

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ



سیگنال‌ها و سیستم‌ها

درس ۲۹

تبدیل Z (۲)

The z-Transform (2)

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<http://courses.fouladi.ir/sigsys>

طرح درس

COURSE OUTLINEخصوصیات تبدیل z Properties of the z -Transform

توابع سیستم سیستم‌های LTI گسسته-زمان؛ علی بودن، پایداری

System Functions of DT LTI Systems; Causality, Stability

ارزیابی هندسی تبدیل‌های z و پاسخ‌های فرکانسی گسسته-زمانGeometric Evaluation of z -Transforms and DT Frequency Responses

سیستم‌های مرتبه-اول و مرتبه-دوم

First- and Second-Order Systems

جبر تابع سیستم و نمودارهای بلوکی

System Function Algebra and Block Diagrams

تبدیل‌های z یک‌طرفهUnilateral z -Transforms

تبدیل Z

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خصوصیات
تبدیل Z

Properties of z-Transforms

(1) Time Shifting $x[n - n_0] \longleftrightarrow z^{-n_0} X(z),$

The rationality of $X(z)$ unchanged, *different* from LT. ROC unchanged except for the possible addition or deletion of the origin or infinity

$$n_0 > 0 \Rightarrow \text{ROC } z \neq 0 \text{ (maybe)}$$

$$n_0 < 0 \Rightarrow \text{ROC } z \neq \infty \text{ (maybe)}$$

(2) z-Domain Differentiation $nx[n] \longleftrightarrow -z \frac{dX(z)}{dz}, \text{ same ROC}$

Derivation:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = - \sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$$

$$-z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

تبدیل Z

خصوصیات

THE Z-TRANSFORM

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Delay	$x[n - 1]$	$z^{-1}X(z)$	R
Multiply by n	$nx[n]$	$-z \frac{dX(z)}{dz}$	R
Convolve in n	$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n - m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

تبدیل Z

مثال

THE Z-TRANSFORM

$$Y(z) = \left(\frac{z}{z-1} \right)^2 \leftrightarrow y[n] = ?$$

$$\frac{z}{z-1} \leftrightarrow u[n]$$

$$-z \frac{d}{dz} \left(\frac{z}{z-1} \right) = z \left(\frac{1}{z-1} \right)^2 \leftrightarrow nu[n]$$

$$z \times \left(-z \frac{d}{dz} \left(\frac{z}{z-1} \right) \right) = \left(\frac{z}{z-1} \right)^2 \leftrightarrow (n+1)u[n+1] = (n+1)u[n]$$

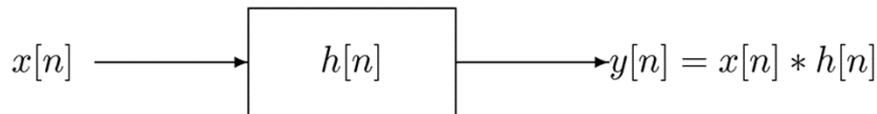
تبدیل Z

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توابع سیستم
سیستم‌های
LTI

گسسته-زمان؛
علی بودن،
پایداری

Convolution Property and System Functions



$Y(z) = H(z)X(z)$, ROC at least the intersection of the ROCs of $H(z)$ and $X(z)$,
can be bigger if there is pole/zero
cancellation. *e.g.*

$$H(z) = \frac{1}{z - a}, \quad |z| > a$$

$$X(z) = z - a, \quad z \neq \infty$$

$$Y(z) = 1 \quad \text{ROC all } z$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad \text{— The System Function}$$

$H(z)$ + ROC tells us everything about system

CAUSALITY

- (1) $h[n]$ right-sided $\Rightarrow$ ROC is the exterior of a circle *possibly* including $z = \infty$:

$$H(z) = \sum_{n=N_1}^{\infty} h[n]z^{-n}$$

If $N_1 < 0$, then the term $h[N_1]z^{-N_1} \rightarrow \infty$ at $z = \infty$
 $\Rightarrow$ ROC outside a circle, but does *not* include ∞ .

Causal $\Leftrightarrow N_1 \geq 0$

$\Downarrow$

No z^m terms with $m > 0$

$\Rightarrow z = \infty \in \text{ROC}$

A DT LTI system with system function $H(z)$ is causal $\Leftrightarrow$ the ROC of $H(z)$ is the exterior of a circle *including* $z = \infty$

Causality for Systems with Rational System Functions

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \dots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \dots + a_1 z + a_0}$$

$\Downarrow$ No poles at ∞ , if $M \leq N$

A DT LTI system with rational system function $H(z)$ is causal

$\Leftrightarrow$ (a) the ROC is the exterior of a circle outside the outermost pole;

and (b) if we write $H(z)$ as a ratio of polynomials

$$H(z) = \frac{N(z)}{D(z)}$$

then

$$\text{degree } N(z) \leq \text{degree } D(z)$$

Stability

- LTI System Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow$ ROC of $H(z)$ includes the unit circle $|z| = 1$

$\Rightarrow$ Frequency Response $H(e^{j\omega})$ (DTFT of $h[n]$) exists.

- A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1

تبدیل Z

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ارزیابی
هندسی
تبدیل‌های Z و
پاسخ‌های
فرکانسی
گسسته-زمان

Geometric Evaluation of a Rational z-Transform

Example #1:

$$X_1(z) = z - a - \text{A first-order zero}$$

Example #2:

$$X_2(z) = \frac{1}{z - a} - \text{A first-order pole}$$

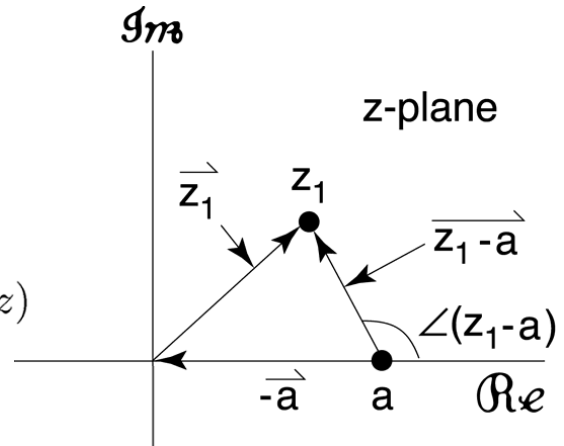
$$|X_2(z)| = \frac{1}{|X_1(z)|}, \quad \angle X_2(z) = -\angle X_1(z)$$

Example #3:

$$X(z) = M \frac{\prod_{i=1}^R (z - \beta_i)}{\prod_{j=1}^P (z - \alpha_j)}$$

$$|X(z)| = |M| \frac{\prod_{i=1}^R |z - \beta_i|}{\prod_{j=1}^P |z - \alpha_j|}$$

$$\angle X(z) = \angle M + \sum_{i=1}^R \angle(z - \beta_i) - \sum_{j=1}^P \angle(z - \alpha_j)$$



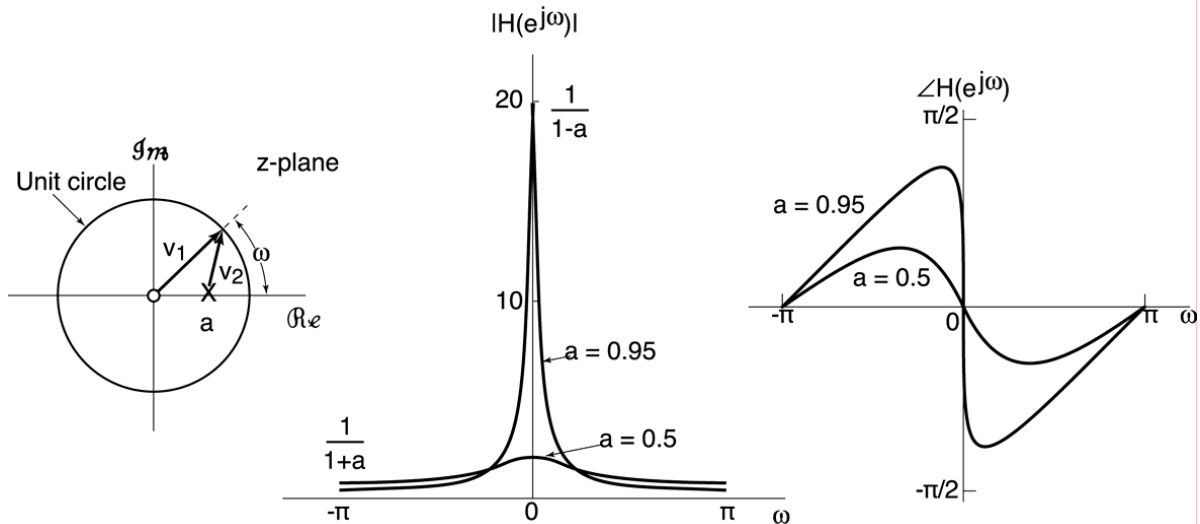
All same as
in s-plane

Geometric Evaluation of DT Frequency Responses

First-Order System
— one *real* pole

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$

$$h[n] = a^n u[n], \quad |a| < 1$$



$$H(e^{j\omega}) = \frac{v_1}{v_2}, \quad |H(e^{j\omega})| = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}, \quad \angle H(e^{j\omega}) = \angle v_1 - \angle v_2 = \omega - \angle v_2$$

تبدیل Z

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سیستم‌های
مرتبه-اول
و
مرتبه-دوم

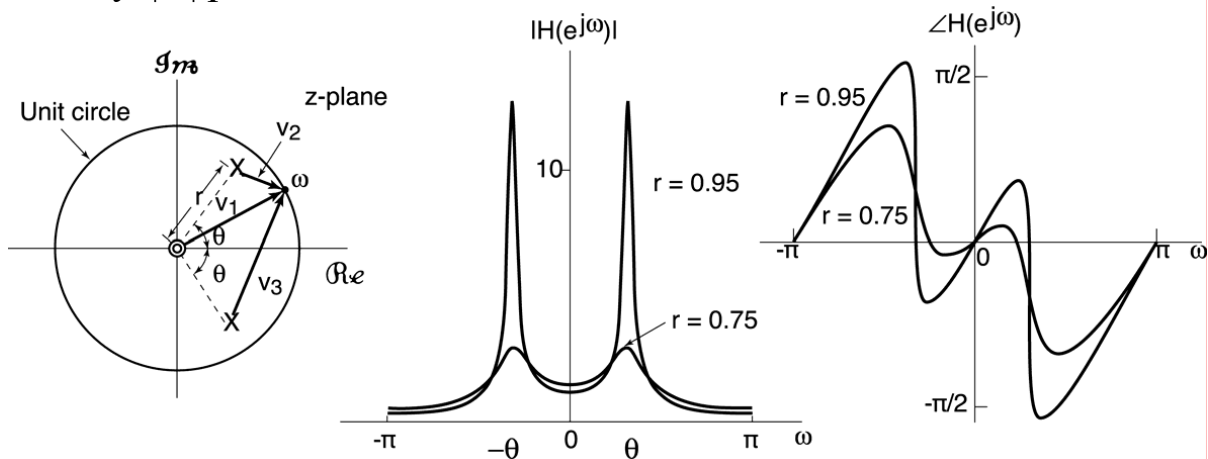
Second-Order System

Two poles that are a complex conjugate pair ($z_1 = re^{j\theta} = z_2^*$)

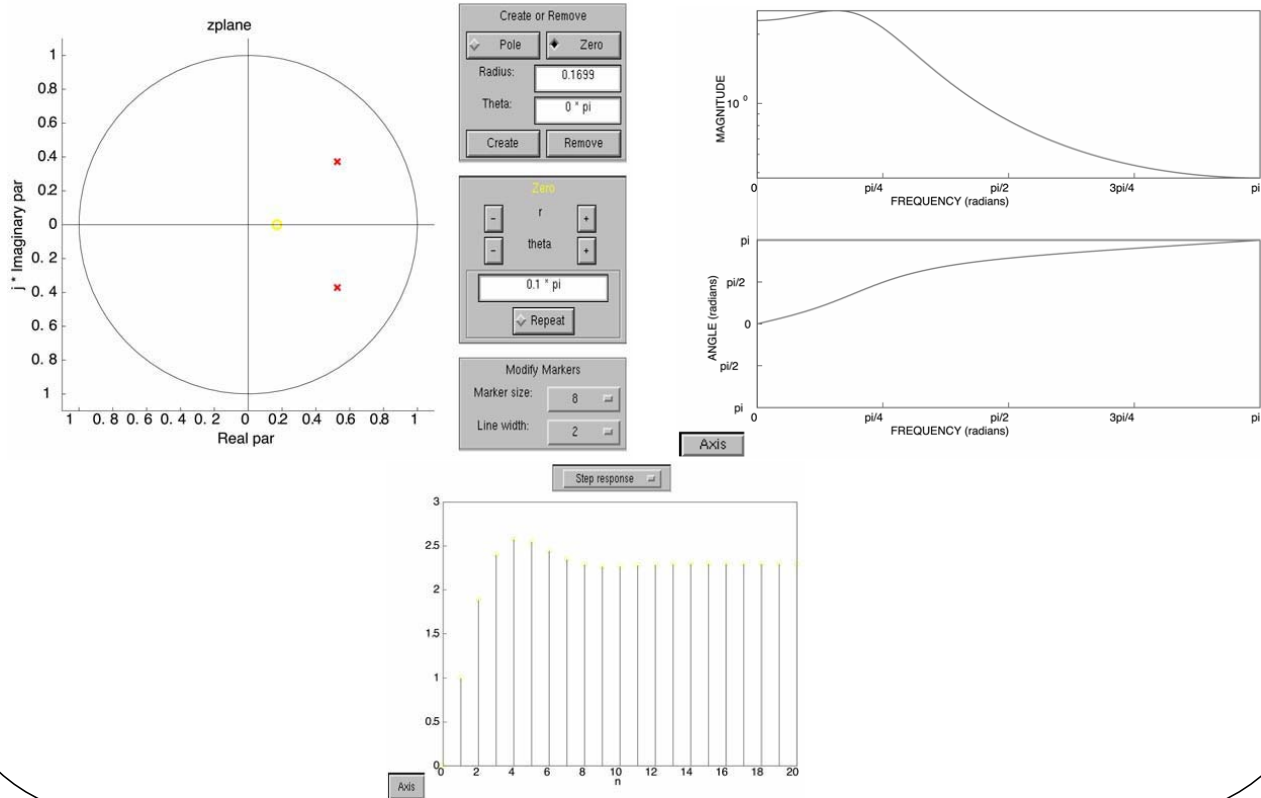
$$H(z) = \frac{z^2}{(z - z_1)(z - z_2)} = \frac{1}{1 - (2r \cos \theta)z^{-1} + r^2 z^{-2}}, \quad 0 < r < 1, \quad 0 \leq \theta \leq \pi$$

$$|H(e^{j\omega})| = \frac{1}{|(e^{j\omega} - re^{j\theta})(e^{j\omega} - re^{-j\theta})|}, \quad h[n] = r^n \frac{\sin[(n+1)\theta]}{\sin \theta} u[n]$$

Clearly, $|H|$ peaks near $\omega = \pm \theta$



Demo: DT pole-zero diagrams, frequency response, vector diagrams, and impulse- & step-responses



DT LTI Systems Described by LCCDEs

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Use the time-shift property

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

$\Downarrow$

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} \quad \text{— Rational}$$

ROC: Depends on Boundary Conditions, left-, right-, or two-sided.

For Causal Systems $\Rightarrow$ ROC is outside the outermost pole

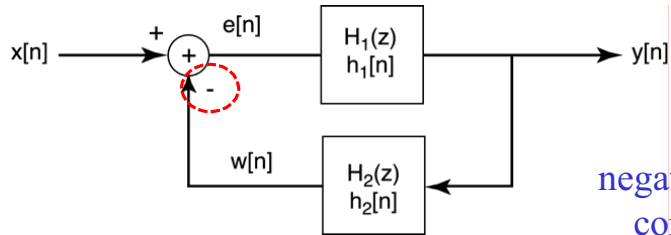
تبدیل Z

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جبر
تابع سیستم
و
نمودارهای
بلوکی

System Function Algebra and Block Diagrams

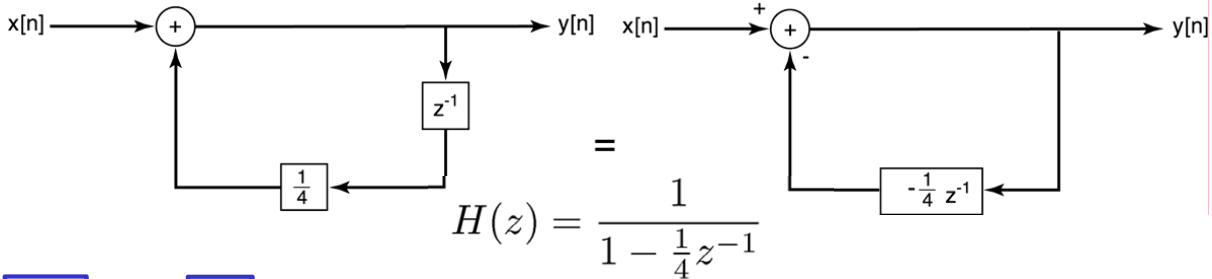
Feedback System
(causal systems)



negative feedback
configuration

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

Example #1:



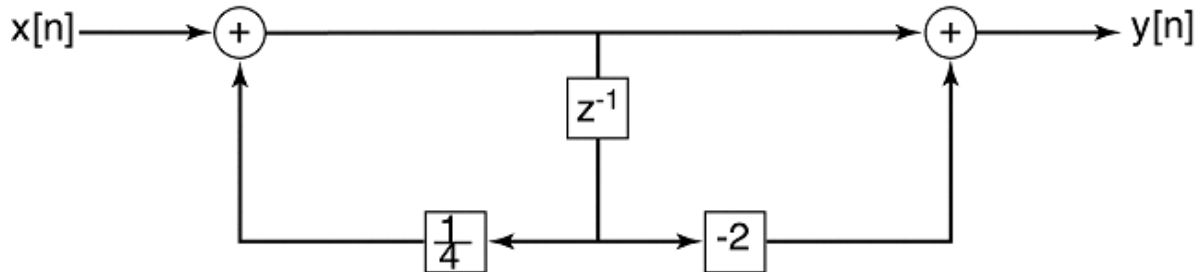
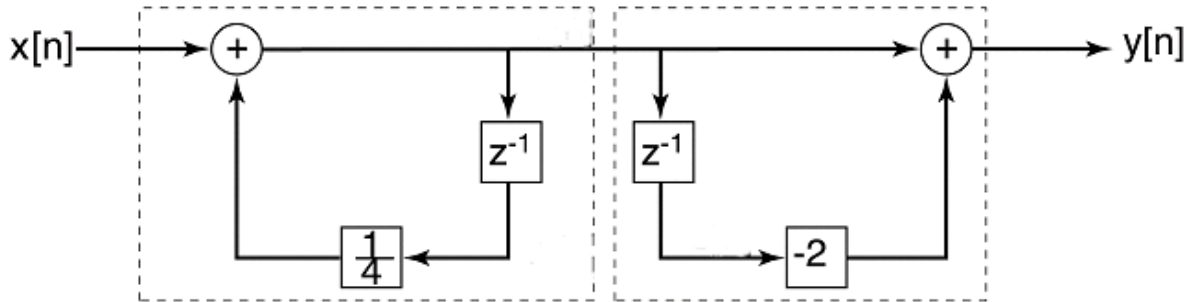
$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$z^{-1} \Leftrightarrow \boxed{D}$
Delay

$$y[n] = \frac{1}{4}y[n-1] + x[n]$$

Example #2:

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right) (1 - 2z^{-1}) \quad \text{--- Cascade of two systems}$$



تبدیل Z

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تبدیل‌های Z
یک‌طرفه

Unilateral z-Transform

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Note:

- (1) If $x[n] = 0$ for $n < 0$, then $\mathcal{X}(z) = X(z)$
- (2) UZT of $x[n] = \text{BZT of } x[n]u[n] \Rightarrow \text{ROC } \textcolor{red}{always} \text{ outside a circle}$
and $\textcolor{blue}{includes} z = \infty$
- (3) For causal LTI systems, $\mathcal{H}(z) = H(z)$

Properties of Unilateral z-Transform

Many properties are analogous to properties of the BZT e.g.

- Convolution property (for $x_1[n < 0] = x_2[n < 0] = 0$)

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{U}\mathcal{Z}} \mathcal{X}_1(z)\mathcal{X}_2(z)$$

- But there are important differences. For example, *time-shift*

$$y[n] = x[n - 1] \longleftrightarrow \mathcal{Y}(z) = \underbrace{x[-1]}_{\text{Initial condition}} + z^{-1}\mathcal{X}(z)$$

Derivation:

$$\begin{aligned}\mathcal{Y}(z) &= \sum_{n=0}^{\infty} y[n]z^{-n} = \sum_{n=0}^{\infty} x[n - 1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n - 1]z^{-n} \\ &= x[-1] + z^{-1} \underbrace{\sum_{m=0}^{\infty} x[m]z^{-m}}_{\mathcal{X}(z)}\end{aligned}$$

Use of UZTs in Solving Difference Equations with Initial Conditions

$$y[n] + 2y[n-1] = x[n]$$

$$y[-1] = \beta, \quad x[n] = \alpha u[n] \longleftrightarrow \frac{\alpha}{1 - z^{-1}}$$

UZT of Difference Equation

$$\mathcal{Y}(z) + 2 \overbrace{[\beta + z^{-1} \mathcal{Y}(z)]}^{\mathcal{UZ}\{y[n-1]\}} = \frac{\alpha}{1 - z^{-1}}$$

$$\Downarrow$$
$$\mathcal{Y}(z) = - \underbrace{\frac{2\beta}{1 + 2z^{-1}}}_{ZIR} + \underbrace{\frac{\alpha}{(1 + 2z^{-1})(1 - z^{-1})}}_{ZSR}$$

ZIR — Output purely due to the initial conditions,

ZSR — Output purely due to the input.

Example (continued)

$\beta = 0 \Rightarrow$ System is initially at rest:

$$\text{ZSR} \quad \mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) = \underbrace{\frac{1}{1+2z^{-1}}}_{\mathcal{H}(z)} \underbrace{\frac{\alpha}{1-z^{-1}}}_{\mathcal{X}(z)}$$

$$\mathcal{H}(z) = H(z) = \frac{1}{1+2z^{-1}}$$

$\alpha = 0 \Rightarrow$ Get response to initial conditions

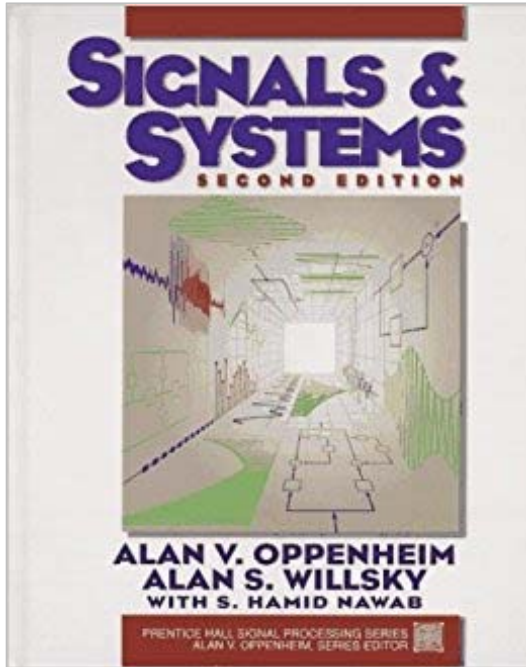
$$\text{ZIR} \quad \mathcal{Y}(z) = -\frac{2\beta}{1+2z^{-1}}$$

$$y[n] = -2\beta(-2)^n u[n]$$

تبدیل Z



منابع



A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
Signals and Systems,
Second Edition, Prentice Hall, 1997.

Chapter 10