.

Problem 5 O&W 7.23

Solution:

(a) We need to sketch $X_p(j\omega)$ and $Y(j\omega)$. In the frequency domain, $X_p(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega)$. We need to determine $P(j\omega)$. Since $P(j\omega)$ is periodic, we need to use the periodic Fourier transform formula. That is

$$P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_o).$$

Here, $\omega_o = \frac{2\pi}{T} = \frac{\pi}{\Delta}$. We need to determine the a_k's using the formula $a_k = \frac{1}{T}\int_T p(t)\mathrm{e}^{-jk\omega_o t}$. A few are shown below:

$$a_0 = \frac{1}{2\Delta}\int_0^{2\Delta} (\delta(t) - \delta(t-\Delta))\mathrm{d}t = 0$$

$$a_1 = \frac{1}{2\Delta}\int_0^{2\Delta} (\delta(t) - \delta(t-\Delta))\mathrm{e}^{-j\frac{\pi}{\Delta}t}\mathrm{d}t = \frac{1}{2\Delta}(1 - 1\cdot \mathrm{e}^{-j\pi}) = \frac{1}{\Delta}$$

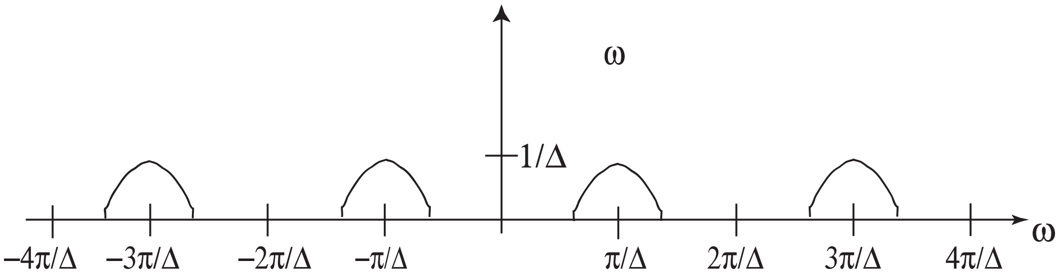
$$a_2 = \frac{1}{2\Delta}\int_0^{2\Delta} (\delta(t) - \delta(t-\Delta))\mathrm{e}^{-j2\frac{\pi}{\Delta}t}\mathrm{d}t = \frac{1}{2\Delta}(1 - 1\cdot \mathrm{e}^{-j2\pi}) = 0$$

$$a_3 = \frac{1}{2\Delta}\int_0^{2\Delta} (\delta(t) - \delta(t-\Delta))\mathrm{e}^{-j3\frac{\pi}{\Delta}t}\mathrm{d}t = \frac{1}{2\Delta}(1 - 1\cdot \mathrm{e}^{-j3\pi}) = \frac{1}{\Delta}$$

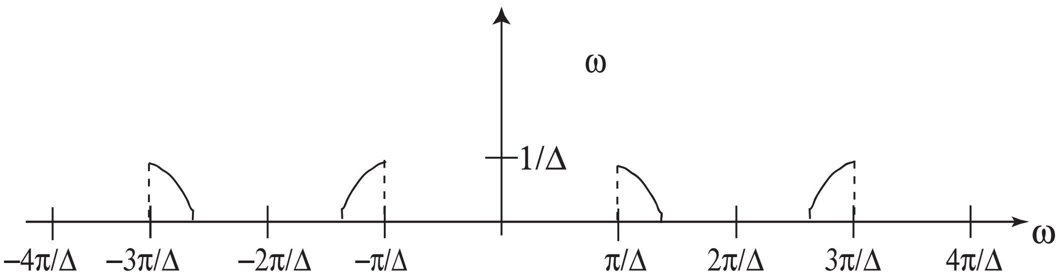
Thus, $a_k = 0$ for k even and $a_k = \frac{1}{\Delta}$ for k odd and

$$P(j\omega) = \sum_{kodd} \frac{2\pi}{\Delta}\delta(\omega - k\frac{\pi}{\Delta}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{\Delta}\delta\left(\omega - (2k+1)\frac{\pi}{\Delta}\right)$$

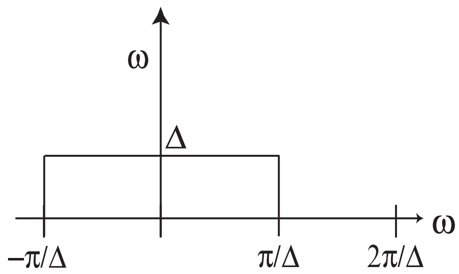
From this Fourier transform for $P(j\omega)$, we can sketch $X_p(j\omega)$ as copies of $X(j\omega)$ scaled by $\frac{1}{\Delta}$ and replicated at intervals of $\omega = (2k+1)\frac{\pi}{\Delta}$. for all k. This can be seen in the figure below:



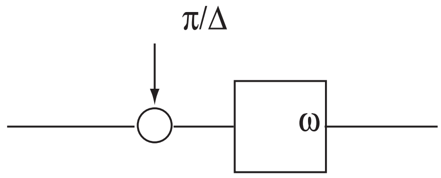
$H(j\omega)$ is a sum of two ideal unity gain bandpass filters. Thus, $Y(j\omega)$ is the part of $X_p(j\omega)$ that is passed through $H(j\omega)$. This is shown below:



(b) To recover $x(t)$ from $x_p(t)$ we need to do two things. First, we need to multiply $x_p(t)$ with a cosine function,cos $\frac{\pi}{\Delta}t$. This will shift $X_p(j\omega)$ such that one of the copies of $X(j\omega)$ is centered around $\omega = 0$. Second, we send the shifted signal through a lowpass filter, $R(j\omega)$, to eliminate the extra copies of $X(j\omega)$. To achieve this we have a filter, $R(j\omega)$ with gain $= \Delta$, bandwidth $\frac{2\pi}{\Delta}$ and centered around $\omega = 0$. This is shown below:

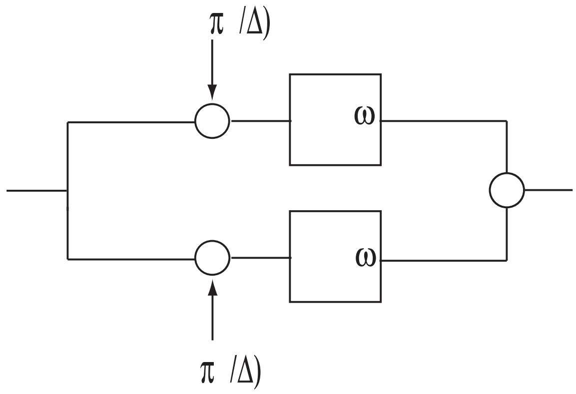


The overall system is shown below:



(c) To recover $x(t)$ from $y(t)$ we need to run $Y(j\omega)$ through two parallel filter systems. The top parallel path will multiply $y(t)$ by $\cos\frac{\pi}{\Delta}t$ which will shift the demi-replicate of $X(j\omega)$ that is centered at $\omega = \frac{\pi}{\Delta}$ over to $\omega = 0$. The shifted signal then passes through the lowpass filter, $R(j\omega)$ described above in part (b) to eliminate the extra copies.

The bottom parallel path will multiply $y(t)$ by $\cos\frac{3\pi}{\Delta}t$ which will shift the demi-replicate of $X(j\omega)$ that is centered at $\omega = \frac{3\pi}{\Delta}$ over to $\omega = 0$. The shifted signal then passes through the lowpass filter, $R(j\omega)$ described above in part (b) to eliminate the extra copies. Thus, the two halves combine together to form a complete $X(j\omega)$ and $x(t)$ is recovered. The overall system is shown below:



(d) To recover x(t) from $x_p(t)$ and y(t), $X_p(j\omega)$ cannot have any overlap in the copies of X$(j\omega)$. Because of this particular p(t), the copies of $X(j\omega)$ are at $\omega = (2k+1)\frac{\pi}{\Delta}$ for all k. Thus, just looking at one interval to make sure the copies of $X(j\omega)$ don't overlap, we have one copy of $X(j\omega)$ centered at $\omega = \frac{\pi}{\Delta}$ and one copy of $X(j\omega)$ centered at $\omega = \frac{3\pi}{\Delta}$. (See Figure of $X_p(j\omega)$ above). For no overlap between these copies,

$$\frac{\pi}{\Delta} + \omega_m < \frac{3\pi}{\Delta} - \omega_m,$$

which gives

$$\Delta < \frac{\pi}{\omega_m} \text{ or } \Delta_{max} = \frac{\pi}{\omega_m}.$$