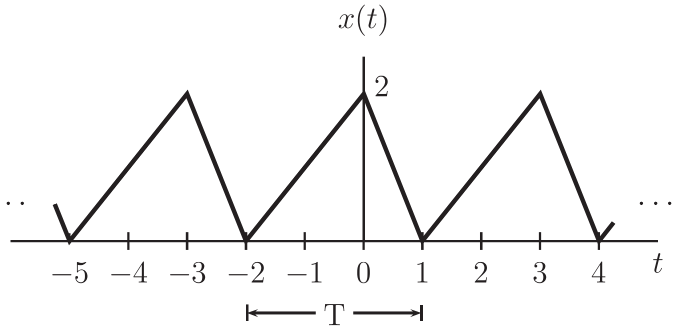
Problem 1 (O&W 3.22 (a) - only the signal in Figure p3.22 (c)) Determine the Fourier series representation for the signal $x(t)$.



$$x(t) = \begin{cases} 2+t, & \text{for } -2 \leq t \leq 0 \\ 2-2t, & \text{for } 0 \leq t \leq 1. \end{cases}$$

$x(t)$ periodic with period $T = 3 \rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$

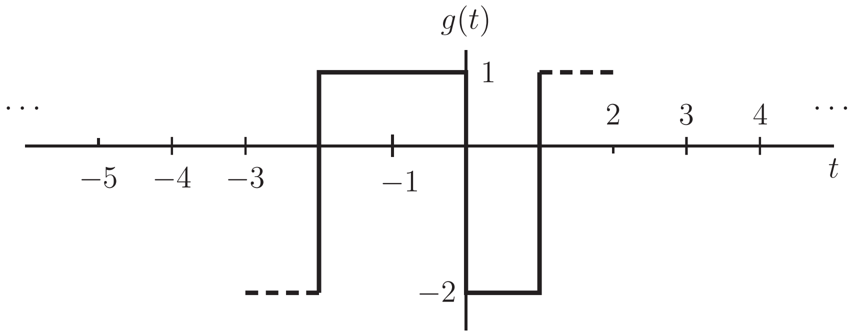
A goal of this problem solution is to show different ways to reaching the same answer. Finding the Fourier series coefficients of a signal using the analysis equation usually requires the most effort, but can be reverted to if everything else fails. Oftentimes, a signal can be dissected into simpler signals that are easier to analyze or can be derived from a simpler signal by integration, differentiation, time shifting, or any combination of the properties of the Fourier series (see Table 3.1, O&W, p.206).

We will start with finding a_0, which is usually straight-forward and doesn't require much effort, and then explore the different methods for finding $a_{k\neq 0}$:

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{3} \text{ (the total area under the curve for one period)} = \frac{1}{3}(2+1) = 1.$$

The following are four possible methods to calculate $a_{k\neq 0}$, the Fourier series coefficients of $x(t)$ for $k \neq 0$:

- Method (a): Using the integration property:
  Let $g(t) = \frac{dx(t)}{dt} \rightarrow x(t) = \int g(t)dt + p$, where p is the value of $x(t)$ at the beginning of the period, and it equals to zero for the period we selected that starts at $t = -2$. Note that, since we are trying to find $a_{k\neq 0}$, the value of p is not important because it only affects the DC level of $x(t)$ and we have already calculated it by finding a_0.



Note that $g(t)$ must have a zero DC level, otherwise a ramping signal will be included in $x(t)$ making it non-periodic, and unbounded. By definition, $g(t)$ should have a zero DC level because the derivative operation eliminates it, so this can be used as a double-check.

After finding b_k, the Fourier series coefficients for $g(t)$, we can use the Fourier series properties to find a_k, the Fourier series coefficients for $x(t)$

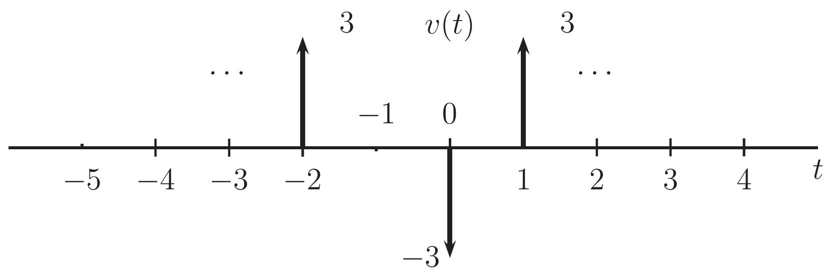
$$\begin{aligned}
b_k &= \frac{1}{T}\int_T g(t)e^{-jk\omega_0 t}dt = \frac{1}{3}\left(\int_{-2}^{0}(1)e^{-jk\omega_0 t}dt + \int_0^1 (-2)e^{-jk\omega_0 t}dt\right) \\
&= \frac{1}{3}\left(\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\Big|_{-2}^{0} - 2\,\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\Big|_{0}^{1}\right) = \frac{-1}{3jk\omega_0}\left(1 - e^{jk\omega_0 2} - 2e^{-jk\omega_0} + 2\right) \\
&= \frac{1}{3jk\omega_0}\left(e^{jk\omega_0 2} + 2e^{-jk\omega_0} - 3\right). \\
a_k &= \frac{1}{jk\omega_0}b_k \quad \text{(from the Integration property, Table 3.1, O \&W, p.206)} \\
&= \frac{1}{jk\omega_0}\frac{1}{3jk\omega_0}\left(e^{jk\omega_0 2} + 2e^{-jk\omega_0} - 3\right) = \frac{1}{3k^2\omega_0^2}\left(3 - 2e^{-jk\omega_0} - e^{jk\omega_0 2}\right) \\
&= \frac{1}{k^2\omega_0^2}\left(1 - e^{jk\omega_0 2}\right) \qquad \text{(remember that } e^{-jk\omega_0} = e^{jk\omega_0 2} \text{ for } T = 3) \\
&= \frac{1}{k^2\omega_0^2}\left(1 - e^{jk\frac{4\pi}{3}}\right) = \frac{1}{k^2\omega_0^2}\left(1 - e^{-jk\frac{2\pi}{3}}\right).
\end{aligned}$$

- Method (b): Using the integration property twice:
  Let's define $v(t)$ as the following:

$$v(t) = \frac{d^2x(t)}{dt^2} = \frac{dg(t)}{dt} \Rightarrow x(t) = \int\int v(t)\,dt\,dt + p = \int g(t)\,dt + p$$

Similar to the discussion in Method(a) of the DC level of $g(t)$, $v(t)$ must have a zero DC level. In addition, its limited integration over one period must also have a zero DC level.

We can find $v(t)$ by differentiating $g(t)$. However, in our case, but not always, we can find $v(t)$ directly from $x(t)$ in one step, by placing an impulse at each point of time where the slope of $x(t)$ changes abruptly. The value of that impulse (i.e its area) is the change in slope of $x(t)$ at that point.



To find c_k, the Fourier series coefficients of $v(t)$, let's take the period between -1 and 2, which contains two impulses.
Note that we can also take the period between -2 and 1, but we have to be careful not include the impulses at both -2 and 1. In other words, we can take the period between $-2+\delta$ and $1+\delta$ or the period between $-2-\delta$ and $1-\delta$.

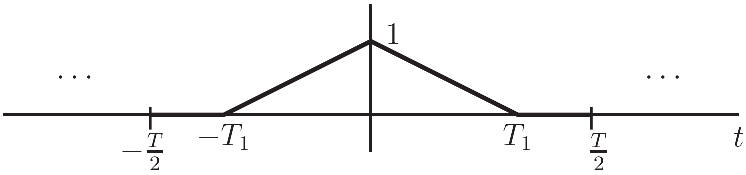
$$\begin{aligned} c_k &= \frac{1}{T}\int_T v(t)e^{-jk\omega_0 t}dt = \frac{1}{3}\left(\int_{-1}^{2}\left[-3\delta(t)+3\delta(t-1)\right]e^{-jk\omega_0 t}dt\right) \\ &= \int_{-1}^{2}\left[-\delta(t)+\delta(t-1)\right]e^{-jk\omega_0 t}dt \\ &= -e^{-jk\omega_0(0)} + e^{-jk\omega_0(1)} = e^{-jk\omega_0} - 1. \end{aligned}$$

Now to find a_k, we just need to use the integration property two times:

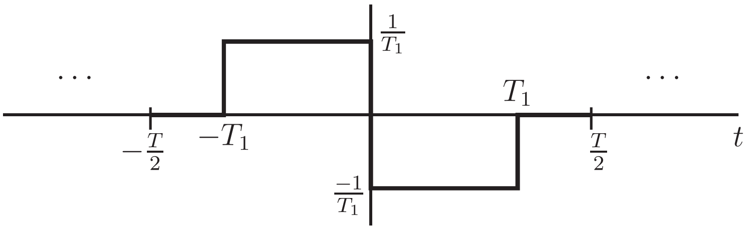
$$\begin{aligned} a_k &= \frac{1}{jk\omega_0}\frac{1}{jk\omega_0}c_k \quad \text{(from the Integration property, Table 3.1, O \&W, p.206)} \\ &= \frac{1}{(jk\omega_0)^2}\left(e^{-jk\omega_0}-1\right) \\ &= \frac{1}{k^2\omega_0^2}\left(1-e^{-jk\omega_0}\right) \\ &= \frac{1}{k^2\omega_0^2}\left(1-e^{-jk\frac{2\pi}{3}}\right), \text{ which is the same answer found in Method(a).} \end{aligned}$$

Before exploring the other methods, let's first find the Fourier series for $y(t)$, shown below, which is a periodic triangular function with a period of T. $y(t)$ will be useful for the Method(c):

$y(t)$



$z(t) = \frac{dy(t)}{dt}$



Let $z(t) = \frac{dy(t)}{dt}$, $z(t) \overset{\mathcal{F}}{\longleftrightarrow} e_k$, and $y(t) \overset{\mathcal{F}}{\longleftrightarrow} d_k = \left(\frac{1}{jk\omega_0}\right) e_k$

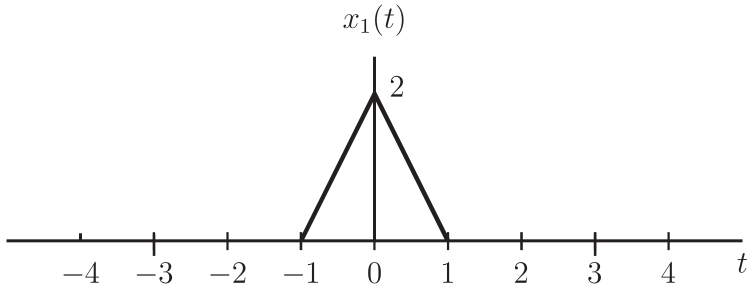
We will find the Fourier series for $z(t)$ and from it, we will find the Fourier series for $y(t)$, as follows:

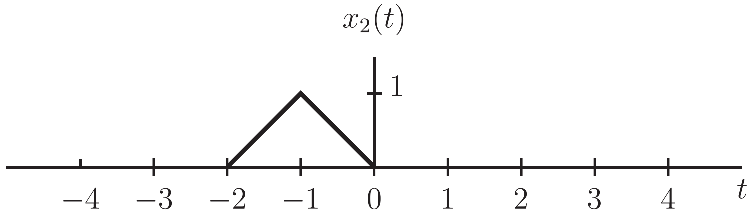
$$\begin{aligned} e_k &= \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}dt = \frac{1}{T}\left[\int_{-T_1}^{0}(\frac{1}{T_1})e^{-jk\omega_0 t}dt + \int_0^{T_1}(\frac{-1}{T_1})e^{-jk\omega_0 t}dt\right] \\ &= \frac{1}{TT_1}\frac{1}{(-jk\omega_0)}\left(e^{-jk\omega_0 t}|_{-T_1}^{0} - e^{-jk\omega_0 t}|_0^{T_1}\right) = \frac{1}{TT_1}\frac{1}{(-jk\omega_0)}\left(1 - e^{jk\omega_0 T_1} + 1 - e^{-jk\omega_0 T_1}\right) \\ &= \frac{-1}{TT_1 jk\omega_0}\left[2 - (e^{jk\omega_0 T_1} + e^{-jk\omega_0 T_1})\right] = \frac{-1}{TT_1 jk\omega_0}\left[2 - 2\cos\left(jk\omega_0 T_1\right)\right]. \end{aligned}$$

Thus,

$$d_k = e_k(\frac{1}{jk\omega_0}) = \frac{2 - 2\cos\left(k\omega_0 T_1\right)}{TT_1 k^2 \omega_0^2}.$$

- Method (c): By dissecting the signal into simpler components:
  Here, we will dissect $x(t)$ into $x_1(t)$ and $x_2(t)$ which we know their Fourier Series (using the result of d_k above and the time-shifting property).





$x(t) = x_1(t) + x_2(t)$, and let $x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} b_k$ and $x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} c_k$

$$\begin{aligned} \therefore a_k &= b_k + c_k = (2)\frac{2 - 2\cos(k\omega_0(1))}{(3)(1)k^2\omega_0^2} + (1)\frac{2 - 2\cos(k\omega_0(1))}{(3)(1)k^2\omega_0^2}e^{-jk\omega_0(-1)} \\ &= \frac{2 - 2\cos k\omega_0}{3k^2\omega_0^2}(2 + e^{jk\omega_0}). \end{aligned}$$

Although this result looks different from those found in the previous methods, further simplification will show that they are identical:

$$\begin{aligned} a_k &= \frac{2 - 2\cos k\omega_0}{3k^2\omega_0^2}(2 + e^{jk\omega_0}) = \frac{1}{3k^2\omega_0^2}(2 - 2\cos k\omega_0)(2 + e^{jk\omega_0}) \\ &= \frac{1}{3k^2\omega_0^2}(2 - e^{jk\omega_0} - e^{-jk\omega_0})(2 + e^{jk\omega_0}) \\ &= \frac{1}{3k^2\omega_0^2}\left(4 + 2e^{jk\omega_0} - 2e^{jk\omega_0} - e^{jk\omega_0 2} - 2e^{-jk\omega_0} - e^0\right) \\ &= \frac{1}{3k^2\omega_0^2}\left(4 - e^{jk\omega_0 2} - 2e^{-jk\omega_0} - 1\right) = \frac{1}{3k^2\omega_0^2}\left(3 - e^{jk\omega_0 2} - 2e^{-jk\omega_0}\right) \\ &= \frac{1}{k^2\omega_0^2}\left(1 - e^{jk\omega_0 2}\right) \qquad \text{(remember that } e^{-jk\omega_0} = e^{jk\omega_0 2} \text{ for } T = 3) \\ &= \frac{1}{k^2\omega_0^2}\left(1 - e^{jk\frac{4\pi}{3}}\right), \text{ which is the same answer found in previous methods.} \end{aligned}$$

- Method (d): using the analysis equation:
  In the process of evaluating the analysis equation, the following integral will save us a lot of derivation steps:

$$\int te^{at}dt = \left(\frac{t}{a} - \frac{1}{a^2}\right)e^{at} \qquad \text{, for any } a \neq 0$$

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt = \frac{1}{3}\int_{-2}^{1} x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{3}\left[\int_{-2}^{0}(2+t)e^{-jk\omega_0 t}dt + \int_{0}^{1}(2-2t)e^{-jk\omega_0 t}dt\right] \\
&= \frac{1}{3}\left[2\int_{-2}^{1} e^{-jk\omega_0 t}dt + \int_{-2}^{0} te^{-jk\omega_0 t}dt - 2\int_{0}^{1} te^{-jk\omega_0 t}dr\right] \\
&= \frac{1}{3}\left[2\left.\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right|_{-2}^{1} + \left.\left(\frac{1}{k^2\omega_0^2} - \frac{t}{jk\omega_0}\right)e^{-jk\omega_0 t}\right|_{-2}^{0} - 2\left.\left(\frac{1}{k^2\omega_0^2} - \frac{t}{jk\omega_0}\right)e^{-jk\omega_0 t}\right|_{0}^{1}\right] \\
&= \frac{1}{3}\left\{\frac{-2}{jk\omega_0}\left(e^{-jk\omega_0(1)} - e^{-jk\omega_0(-2)}\right) + \frac{1}{k^2\omega_0^2} - \left(\frac{1}{k^2\omega_0^2} - \frac{(-2)}{jk\omega_0}\right)e^{-jk\omega_0(-2)}\right. \\
&\quad \left. -2\left[\left(\frac{1}{k^2\omega_0^2} - \frac{(1)}{jk\omega_0}\right)e^{-jk\omega_0(1)} - \frac{1}{k^2\omega_0^2}\right]\right\} \\
&= \frac{1}{3}\left(\frac{-2}{jk\omega_0}e^{-jk\omega_0} + \frac{2}{jk\omega_0}e^{jk\omega_0 2} + \frac{1}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}e^{jk\omega_0 2} - \frac{2}{jk\omega_0}e^{jk\omega_0 2}\right. \\
&\quad \left. -\frac{2}{k^2\omega_0^2}e^{-jk\omega_0} + \frac{2}{jk\omega_0}e^{-jk\omega_0} + \frac{2}{k^2\omega_0^2}\right) \\
&= \frac{1}{3}\left(\frac{-2}{jk\omega_0}e^{-jk\omega_0} - \frac{2}{k^2\omega_0^2}e^{-jk\omega_0} + \frac{2}{jk\omega_0}e^{-jk\omega_0} + \frac{1}{k^2\omega_0^2} + \frac{2}{k^2\omega_0^2}\right. \\
&\quad \left. +\frac{2}{jk\omega_0}e^{jk\omega_0 2} - \frac{1}{k^2\omega_0^2}e^{jk\omega_0 2} - \frac{2}{jk\omega_0}e^{jk\omega_0 2}\right) \\
&= \frac{1}{3}\left(-\frac{2}{k^2\omega_0^2}e^{-jk\omega_0} + \frac{3}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}e^{jk\omega_0 2}\right) \\
&= \frac{1}{3k^2\omega_0^2}\left(3 - 2e^{-jk\omega_0} - e^{jk\omega_0 2}\right) \\
&= \frac{1}{k^2\omega_0^2}\left(1 - e^{jk\omega_0 2}\right) \qquad \text{(remember that } e^{-jk\omega_0} = e^{jk\omega_0 2} \text{ for } T = 3\text{)} \\
&= \frac{1}{k^2\omega_0^2}\left(1 - e^{jk\frac{4\pi}{3}}\right) \text{, which is the same answer found in previous methods.} \\
&= \frac{9}{4k^2\pi^2}\left(1 - e^{jk\frac{4\pi}{3}}\right).
\end{aligned}$$

Problem 2 O & W 3.23 (a)
Given a_k , the Fourier series coefficients of a periodic continuous time signal with period 4, determine the signal $x(t)$.
The Fourier series coefficients a_k are given as follows:

$$a_k = \begin{cases} 0, & k = 0 \\ (j)^k \dfrac{\sin k\pi/4}{k\pi}, & \text{otherwise.} \end{cases}$$

Here are some of the facts we know about $x(t)$:

- $a_0 = 0 \rightarrow$ no DC component in $x(t)$
- $T = 4 \rightarrow \omega_0 = 2\pi/4 = \pi/2$
-

$$\begin{aligned} a_{-k} &= (j)^{-k} \frac{\sin(-k\pi/4)}{-k\pi} = \left(\frac{1}{j}\right)^k \frac{-\sin(k\pi/4)}{-k\pi} \\ &= (-j)^k \frac{\sin(k\pi/4)}{k\pi} = a_k^*. \end{aligned}$$

Thus $x(t)$ is a real signal (O&W, Section 3.5.6, p.204).

Noting that $j = e^{j\pi/2} \rightarrow (j)^k = \left(e^{j\pi/2}\right)^k = e^{jk\pi/2} = e^{jk\omega_0} = e^{-jk\omega_0(-1)}$, we can consider $x(t)$ to be a time-shifted version of another signal y(t) such that:

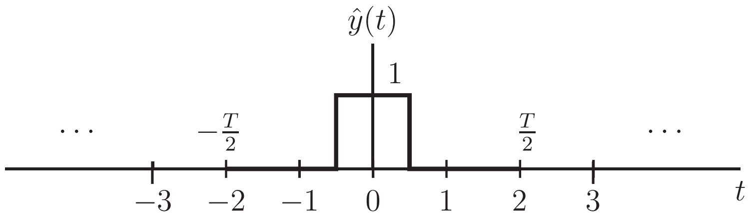
$$x(t) = y(t+1), \text{where } \; y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} b_0 = 0, b_{k\neq 0} = \frac{\sin k\pi/4}{k\pi} \text{ and } a_k = b_k e^{jk\omega_0(1)}$$

By backtracking the derivation equation of b_k, we can find the signal $\hat{y}(t)$ which has the same b_k but can have a different DC level (i.e. b_0) :

$$\begin{aligned} \hat{b}_{k\neq 0} = b_{k\neq 0} &= \frac{\sin k\pi/4}{k\pi} = \frac{1}{k\pi}\left(\frac{e^{jk\pi/4} - e^{-jk\pi/4}}{2j}\right) \\ &= \frac{1}{(4)}\frac{1}{jk(\frac{\pi}{2})}\left(e^{jk\pi/4} - e^{-jk\pi/4}\right) \\ &= \frac{1}{T}\frac{1}{jk\omega_0}\left(e^{jk\omega_0(\frac{1}{2})} - e^{jk\omega_0(-\frac{1}{2})}\right) = \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} (1)e^{jk\omega_0 t}dt. \end{aligned}$$

The integration above suggests that

$$\hat{y}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{elsewhere in the same period T=4.} \end{cases}$$



Note that the same conclusion can be reached by noticing that $\hat{y}(t)$ is the same signal in Example 3.5 (O& W, p.193) with $T_1 = \frac{1}{2}$ and $T = 4$.

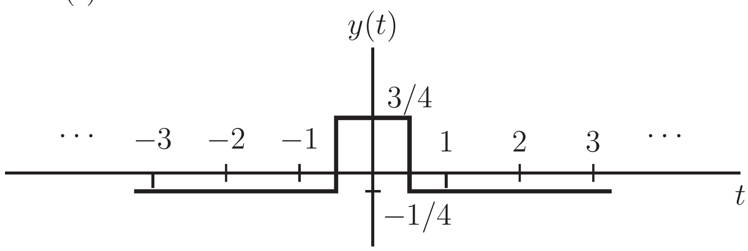
To find $y(t)$, which has $b_0 = 0$, we first calculate $\hat{b}_0$ and then subtract it from $\hat{y}(t)$:

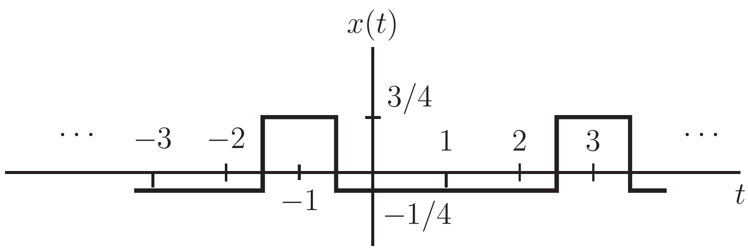
$$\hat{b}_0 = \frac{1}{T}\int_T \hat{y}(t)dt = \frac{1}{4}\int_{-1/2}^{1/2}(1)dt = \frac{1}{4}$$

$$\rightarrow y(t) = \hat{y}(t) - \frac{1}{4} \Rightarrow y(t) = \begin{cases} \frac{3}{4}, & -\frac{1}{2} < t < \frac{1}{2} \\ -\frac{1}{4}, & \frac{1}{2} < |t| < 2. \end{cases}$$

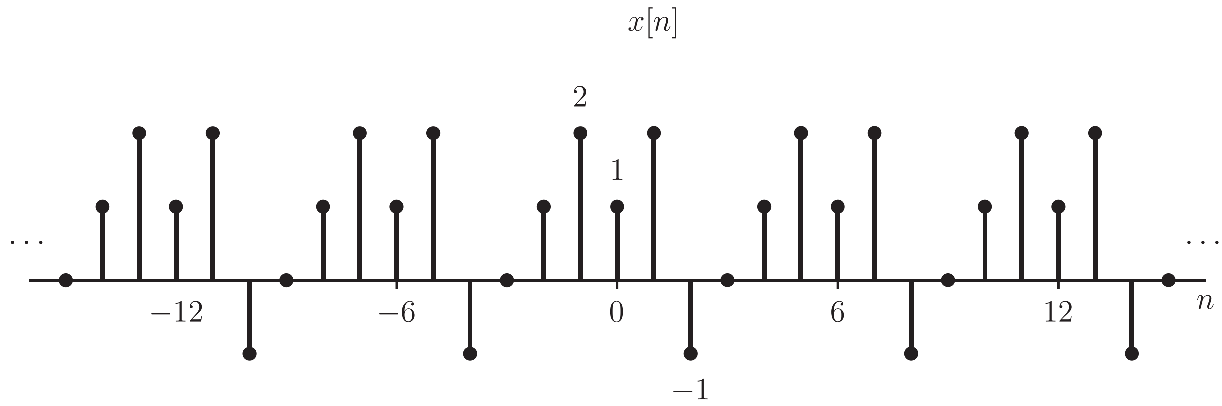
$$\rightarrow x(t) = y(t+1) = \begin{cases} \frac{3}{4}, & -1.5 < t < -0.5 \\ -\frac{1}{4}, & -0.5 < t < 2.5 \end{cases}$$

Sketches of $y(t)$ and $x(t)$ are shown below:





Problem 3 Determine the Fourier series coefficients for the periodic signal $x[n]$ depicted below. Plot the magnitude and phase of these coefficients.



Fundamental period $N = 6 \rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$.

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{6}\sum_{n=0}^{5} x[n]e^{-jk\omega_0 n} = \frac{1}{6}\sum_{n=-3}^{2} x[n]e^{-jk\omega_0 n}$$

Notice that the last two expressions will give the same result, but the latter would take advantage of the symmetry of some of the samples to combine them into sinusoids.

$$\begin{aligned}
a_k &= \frac{1}{6}\left[(0)e^{-jk\omega_0(-3)} + (1)e^{-jk\omega_0(-2)} + (2)e^{-jk\omega_0(-1)} + (1)e^{-jk\omega_0(0)} + \right.\\
&\qquad\left. +(2)e^{-jk\omega_0(1)} + (-1)e^{-jk\omega_0(2)}\right]\\
&= \frac{1}{6}\left[e^{-jk\omega_0(-2)} - e^{-jk\omega_0(2)} + 2e^{-jk\omega_0(-1)} + 2e^{-jk\omega_0(1)} + 1\right]\\
&= \frac{1}{6}\left[(2j)\sin k\omega_0 2 + 2(2)\cos k\omega_0 + 1\right]\\
&= \frac{1}{6} + \frac{2}{3}\cos k\omega_0 + \frac{j}{3}\sin k\omega_0 2\\
\therefore a_k &= \frac{1}{6} + \frac{2}{3}\cos\left(k\frac{\pi}{3}\right) + j\frac{1}{3}\sin\left(k\frac{2\pi}{3}\right)\\
&= \frac{1}{6}\left[1 + 4\cos\left(k\frac{\pi}{3}\right) + j2\sin\left(k\frac{2\pi}{3}\right)\right]
\end{aligned}$$

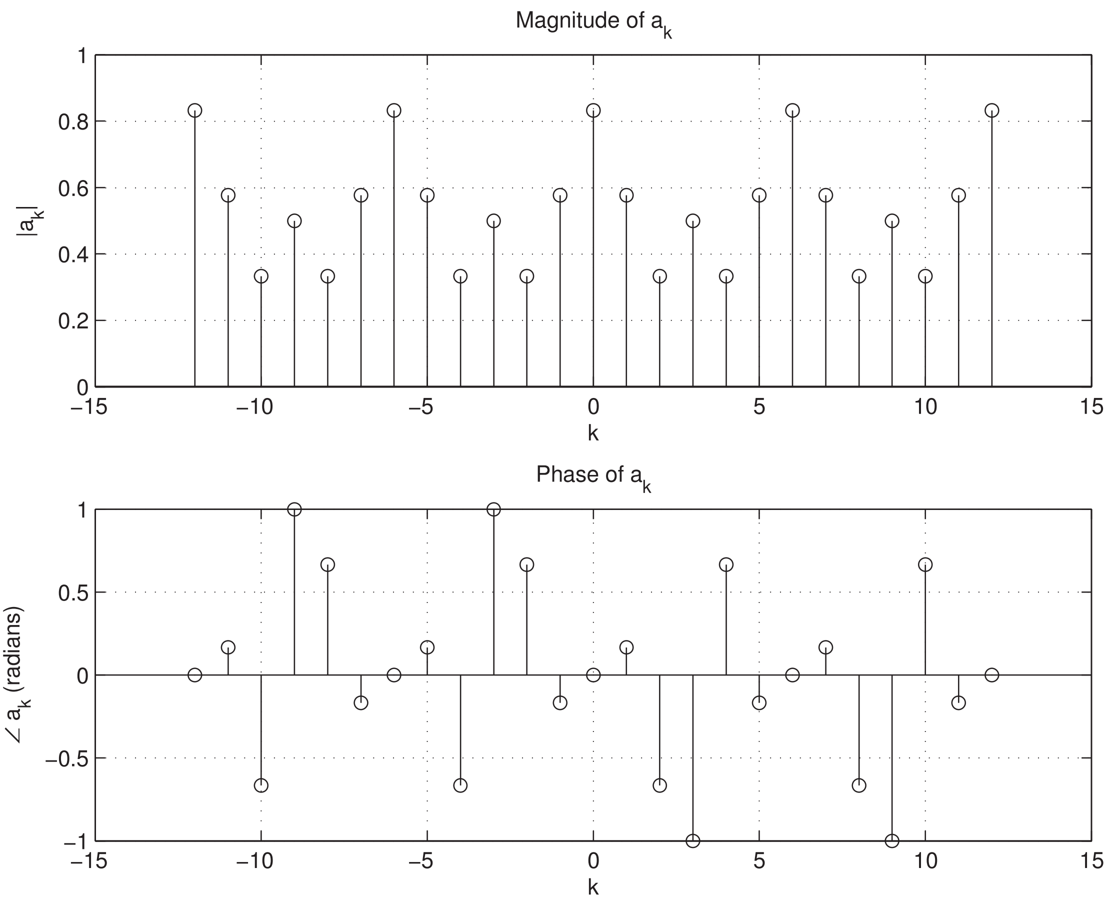
Here are the values of a_k for one period of six consecutive points (from k=-2 to k=3):

$$\begin{aligned}
a_{-2} &= \frac{1}{6}\left[1+4\cos\left((-2)\frac{\pi}{3}\right)+2j\sin\left((-2)\frac{2\pi}{3}\right)\right] \\
&= \frac{1}{6}\left[1+4\cos\left(\frac{2\pi}{3}\right)-2j\sin\left(\frac{4\pi}{3}\right)\right]=\frac{1}{6}\left[1+4\left(\frac{-1}{2}\right)-2j\left(\frac{-\sqrt{3}}{2}\right)\right] \\
&= \frac{1}{6}(-1+j\sqrt{3})=-0.1667+j0.2887 \\
&= \frac{\sqrt{(-1)^2+(\sqrt{3})^2}}{6}e^{j\ \mathrm{atan}(\sqrt{3},-1)}=\frac{1}{3}e^{j\frac{2\pi}{3}}\rightarrow|a_{-2}|=\frac{1}{3},\angle a_{-2}=\frac{2\pi}{3}.
\end{aligned}$$

Similarly, for the magnitude and phase of a_k for $k=-1\rightarrow 3$ which are summarized in the table below:

k	a_k	$\lvert a_k\rvert$	$\angle a_k$
-2	-0.1667 + j 0.2887	$1/3$	$2\pi/3$
-1	0.5000 - j 0.2887	$1/\sqrt{3}$	$-\pi/6$
0	0.8333	$5/6$	0
1	0.5000 + j 0.2887	$1/\sqrt{3}$	$\pi/6$
2	-0.1667 -j 0.2887	$1/3$	$-2\pi/3$
3	-0.5000	$3/2$	$-\pi$

The magnitude and phase of the Fourier series coefficients were plotted below, using MATLAB:



For your reference, the MATLAB code used to compute and plot the magnitude and phase of the Fourier series coefficients is shown below:

```
MATLAB Code:
A=inline('1/6 +2/3*cos(k*pi/3)+j/3*sin(k*2*pi/3)');
k=-12:12;a=A(k);am=abs(a);ap=angle(a); subplot(2,1,1);stem(k,am);grid
on;xlabel('k');ylabel('|a_k|');title('Magnitude of a_k');
subplot(2,1,2);stem(k,ap/pi);grid on; xlabel('k');ylabel('\angle a_k
(radians)');title('Phase of a_k');
```

MATLAB tip: you can use TEX expressions in the text of figures.

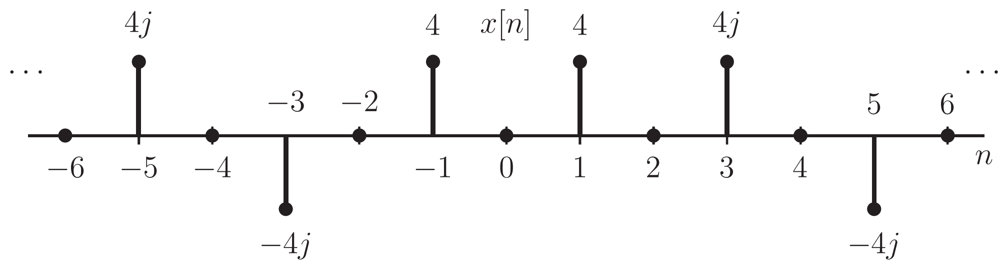
Problem 4 O & W 3.29 (a)
Given a_k , the Fourier series coefficients of a periodic discrete time signal with period 8, determine the signal $x[n]$.
The Fourier coefficients are given as follows:

$$a_k = \cos\frac{k\pi}{4} + \sin\frac{3k\pi}{4}.$$

N=8 $\rightarrow$ there are only 8 samples to compute in $x[n]$, some of which can have a zero value, $\omega_0 = 2\pi/8 = \pi/4$.

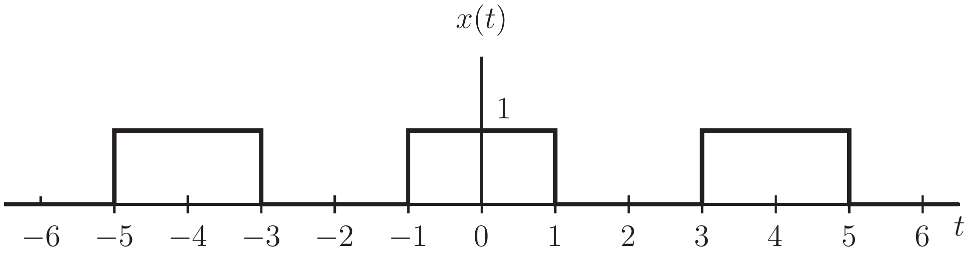
$$\begin{aligned}
a_k &= \cos\frac{k\pi}{4} + \sin\frac{3k\pi}{4} = \cos k\omega_0 + \sin 3k\omega_0 \\
&= \frac{1}{2}e^{jk\omega_0} + \frac{1}{2}e^{-jk\omega_0} + \frac{1}{2j}e^{j3k\omega_0} - \frac{1}{2j}e^{-j3k} \\
&= \frac{1}{8}\left[4\,e^{-jk\omega_0(-1)} + 4\,e^{-jk\omega_0(1)} + \frac{4}{j}\,e^{-jk(-3)\omega_0} - \frac{4}{j}\,e^{-jk(3)}\right] \\
&= \frac{1}{8}\left[(4)\,e^{-jk\omega_0(-1)} + (4)\,e^{-jk\omega_0(1)} + (-4j)\,e^{-jk(-3)\omega_0} + (4j)\,e^{-jk(3)}\right] \\
&= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{8}\sum_{n=-3}^{4} x[n]e^{-jk\omega_0 n}
\end{aligned}$$

By matching the expressions of $a_k \rightarrow x[n] = \begin{cases} -4j, & n=-3 \\ 4j, & n=3 \\ 4, & n=\pm 1 \\ 0, & n=0,\pm 2, 4 \end{cases}$



Problem 5 Consider the following CT periodic signals, $x(t)$, $y(t)$, and $z(t)$.

(a) Determine the fundamental frequency, period, and Fourier series coefficients, a_k, for $x(t)$.

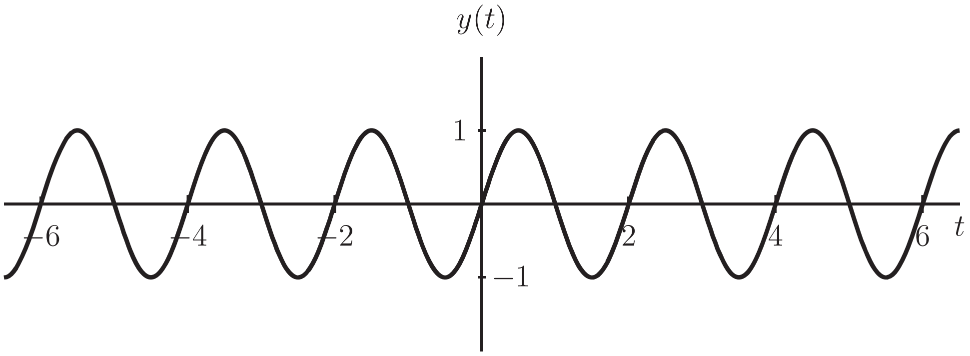


Fundamental period of $x(t) = T = 4 \longrightarrow \omega_0 = 2\pi/4 = \pi/2$.

$$\begin{aligned} a_0 &= \frac{1}{T}\int_T x(t)\,dt = \frac{1}{4}\int_{-1}^{1} x(t)\,dt = \frac{2}{4} = \frac{1}{2}. \\ a_k &= \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}dt = \frac{1}{4}\int_{-1}^{1}(1)\,e^{-jk\omega_0 t}dt = \frac{1}{-jk\omega_0 4}\,e^{-jk\omega_0 t}\Big|_{-1}^{1} \\ &= \frac{1}{jk2\pi}\left(e^{jk\omega_0} - e^{-jk\omega_0}\right) = \frac{\sin(k\omega_0)}{k\pi} = \frac{\sin(k\frac{\pi}{2})}{k\pi}. \end{aligned}$$

The same result can also be found directly using Example 3.5 (O & W, P.193).

(b) Determine the fundamental frequency,period, and Fourier series coefficients, b_k, for $y(t)$.



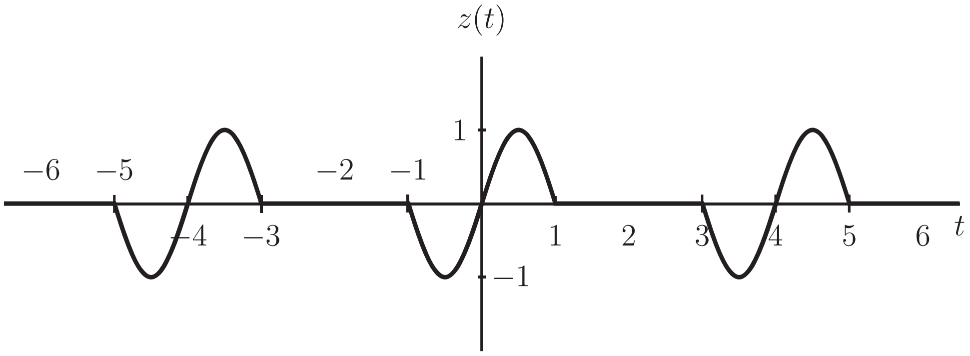
Fundamental period of $y(t) = \hat{T} = 2 \longrightarrow \hat{\omega}_0 = 2\pi/2 = \pi$.

$$b_0 = \frac{1}{T}\int_T y(t)dt = 0 \quad (\because y(t) \text{ has no DC component}).$$

$$y(t) = \sin \hat{\omega}_0 t = \frac{e^{j\hat{\omega}_0 t} - e^{-j\hat{\omega}_0 t}}{2j} = \frac{-j}{2}\, e^{j(1)\hat{\omega}_0 t} - \frac{-j}{2}\, e^{j(-1)\hat{\omega}_0 t} = \sum_{k=-\infty}^{+\infty} b_k e^{jk\hat{\omega}_0 t}$$

$$\rightarrow b_k = \begin{cases} \frac{-j}{2}, & k = 1 \\ \frac{j}{2}, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

(c) Determine the fundamental frequency and period for $z(t)$. Also, using the results of parts (a) and (b), determine the Fourier series coefficients, c_k for $z(t)$.



Fundamental period of $z(t)$=Fundamental period of $x(t) = T = 4 \rightarrow \omega_0 = 2\pi/4 = \frac{\pi}{2}$.

$$c_0 = \frac{1}{T}\int_T z(t)dt = 0 \quad (\because z(t) \text{ has no DC component}).$$

Noticing that $z(t) = x(t)y(t)$, we can find c_k using the multiplication property. However, the fundamental frequencies of $x(t)$ and $y(t)$ must be identical in order for the Fourier coefficients to match (i.e. to represent the same frequencies). The fundamental period of $y(t)$ is 2, but if we define it to be 4, then we only need to scale the frequency components accordingly to keep the value of $k\omega_0$ constant. In our case, for $y(t)$: $\omega_0 = \frac{\pi}{2} = \hat{\omega}_0/2$

$$b'_k = \begin{cases} \frac{-j}{2}, & k = 2 \\ \frac{j}{2}, & k = -2 \\ 0, & \text{otherwise} \end{cases}$$

Using the multiplication property:

$$c_k = \sum_{n=-\infty}^{+\infty} a_n b'_{k-n}, \text{ which looks like the discrete-time convolution, in frequency.}$$

Note that $a_n b'_{k-n} \neq 0$ only when $k - n = \pm 2$

$$\rightarrow c_k = a_{k-2} b'_2 + a_{k+2} b'_{-2} = a_{k-2}\frac{-j}{2} + a_{k+2}\frac{j}{2} = \frac{j}{2}(a_{k+2} - a_{k-2})$$

$$a_{k+2} = \begin{cases} \frac{\sin{(k+2)\frac{\pi}{2}}}{(k+2)\pi}, & k \neq -2 \\ \frac{1}{2}, & k = -2 \end{cases} = \begin{cases} \frac{-\sin{(k\frac{\pi}{2})}}{(k+2)\pi}, & k \neq -2 \\ \frac{1}{2}, & k = -2 \end{cases}$$

$$a_{k-2} = \begin{cases} \frac{\sin{(k-2)\frac{\pi}{2}}}{(k-2)\pi}, & k \neq 2 \\ \frac{1}{2}, & k = 2 \end{cases} = \begin{cases} \frac{-\sin{(k\frac{\pi}{2})}}{(k-2)\pi}, & k \neq 2 \\ \frac{1}{2}, & k = 2 \end{cases}$$

Problem 6 Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefficients of each of the following signals in terms of a_k:

(a) $\mathcal{O}d\{x(t-T/2)\}$

$$\begin{aligned} x(t-T/2) \longleftrightarrow b_k &= a_k e^{-jk\omega_0 \frac{T}{2}} \quad \text{(Time Shifting Property)} \\ &= a_k e^{-jk\pi} = a_k (e^{-j\pi})^k \\ &= a_k (-1)^k \end{aligned}$$

If we assume that $x(t)$ is real, then:

$$\begin{aligned} \mathcal{O}d\{x(t-T/2)\} \longleftrightarrow c_k &= j\,\Im m\{b_k\} \quad \text{(Even-Odd Decomposition of Real Signals Propriety, Table 3.1, O \& W, p. 206)} \\ &= j\,\Im m\{a_k(-1)^k\} = (-1)^k j \Im m\{a_k\}. \end{aligned}$$

However, the question didn't specify $x(t)$ to be real, so assuming that $x(t)$ is complex, we will just use the general formula for finding the Odd part of a signal:

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)] \quad \text{(O \&W, Sec. 1.2.3, and specifically eq.(1.19), p.14)}$$

$$\begin{aligned} \mathcal{O}d\{x(t-T/2)\} = \frac{1}{2}\left[x(t-T/2) - x(-t-T/2)\right] \longleftrightarrow d_k &= \frac{1}{2}[a_k(-1)^k - a_{-k}(-1)^{-k}] \\ &= \frac{1}{2}(-1)^k(a_k - a_{-k}). \end{aligned}$$

Note that for real $x(t)$: $a_{-k} = a_k^* \rightarrow a_k - a_{-k} = 2j\,\Im m\{a_k\} \rightarrow c_k = d_k$.

(b) $x(T/4-t)$

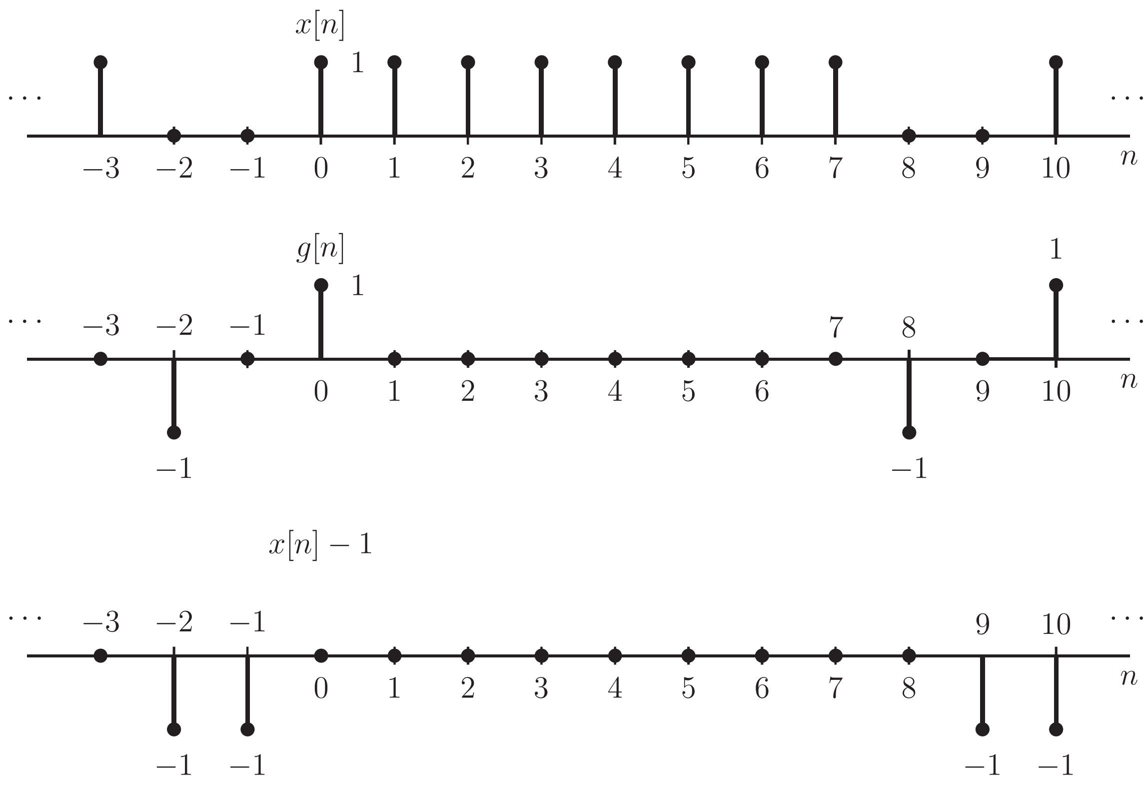
$$x(-t) \longleftrightarrow c_k = a_{-k} \quad \text{(Time-Reversal Property)}$$

$$\begin{aligned} x(T/4-t) \longleftrightarrow d_k = c_k e^{-jk\omega_0 T/4} \quad &\text{(Time-Shift in the positive time direction, i.e. delay)} \\ = c_k e^{-jk\frac{\pi}{2}} = c_k(-j)^k = a_{-k}(-j)^k. \end{aligned}$$

Problem 7 O & W 3.31 (also determine a_0)

Let $x[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & 8 \le n \le 9 \end{cases}$, $x[n]$: periodic, $N = 10$, Fourier series coefficients: a_k.

Also, let $g[n] = x[n] - x[n-1]$.



Fundamental Period $= N = 10 \rightarrow \omega_0 = \frac{\pi}{5}$.

$$a_0 = \frac{1}{N} \sum_{n=<N>} x[n] = \frac{1}{10} \sum_{n=0}^{9} x[n] = \frac{1}{10}[1(8) + 0(2)] = 8/10 = 4/5.$$

(a) Show that $g[n]$ has a fundamental period of 10.

$$g[n+N] = x[n+N] - x[n+N-1]$$

$$\because x[n+N] = x[n] \rightarrow g[n+N] = x[n] - x[n-1] = g[n]$$

$$\rightarrow g[n] \text{ has a fundamental period of } N = 10.$$

(b) Determine the Fourier series coefficients of $g[n]$.

$$g[n] = x[n] - x[n-1] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le 7 \\ -1, & n = 8. \end{cases}$$

$$\begin{aligned} b_k &= \frac{1}{N} \sum_{n=<N>} g[n] e^{-jk\omega_0 n} \\ &= \frac{1}{10} \sum_{n=-2}^{7} g[n] e^{-jk\omega_0 n} \text{ , the limits were chosen to use the non-zeros near the origin} \\ &= \frac{1}{10} \left[(-1)e^{-jk\omega_0(-2)} + (1)e^{-jk\omega_0(0)}\right] = \frac{1}{10}\left[1 - e^{jk\omega_0 2}\right] = \frac{1}{10} e^{jk\omega_0} \left[e^{-jk\omega_0} - e^{jk\omega_0)}\right] \\ &= \frac{-j2}{10} e^{jk\omega_0} \left[\frac{e^{jk\omega_0} - e^{-jk\omega_0}}{2j}\right] = \frac{-j}{5} e^{jk\omega_0} \sin k\omega_0 = \frac{-j}{5} e^{jk\frac{\pi}{5}} \sin k\frac{\pi}{5} \end{aligned}$$

(c) Using the Fourier series coefficients of $g[n]$ and the First-Difference property in Table 3.2, determine a_k for $k \neq 0$.

From Table 3.2 (O & W, p. 221): $x[n] - x[n-1] \longleftrightarrow (1 - e^{-jk(2\pi/N)})\, a_k = b_k$

$$\rightarrow b_k = \frac{1}{10}(1 - e^{jk\omega_0 2}) = (1 - e^{-jk\omega_0})\, a_k$$

$$\begin{aligned} a_k &= \frac{1}{10} \frac{1 - e^{jk\omega_0 2}}{1 - e^{-jk\omega_0}} = \frac{1}{10} \frac{(1 - e^{jk\omega_0})(1 + e^{jk\omega_0})}{e^{-jk\omega_0}(e^{jk\omega_0} - 1)} = \frac{-1}{10} \frac{(1 + e^{jk\omega_0})}{e^{-jk\omega_0}} \\ &= \frac{-1}{10} \frac{e^{jk\omega_0 \frac{1}{2}}}{e^{-jk\omega_0}} (e^{-jk\omega_0 \frac{1}{2}} + e^{jk\omega_0 \frac{1}{2}}) = \frac{-1}{5} e^{jk\omega_0 \frac{3}{2}} \cos\left(k\omega_0 \frac{1}{2}\right) \\ &= \frac{-1}{5} e^{jk\frac{3\pi}{10}} \cos\left(k\frac{\pi}{10}\right). \end{aligned}$$

Let's double check the result, and at the same time use another route to find a_k:

Note that $x[n] - 1$ would have the same a_k (only a_0 changes with a change in the DC level of a signal).

$$x[n] - 1 = \begin{cases} 0, & 0 \le n \le 7 \\ -1, & 8 \le n \le 9 \end{cases} = \begin{cases} 0, & 0 \le n \le 7 \\ -1, & -2 \le n \le -1 \end{cases}$$

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=<N>} (x[n] - 1) e^{-jk\omega_0 n} = \frac{1}{10}\left[(-1)e^{-jk\omega_0(-2)} + (-1)e^{-jk\omega_0(-1)}\right] = \frac{-1}{10}\left[e^{jk\omega_0 2} + e^{jk\omega_0}\right] \\ &= \frac{-1}{10} e^{jk\omega_0(3/2)} \left[e^{jk\omega_0(\frac{1}{2})} + e^{-jk\omega_0(\frac{1}{2})}\right] = \frac{-1}{5} e^{jk\omega_0(3/2)} \cos(k\omega_0 \frac{1}{2}) = \frac{-1}{5} e^{jk\frac{3\pi}{10}} \cos\left(k\frac{\pi}{10}\right). \end{aligned}$$

Problem 8 O & W 3.51

$x[n]$: periodic signal with period $N = 8$ and Fourier series coefficients $a_k = -a_{k-4}$.
$y[n]$: periodic signal with period $N = 8$ and Fourier series coefficients b_k ,

$$y[n] = \left(\frac{1 + (-1)^n}{2}\right) x[n-1]$$

Find a function $f[k]$ such that $b_k = f[k]a_k$.

$$y[n] = \left(\frac{1 + (-1)^n}{2}\right) x[n-1] = \frac{1}{2}x[n-1] + \frac{1}{2}(-1)^n x[n-1].$$

$$x[n-1] \longleftrightarrow a_k e^{-jk\omega_0(1)} \quad \text{(Time Shifting Property)} \qquad (1)$$

Note that $(-1)^n = (e^{j\pi})^n = e^{j4(\frac{2\pi}{8})n} = e^{j4\omega_0 n}, \omega_0 = 2\pi/8 = \frac{\pi}{4}$

$$(-1)^n x[n-1] = e^{j4\omega_0 n} x[n-1] \longleftrightarrow a_{k-4} e^{-j(k-4)\omega_0} \quad \text{(Frequency Shifting Property)} \qquad (2)$$

From (1) and (2): $y[n] = \frac{1}{2}x[n-1] + \frac{1}{2}(-1)^n x[n-1] \longleftrightarrow b_k = \frac{1}{2}\, a_k e^{-jk\omega_0} + \frac{1}{2}\, a_{k-4} e^{-j(k-4)\omega_0}$.
Substituting $a_k = -a_{k-4}$:

$$\begin{aligned} b_k &= \frac{1}{2}\, a_k e^{-jk\omega_0} + \frac{1}{2}\, (-a_k)\, e^{-jk\omega_0} e^{j4\omega_0} \\ &= \frac{1}{2}\, a_k e^{-jk\omega_0} (1 - e^{j4\omega_0}) = \frac{(1 - e^{j4\omega_0})}{2}\, e^{-jk\omega_0} a_k \\ &= \frac{(1 - e^{j4\frac{\pi}{4}})}{2}\, e^{-jk\frac{\pi}{4}} a_k = \frac{1 - (-1)}{2}\, e^{-jk\frac{\pi}{4}} a_k \\ &= e^{-jk\frac{\pi}{4}} a_k \end{aligned}$$

$$\rightarrow f[k] = e^{-jk\frac{\pi}{4}} .$$