# سیگنال‌ها و سیستم‌ها

## درس ۲۵

# تبدیل لاپلاس (۲)

# The Laplace Transform (2)

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# طرح درس

## COURSE OUTLINE

| تبدیل لاپلاس معکوس |
|---|
| Inverse Laplace Transforms |
| خصوصیات تبدیل لاپلاس |
| Laplace Transform Properties |
| تابع سیستم برای یک سیستم خطی تغییرناپذیر با زمان |
| The System Function of an LTI System |
| ارزیابی هندسی تبدیل‌های لاپلاس و پاسخ‌های فرکانسی |
| Geometric Evaluation of Laplace Transforms and Frequency Responses |

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# تبدیل لاپلاس معکوس

# Inverse Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega \in \text{ ROC}$$
$$= \mathcal{F}\{x(t)e^{-\sigma t}\}$$

Fix $\sigma \in$ ROC and apply the inverse Fourier transform

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

But $s = \sigma + j\omega$ ($\sigma$ fixed) $\Rightarrow ds = jd\omega$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}ds$$

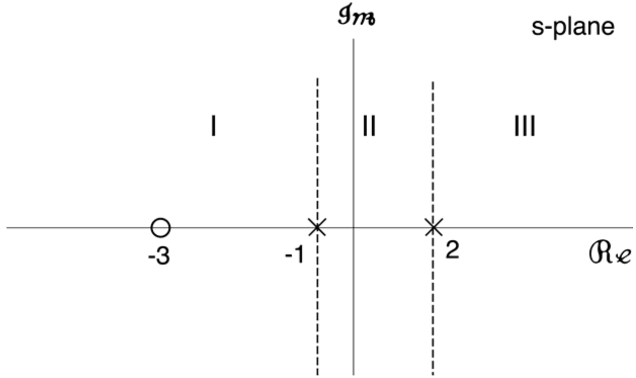
# Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

**Example:**

$$X(s) = \frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = -\frac{2}{3}, \quad B = \frac{5}{3}$$

Three possible ROC's — corresponding to three *different* signals




Recall

$$\frac{1}{s+a}, \quad \Re e\{s\} < -a \longleftrightarrow -e^{-at}u(-t) \text{ left-sided}$$

$$\frac{1}{s+a}, \quad \Re e\{s\} > -a \longleftrightarrow e^{-at}u(t) \text{ right-sided}$$

ROC I: — Left-sided signal.

$$
\begin{aligned}
x(t) &= -Ae^{-t}u(-t) - Be^{2t}u(-t) \\
&= \left[\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t}\right]u(-t) \qquad \text{Diverges as } t \to -\infty
\end{aligned}
$$

ROC II: — Two-sided signal, has Fourier Transform.

$$
\begin{aligned}
x(t) &= Ae^{-t}u(t) - Be^{2t}u(-t) \\
&= -\left[\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t)\right] \quad \to 0 \text{ as } t \to \pm\infty
\end{aligned}
$$

ROC III:— Right-sided signal.

$$
\begin{aligned}
x(t) &= Ae^{-t}u(t) + Be^{2t}u(t) \\
&= \left[-\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t}\right]u(t) \qquad \text{Diverges as } t \to +\infty
\end{aligned}
$$

# تبدیل لاپلاس معکوس

Expand with partial fractions:

$$\frac{2s}{s^2-4} = \underbrace{\frac{1}{s+2}}_{\text{pole at } -2} + \underbrace{\frac{1}{s-2}}_{\text{pole at } 2}$$

| pole | function | right-sided; ROC | left-sided (ROC) |
|---|---|---|---|
| $-2$ | $e^{-2t}$ | $e^{-2t}u(t)$; $\mathrm{Re}(s) > -2$ | $-e^{-2t}u(-t)$; $\mathrm{Re}(s) < -2$ |
| $2$ | $e^{2t}$ | $e^{2t}u(t)$; $\mathrm{Re}(s) > 2$ | $-e^{2t}u(-t)$; $\mathrm{Re}(s) < 2$ |

| | | | |
|---|---|---|---|
| 1. | $e^{-2t}u(t) + e^{2t}u(t)$ | $\mathrm{Re}(s) > -2 \cap \mathrm{Re}(s) > 2$ | $\mathrm{Re}(s) > 2$ |
| 2. | $e^{-2t}u(t) - e^{2t}u(-t)$ | $\mathrm{Re}(s) > -2 \cap \mathrm{Re}(s) < 2$ | $-2 < \mathrm{Re}(s) < 2$ |
| 3. | $-e^{-2t}u(-t) + e^{2t}u(t)$ | $\mathrm{Re}(s) < -2 \cap \mathrm{Re}(s) > 2$ | none |
| 4. | $-e^{-2t}u(-t) - e^{2t}u(-t)$ | $\mathrm{Re}(s) < -2 \cap \mathrm{Re}(s) < 2$ | $\mathrm{Re}(s) < -2$ |

تبدیل لاپلاس (۲)

# ۲

# خصوصیات تبدیل لاپلاس

# Properties of Laplace Transforms

- Many parallel properties of the CTFT, but for Laplace transforms we need to determine implications for the ROC
- For example:

Linearity

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(s) + b X_2(s)$$

ROC at least the intersection of ROCs of $X_1(s)$ and $X_2(s)$

ROC can be bigger (due to pole-zero cancellation)

E.g. $\quad x_1(t) = x_2(t)$ and $a = -b$

Then $\quad a x_1(t) + b x_2(t) = 0 \longrightarrow X(s) = 0$

$\Rightarrow$ROC entire *s*plane-

## Time Shift

$$x(t-T) \longleftrightarrow e^{-sT}X(s), \text{ same ROC as } X(s)$$

Example:

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad ?$$

$$\frac{e^{-sT}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2t}u(t)|_{t \to t-T}$$

$$\downarrow T = -3$$

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2(t+3)}u(t+3)$$

# Time-Domain Differentiation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds, \quad \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} sX(s)e^{st}ds$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s), \text{ with ROC containing the ROC of } X(s)$$

ROC could be bigger than the ROC of $X(s)$, if there is pole-zero cancellation. E.g.,

$$\begin{aligned} x(t) &= u(t) \leftrightarrow \tfrac{1}{s}, & \Re e\{s\} > 0 \\ \frac{dx(t)}{dt} &= \delta(t) \leftrightarrow 1 = s \cdot \tfrac{1}{s} & \text{ROC} = \text{entire s-plane} \end{aligned}$$

# *s*-Domain Differentiation

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds}, \text{ with same ROC as } X(s)$$

(Derivation is similar to $\frac{d}{dt} \leftrightarrow s$)

E.g., $$te^{-at}u(t) \leftrightarrow -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \quad \Re e\{s\} > -a$$

# تبدیل لاپلاس مشتق

## Laplace Transform of a Derivative

Assume that $X(s)$ is the Laplace transform of $x(t)$:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Find the Laplace transform of $y(t) = \dot{x}(t)$.

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt = \int_{-\infty}^{\infty} \underbrace{\dot{x}(t)}_{\dot{v}}\,\underbrace{e^{-st}}_{u}dt$$

$$= \underbrace{x(t)}_{v}\,\underbrace{e^{-st}}_{u}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{x(t)}_{v}(\underbrace{-se^{-st}}_{\dot{u}})dt$$

The first term must be zero since $X(s)$ converged. Thus

$$Y(s) = s\int_{-\infty}^{\infty} x(t)e^{-st}dt = sX(s)$$

## تبدیل لاپلاس تابع ضربه

### LAPLACE TRANSFORM OF THE IMPULSE FUNCTION

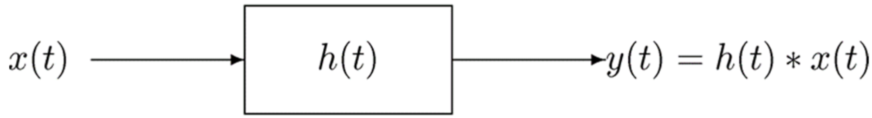
Let $x(t) = \delta(t)$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t)\; e^{-st}\Big|_{t=0} dt \\ &= \int_{-\infty}^{\infty} \delta(t)\; 1\; dt \\ &= 1 \end{aligned}$$

**Sifting property**:

$\delta(t)$ **sifts** out the value of $e^{-st}$ at $t = 0$.

# Convolution Property




For $x(t) \longleftrightarrow X(s), y(t) \longleftrightarrow Y(s), h(t) \longleftrightarrow H(s)$

Then $$Y(s) = H(s) \cdot X(s)$$

- ROC of $Y(s) = H(s)X(s)$: at least the overlap of the ROCs of $H(s)$ & $X(s)$
- ROC could be empty if there is no overlap between the two ROCs

  E.g.

  $$x(t) = e^t u(t) \quad , \text{ and } \quad h(t) = -e^{-t} u(-t)$$

- ROC could be larger than the overlap of the two.

  E.g.

  $$x(t) * h(t) = \delta(t)$$

# خواص تبدیل لاپلاس

## PROPERTIES OF LAPLACE TRANSFORMS

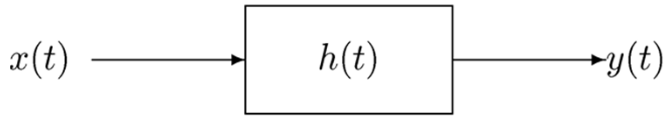
| Property | $x(t)$ | $X(s)$ | ROC |
|---|---|---|---|
| Linearity | $ax_1(t) + bx_2(t)$ | $aX_1(s) + bX_2(s)$ | $\supset (R_1 \cap R_2)$ |
| Delay by $T$ | $x(t-T)$ | $X(s)e^{-sT}$ | $R$ |
| Multiply by $t$ | $tx(t)$ | $-\frac{dX(s)}{ds}$ | $R$ |
| Multiply by $e^{-\alpha t}$ | $x(t)e^{-\alpha t}$ | $X(s+\alpha)$ | shift $R$ by $-\alpha$ |
| Differentiate in $t$ | $\frac{dx(t)}{dt}$ | $sX(s)$ | $\supset R$ |
| Integrate in $t$ | $\int_{-\infty}^{t} x(\tau)\, d\tau$ | $\frac{X(s)}{s}$ | $\supset \left(R \cap (\mathrm{Re}(s) > 0)\right)$ |
| Convolve in $t$ | $\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau$ | $X_1(s)X_2(s)$ | $\supset (R_1 \cap R_2)$ |

تبدیل لاپلاس (۲)

# ۳

# تابع سیستم برای یک سیستم خطی تغییرناپذیر با زمان

# The System Function of an LTI System




$h(t) \quad \longleftrightarrow \quad H(s)$ – the system function

The system function characterizes the system

$\Downarrow$

System properties correspond to properties of $H(s)$ and its ROC

A first example:

$$\text{System is stable} \Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \quad \Leftrightarrow \quad \begin{array}{l}\text{ROC of } H(s) \\ \text{includes the } j\omega \text{ axis}\end{array}$$

# تابع تبدیل سیستم

## محاسبه‌ی تابع تبدیل سیستم به کمک تبدیل لاپلاس

Start with differential equation:

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

Take the Laplace transform of a each term:

$$2s^2Y(s) + 3sY(s) + Y(s) = 2X(s)$$

Solve for system function:

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

# تابع تبدیل سیستم

محاسبه‌ی پاسخ ضربه‌ی سیستم به کمک تبدیل لاپلاس

Differential equation:

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

If $x(t) = \delta(t)$ then $y(t)$ is the impulse response $h(t)$.

If $X(s) = 1$ then $Y(s) = H(s)$.

تابع سیستم، تبدیل لاپلاس پاسخ ضربه است.

تبدیل لاپلاس (۲)

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# ارزیابی هندسی تبدیل‌های لاپلاس و پاسخ‌های فرکانسی

# Geometric Evaluation of Rational Laplace Transforms

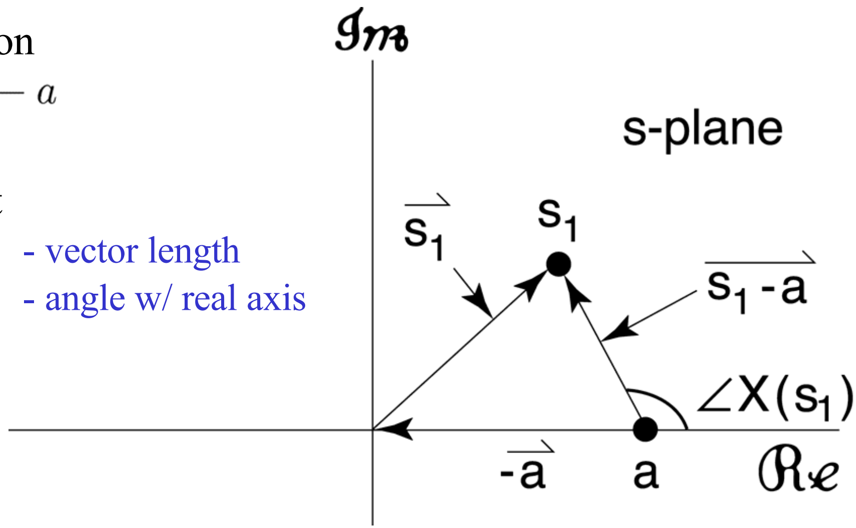
**Example #1:** $X_1(s) = s - a$ A first-order zero

Graphic evaluation of $X_1(s) = s_1 - a$

Can reason about

$|X_1(s)|$ - vector length

$\angle X_1(s)$ - angle w/ real axis



**Example #2:** A first-order pole

$$X_2(s) = \frac{1}{s-a} = \frac{1}{X_1(s)}$$

$$\Rightarrow |X_2(s)| = \frac{1}{|X_1(s)|} \quad (\text{or } \log|X_2(s)| = -\log|X_1(s)|)$$

$$\angle X_2(s) = -\angle X_1(s)$$

Still reason with vector, but remember to “invert” for poles

**Example #3:** A higher-order rational Laplace transform

$$X(s) = M\frac{\Pi_{i=1}^{R}(s-\beta_i)}{\Pi_{j=1}^{P}(s-\alpha_j)}$$

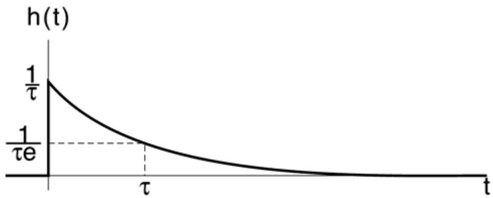
$$|X(s)| = |M|\frac{\Pi_{i=1}^{R}|s-\beta_i|}{\Pi_{j=1}^{P}|s-\alpha_j|}$$

$$\angle X(s) = \angle M + \sum_{i=1}^{R}\angle(s-\beta_i) - \sum_{j=1}^{P}\angle(s-\alpha_j)$$

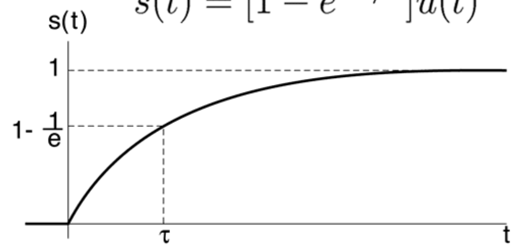
## First-Order System

$$H(s) = \frac{1}{s\tau + 1} = \frac{1/\tau}{s + 1/\tau}, \Re e\{s\} > -\frac{1}{\tau}$$

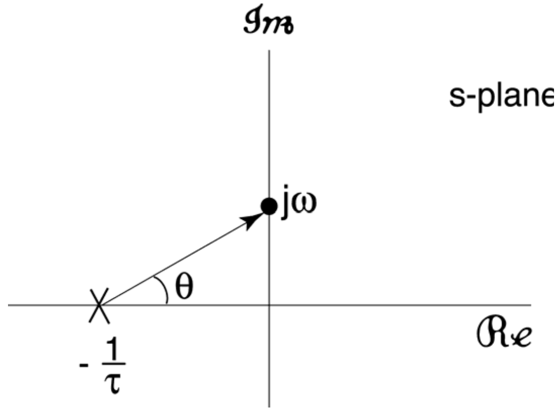
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$




$$s(t) = [1 - e^{-t/\tau}] u(t)$$




Graphical evaluation of $H(j\omega)$: $$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau} = \frac{1}{\tau} \cdot \frac{1}{j\omega + 1/\tau}$$



# Bode Plot of the First-Order System

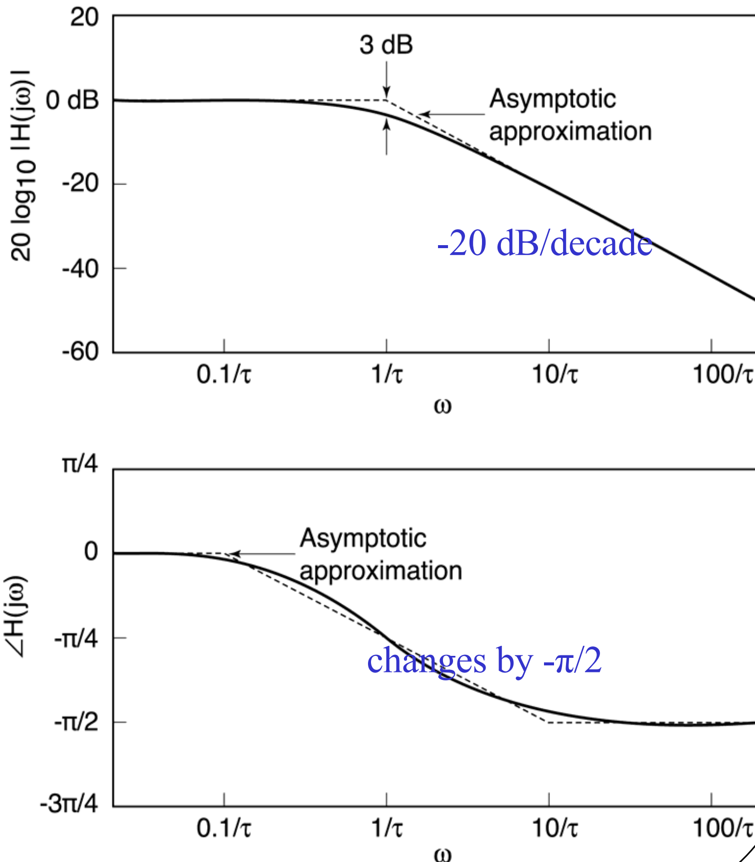
$$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$$

$$|H(j\omega)| = \frac{1/\tau}{\sqrt{\omega^2 + (1/\tau)^2}}$$

$$= \begin{cases} 1 & \omega = 0 \\ 1/\sqrt{2} & \omega = 1/\tau \\ 1/\omega\tau & \omega >> 1/\tau \end{cases}$$

$$\angle H(j\omega) = -\theta = -\tan^{-1}(\omega\tau)$$

$$= \begin{cases} 0 & \omega = 0 \\ -\pi/4 & \omega = 1/\tau \\ -\pi/2 & \omega >> 1/\tau \end{cases}$$



## Second-Order System

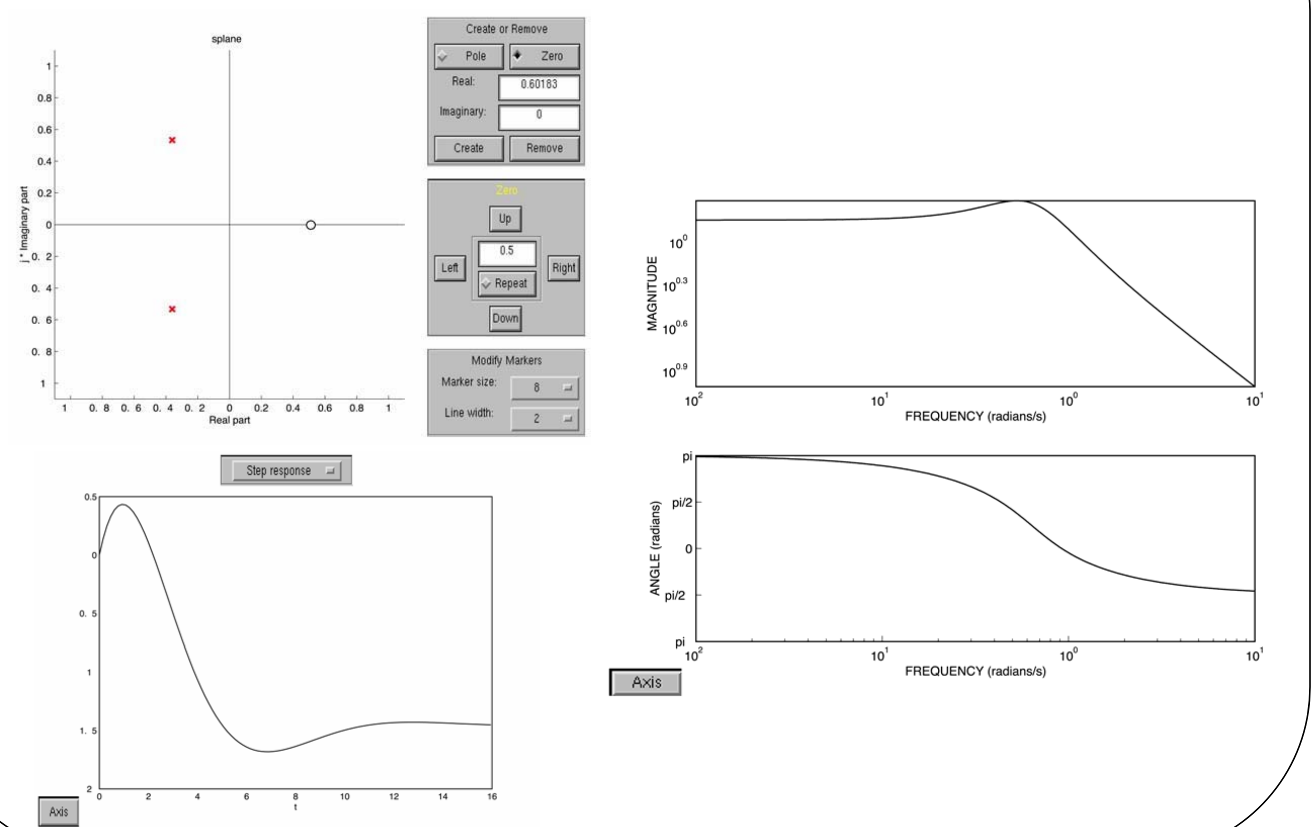
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \text{ROC } \Re e\{s\} > \Re e(\text{pole})$$

$0 < \zeta < 1$ $\Rightarrow$ complex poles
— *Under*damped

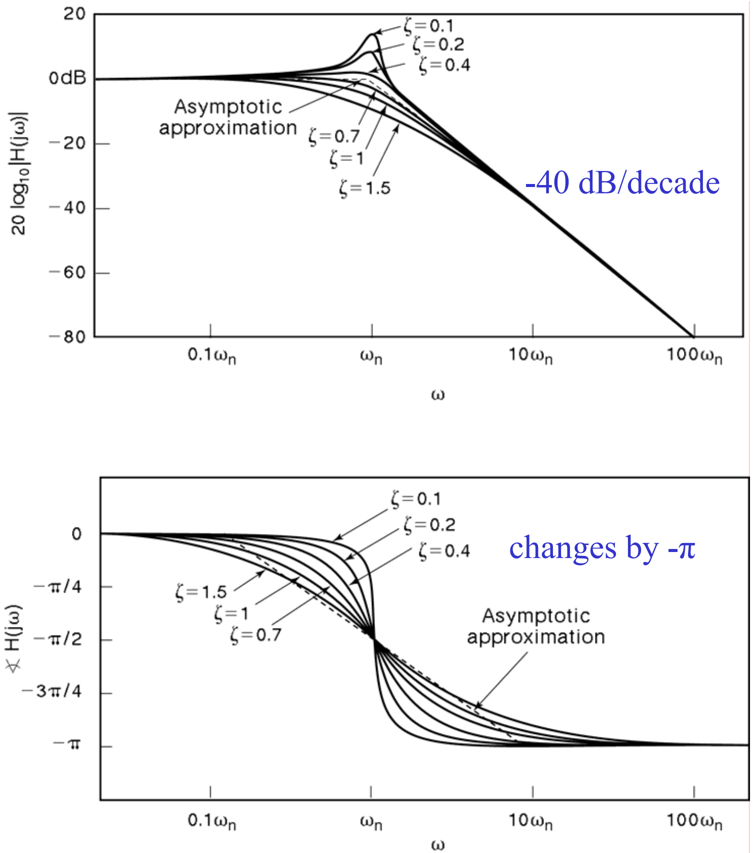
$\zeta = 1$ $\Rightarrow$ double pole at $s = -\omega_n$
— *Critically* damped

$\zeta > 1$ $\Rightarrow$ 2 poles on negative real axis
— *Over*damped

**Demo** Pole-zero diagrams, frequency response, and step response of first-order and second-order CT causal systems



# Bode Plot of a Second-Order System

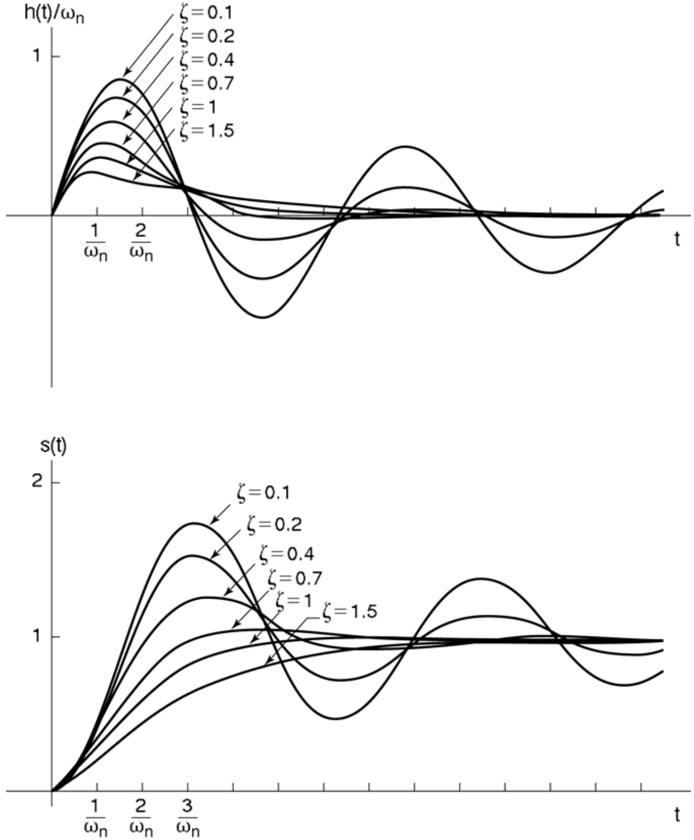



Top is flat when $\zeta = 1/\sqrt{2} = 0.707$

$\Rightarrow$ a LPF for

$\omega < \omega_n$

# Unit-Impulse and Unit-Step Response of a Second-Order System

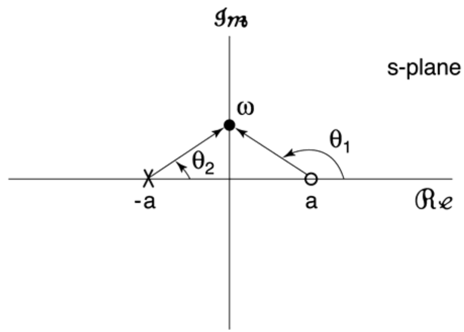



No oscillations when $\zeta \geq 1$

$\Rightarrow$ Critically (=) and over (>) damped.

# First-Order All-Pass System

$$H(s) = \frac{s-a}{s+a}, \quad \Re e\{s\} > -a \quad (a>0)$$

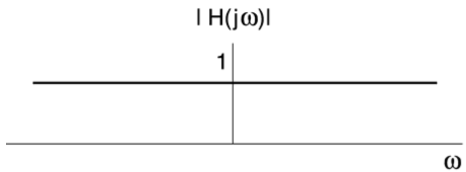


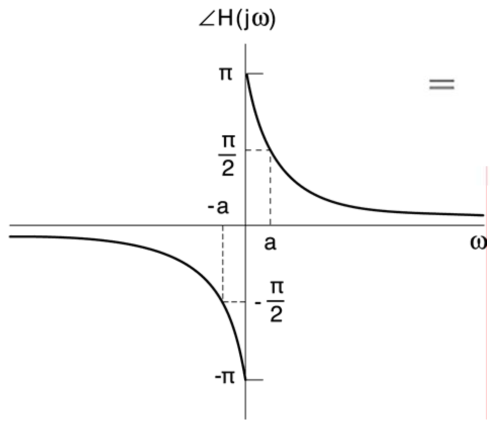

1. Two vectors have the same lengths
2. $\angle H(j\omega) = \theta_1 - \theta_2$

$$= (\pi - \theta_2) - \theta_2$$

$$= \pi - 2\theta_2$$

$$= \begin{cases} \pi & \omega = 0 \\ \pi/2 & \omega = a \\ \backsim 0 & \omega >> a \end{cases}$$



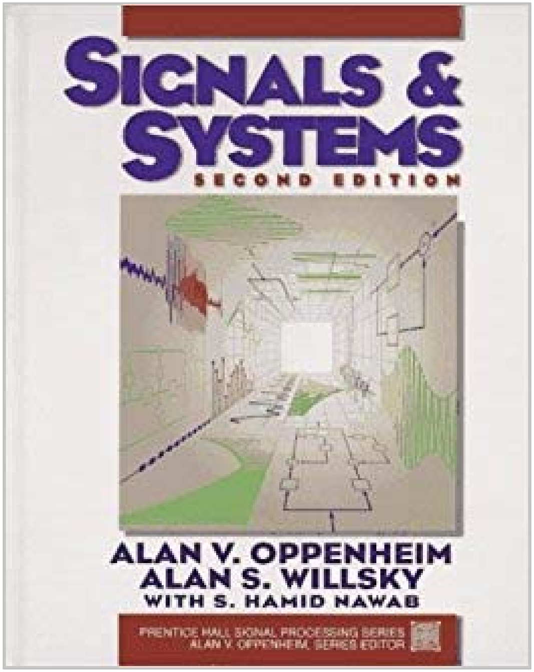



تبدیل لاپلاس (۲)

# ۵

# منابع

# منبع اصلی

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**Chapter 9**