



سیگنال‌ها و سیستم‌ها

درس ۲۴

تبدیل لاپلاس (۱)

The Laplace Transform (1)

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طرح درس

COURSE OUTLINE

بازنمایی‌های چندگانه‌ی سیستم‌های پیوسته-زمان

Multiple Representations of CT Systems

انگیزه و تعریف تبدیل لاپلاس (دوطرفه)

Motivation and Definition of the (Bilateral) Laplace Transform

مثال‌هایی از تبدیل‌های لاپلاس و نواحی همگرایی آنها (ROC)

Examples of Laplace Transforms and Their Regions of Convergence (ROCs)

خصوصیات نواحی همگرایی

Properties of ROCs

تبدیل لاپلاس (۱)

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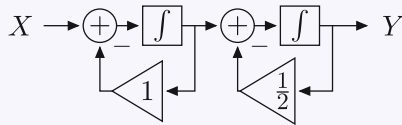
بازنمایی‌های
چندگانه‌ی
سیستم‌های
پیوسته-زمان

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

سیستم‌های پیوسته-زمان را می‌توان با روش‌های مختلفی بازنمایی کرد.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

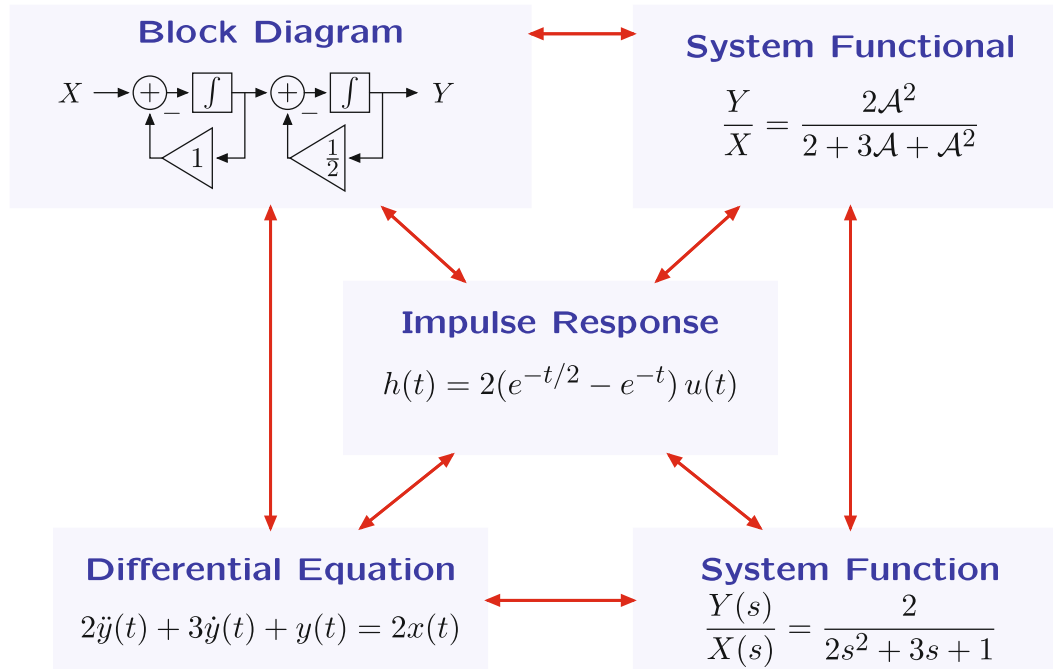
System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

بین بازنمایی‌های مختلف رابطه‌هایی وجود دارد.



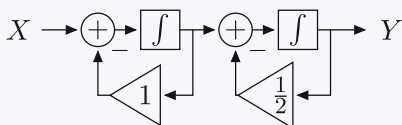
نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

دو تعبیر برای انتگرال:

$$X \rightarrow \left[\int \right] \rightarrow \mathcal{A}X$$

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})u(t)$$

$$\dot{x}(t) \rightarrow \left[\int \right] \rightarrow x(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

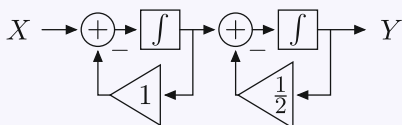
$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

رابطه‌ی میان «معادله‌ی تابعی سیستم» و «تابع سیستم»:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})u(t)$$

$$\mathcal{A} \rightarrow \frac{1}{s}$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

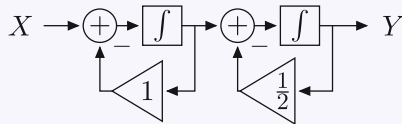
$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

رابطه‌ی میان «معادله‌ی تابعی سیستم» و «پاسخ ضربه»:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

رابطه‌ی میان «معادله‌ی تابعی سیستم» و «پاسخ ضربه»

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

چگونه می‌توان پاسخ ضربه را از روی معادله‌ی تابعی سیستم به دست آورد؟

معادله‌ی تابعی را با استفاده از کسرهای جزئی بسط می‌دهیم:

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2} = \frac{\mathcal{A}^2}{(1 + \frac{1}{2}\mathcal{A})(1 + \mathcal{A})} = \frac{2\mathcal{A}}{1 + \frac{1}{2}\mathcal{A}} - \frac{2\mathcal{A}}{1 + \mathcal{A}}$$

فرم‌های هر جمله را بازشناسی می‌کنیم: هر یک متناظر با یک نمایی هستند؛
یا اینکه هر جمله را در قالب یک سری بسط می‌دهیم:

$$\frac{Y}{X} = 2\mathcal{A}\left(1 - \frac{1}{2}\mathcal{A} + \frac{1}{4}\mathcal{A}^2 - \frac{1}{8}\mathcal{A}^3 + -\dots\right) - 2\mathcal{A}\left(1 - \mathcal{A} + \mathcal{A}^2 - \mathcal{A}^3 + -\dots\right)$$

Let $X = \delta(t)$. Then

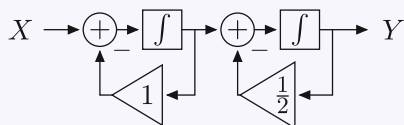
$$\begin{aligned} Y &= 2\left(1 - \frac{1}{2}t + \frac{1}{8}t^2 - \frac{1}{48}t^3 + -\dots\right) u(t) - 2\left(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + -\dots\right) u(t) \\ &= 2\left(e^{-t/2} - e^{-t}\right) u(t) \end{aligned}$$

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

رابطه‌ی میان «معادله‌ی تابعی سیستم» و «پاسخ ضربه»:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

series / partial
fractions

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t}) u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

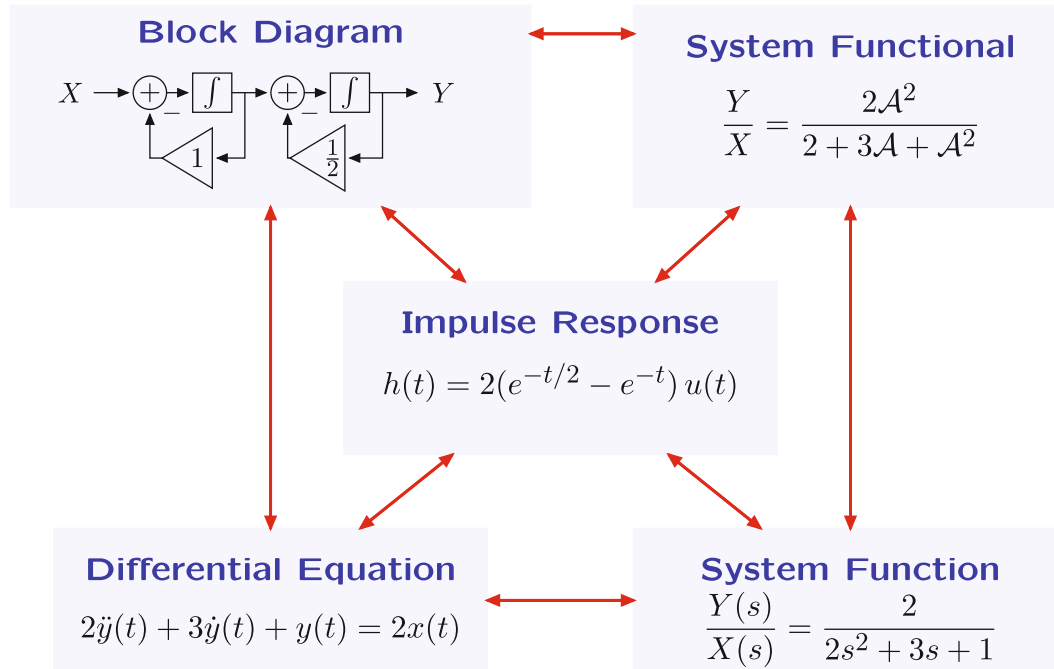
System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

با تبدیل لاپلاس می‌توان رابطه‌های جدیدی را تعیین کرد.

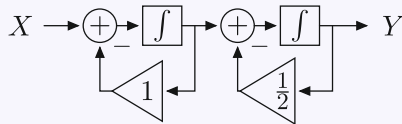


نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

رابطه‌ی میان معادله‌ی دیفرانسیل با تابع سیستم:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

 $\mathcal{L}$

System Function

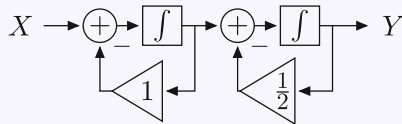
$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

نقشه‌ی مفهومی سیستم‌های پیوسته-زمان

CONCEPT MAP OF CONTINUOUS-TIME SYSTEMS

رابطه‌ی میان پاسخ ضربه و تابع سیستم:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$



تبدیل لاپلاس (۱)

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تعریف
تبدیل
لاپلاس

تبدیل لاپلاس

THE LAPLACE TRANSFORM

تبدیل لاپلاس تابعی از زمان t را به تابعی از s نگاشت می‌دهد.

$$X(s) = \int x(t)e^{-st} dt$$

دوطرفه

Bilateral

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

یک‌طرفه

Unilateral

$$X(s) = \int_0^{\infty} x(t)e^{-st} dt$$

هر دو خواص مهم مشترکی دارند + تفاوت‌ها

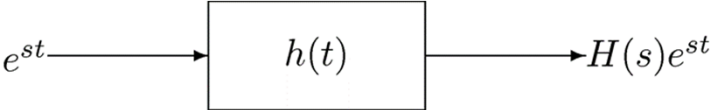
Motivation for the Laplace Transform

- CT Fourier transform enables us to do a lot of things, e.g.
 - Analyze frequency response of LTI systems
 - Sampling
 - Modulation
 - $\vdots$
- Why do we need yet another transform?
- One view of Laplace Transform is as an *extension* of the **Fourier transform** to allow analysis of broader class of signals and systems
- In particular, **Fourier transform** *cannot* handle large (and important) classes of signals and *unstable* systems, i.e. when

$$\int_{-\infty}^{\infty} |x(t)| dt = \infty$$

Motivation for the Laplace Transform (continued)

- In many applications, we do need to deal with *unstable* systems, e.g.
 - Stabilizing an inverted pendulum
 - Stabilizing an airplane or space shuttle
 - $\vdots$
 - Instability is *desired* in some applications, e.g. oscillators and lasers
- How do we analyze such signals/systems?
Recall from Chapter #3, *eigenfunction* property of LTI systems:



A block diagram of an LTI system. An input signal e^{st} enters a rectangular block labeled $h(t)$. The output signal is $H(s)e^{st}$.

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt \quad (\text{assuming this converges})$$

- e^{st} is an eigenfunction of *any* LTI system
- $s = \sigma + j\omega$ can be complex in general

The (Bilateral) Laplace Transform

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \mathcal{L}\{x(t)\}$$

$s = \sigma + j\omega$ is a *complex* variable – Now we explore the full range of s

Basic ideas:

(1) $X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$

 absolute integrability needed

- (2) A critical issue in dealing with Laplace transform is convergence:
— $X(s)$ generally exists only for *some* values of s ,
located in what is called the *region of convergence* (ROC)

$$\text{ROC} = \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{\infty} \underbrace{|x(t)e^{-\sigma t}|}_{\substack{\text{Depends} \\ \text{only on } \sigma \\ \text{not on } \omega}} dt < \infty\}$$

- (3) If $s = j\omega$ is in the ROC (i.e. $\sigma = 0$), then

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

 absolute
integrability
condition

تبدیل لاپلاس (۱)

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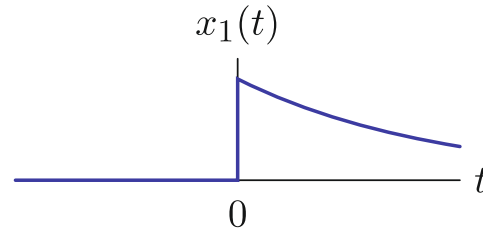
مثال‌هایی از
تبدیل‌های
لاپلاس و
نواحی
همگرایی آنها
(ROC)

تبدیل لاپلاس

مثال

LAPLACE TRANSFORM

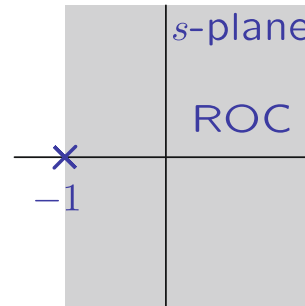
$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_1(s) = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt = \int_0^{\infty} e^{-t} e^{-st} dt = \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^{\infty} = \frac{1}{s+1}$$

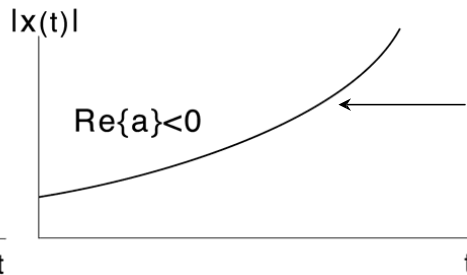
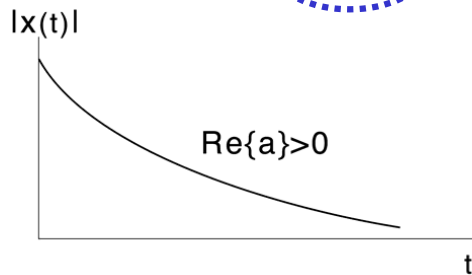
provided $\text{Re}(s+1) > 0$ which implies that $\text{Re}(s) > -1$.

$$\frac{1}{s+1} ; \quad \text{Re}(s) > -1$$



Example #1:

$$x_1(t) = e^{-at}u(t) \quad (a - \text{an arbitrary real or complex number})$$



Unstable:

- no *Fourier Transform*
- but *Laplace Transform* exists

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt \\ &= -\frac{1}{s+a}e^{-(s+a)t} \Big|_0^{\infty} = -\frac{1}{s+a}[e^{-(s+a)\infty} - 1] \end{aligned}$$

This converges only if $\text{Re}(s+a) > 0$, i.e. $\text{Re}(s) > -\text{Re}(a)$

$$\Downarrow$$
$$X_1(s) = \frac{1}{s+a}, \quad \underbrace{\Re\{s\} > -\Re\{a\}}_{\text{ROC}}$$

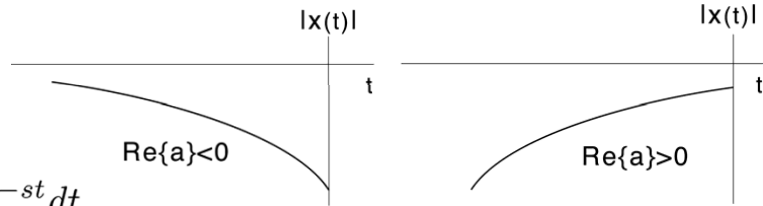
Example #2:

$$x_2(t) = -e^{-at}u(-t)$$

$$X_2(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt$$

$$= -\int_{-\infty}^0 e^{-(s+a)t}dt$$

$$= +\frac{1}{s+a}e^{-(s+a)t}\bigg|_{-\infty}^0 = \frac{1}{s+a}[1 - e^{(s+a)\infty}]$$



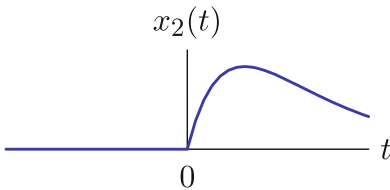
This converges only if $\text{Re}(s+a) < 0$, i.e. $\text{Re}(s) < -\text{Re}(a)$

$$X_2(s) = \frac{1}{s+a}, \quad \underbrace{\text{Re}\{s\} < -\text{Re}\{a\}}_{\text{ROC}} - \text{Same as } X_1(s), \text{ but different ROC}$$

Key Point (and key difference from *FT*): Need *both* $X(s)$ and ROC to uniquely determine $x(t)$. No such an issue for *FT*.

تبدیل لاپلاس

مثال

LAPLACE TRANSFORM

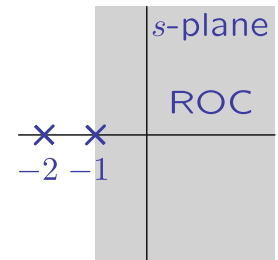
$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} X_2(s) &= \int_0^{\infty} (e^{-t} - e^{-2t}) e^{-st} dt \\ &= \int_0^{\infty} e^{-t} e^{-st} dt - \int_0^{\infty} e^{-2t} e^{-st} dt \\ &= \frac{1}{s+1} - \frac{1}{s+2} = \frac{(s+2) - (s+1)}{(s+1)(s+2)} = \frac{1}{(s+1)(s+2)} \end{aligned}$$

These equations converge if

 $\text{Re}(s+1) > 0$ and $\text{Re}(s+2) > 0$, thus $\text{Re}(s) > -1$.

$$\frac{1}{(s+1)(s+2)} ; \quad \text{Re}(s) > -1$$



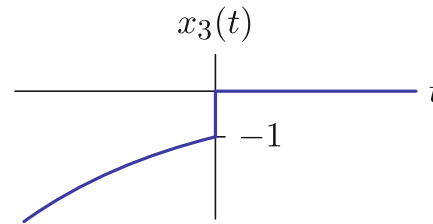
تبدیل لاپلاس

مثال

LAPLACE TRANSFORM

سیگنال‌های سمت چپی، تبدیل‌های لاپلاس سمت چپی دارند (فقط در تبدیل لاپلاس دوطرفه)

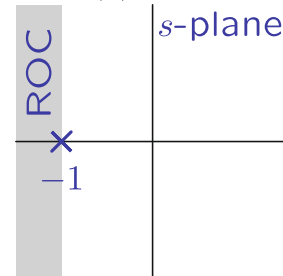
$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_3(s) = \int_{-\infty}^{\infty} x_3(t) e^{-st} dt = \int_{-\infty}^0 -e^{-t} e^{-st} dt = \frac{-e^{-(s+1)t}}{-(s+1)} \Big|_{-\infty}^0 = \frac{1}{s+1}$$

provided $\text{Re}(s+1) < 0$ which implies that $\text{Re}(s) < -1$.

$$\frac{1}{s+1} ; \quad \text{Re}(s) < -1$$



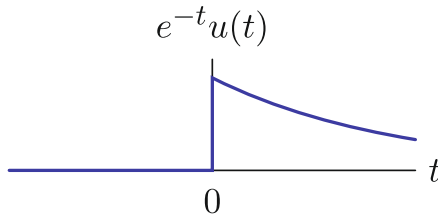
تبدیل لاپلاس

نواحی همگرایی سمت چپی و سمت راستی

LEFT- AND RIGHT-SIDED ROCs

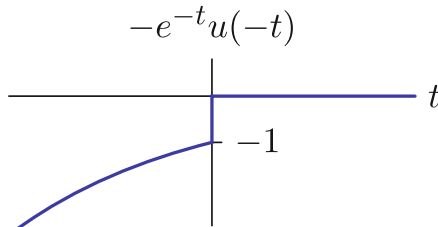
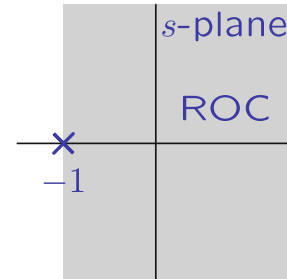
تبدیل‌های لاپلاس نمایی‌های سمت چپی و سمت راستی، شکل‌های مشابهی دارند، اما نواحی همگرایی آنها به ترتیب سمت چپی و سمت راستی است.

time function

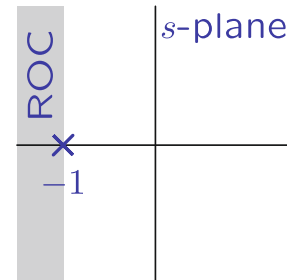


Laplace transform

$$\frac{1}{s+1}$$



$$\frac{1}{s+1}$$

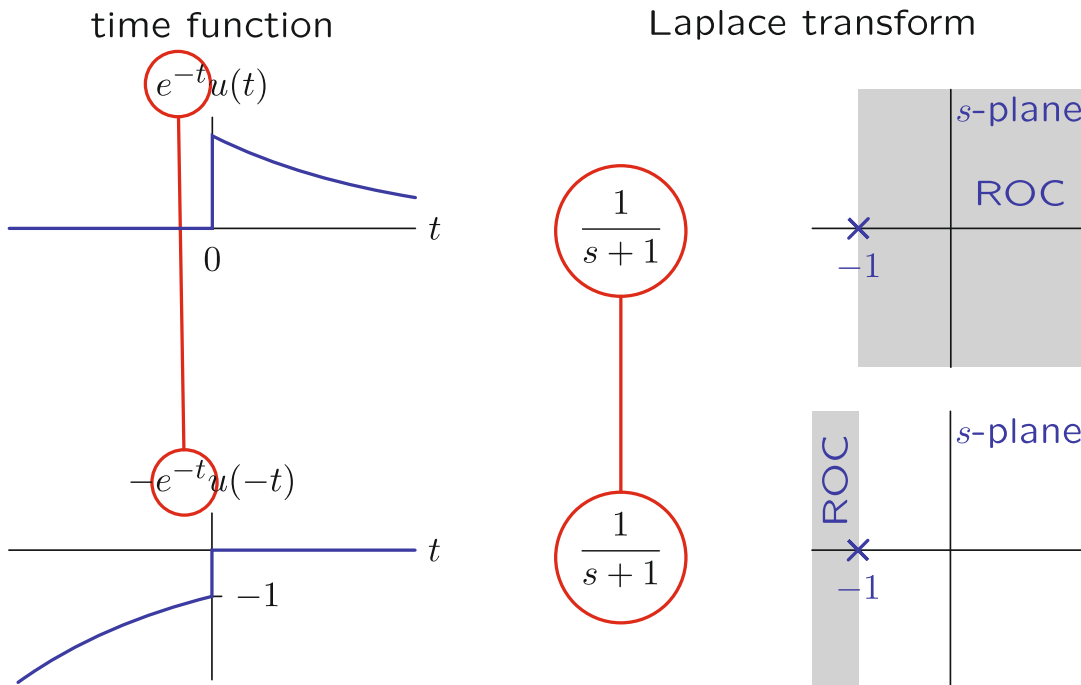


تبدیل لاپلاس

نواحی همگرایی دست چپی و دست راستی

LEFT- AND RIGHT-SIDED ROCs

تبدیل‌های لاپلاس نمایی‌های سمت چپی و سمت راستی، شکل‌های مشابهی دارند، اما نواحی همگرایی آنها به ترتیب سمت چپی و سمت راستی است.

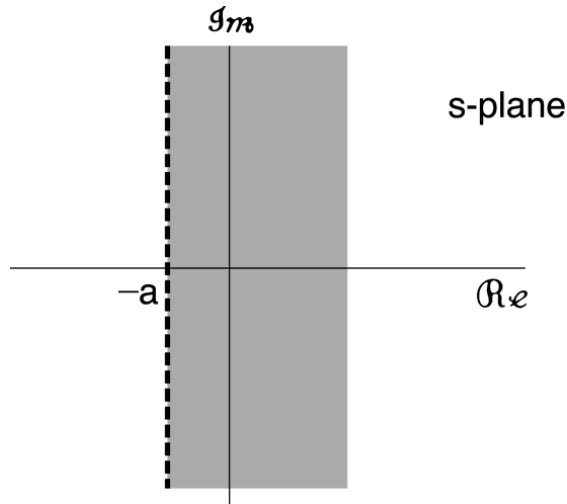


Graphical Visualization of the ROC

Example #1

$$X_1(s) = \frac{1}{s+a}, \quad \Re\{s\} > -\Re\{a\}$$

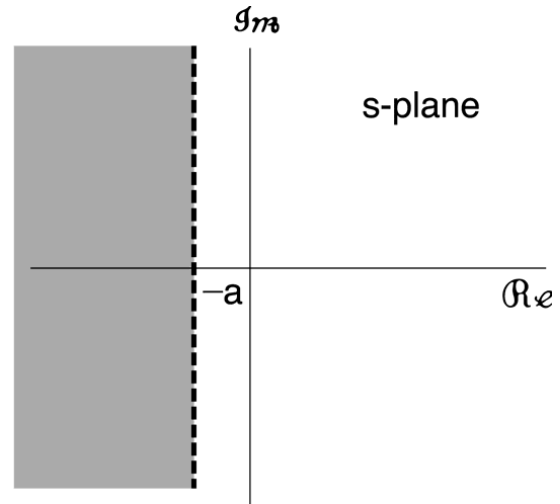
$x_1(t) = e^{-at}u(t)$ - right-sided signal



Example #2

$$X_2(s) = \frac{1}{s+a}, \quad \Re\{s\} < -\Re\{a\}$$

$x_2(t) = -e^{-at}u(-t)$ - left-sided signal



Rational Transforms

- Many (but by no means all) Laplace transforms of interest to us are rational functions of s (e.g., Examples #1 and #2; in general, impulse responses of LTI systems described by LCCDEs), where

$$X(s) = \frac{N(s)}{D(s)}, \quad N(s), D(s) - \text{polynomials in } s$$

- Roots of $N(s)$ = **zeros** of $X(s)$
- Roots of $D(s)$ = **poles** of $X(s)$
- Any $x(t)$ consisting of a linear combination of complex exponentials for $t > 0$ and for $t < 0$ (e.g., as in Example #1 and #2) has a rational Laplace transform.

Example #3 $x(t) = 3e^{2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \int_0^{\infty} [3e^{2t} - 2e^{-t}]e^{-st} dt$$

$$= 3 \underbrace{\int_0^{\infty} e^{-(s-2)t} dt}_{\text{Requires } \Re\{s\} > 2} - 2 \underbrace{\int_0^{\infty} e^{-(s+1)t} dt}_{\text{Requires } \Re\{s\} > -1}$$

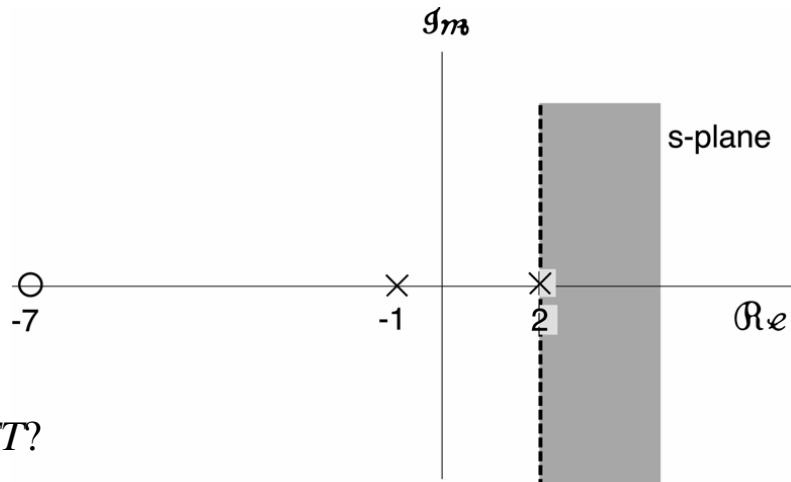
**BOTH required →
ROC intersection**

$$X(s) = \frac{3}{s-2} - \frac{2}{s+1} = \frac{s+7}{(s-2)(s+1)} = \frac{s+7}{s^2 - s - 2} \quad \Re\{s\} > 2$$

Notation:

× — *pole*

○ — *zero*



Q: Does $x(t)$ have FT?

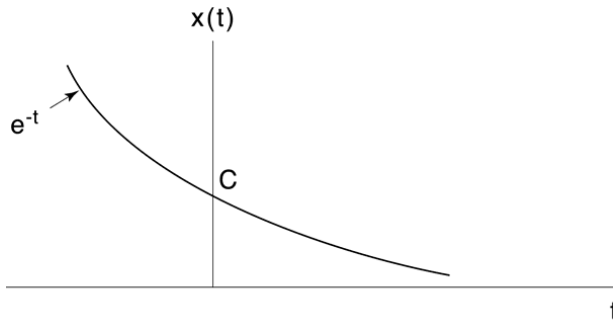
Laplace Transforms and ROCs

- Some signals do not have Laplace Transforms (have no ROC)

(a) $x(t) = Ce^{-t}$ for all t since $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \infty$ for all σ

$$\parallel$$

$$Ce^{-(\sigma+1)t}$$



(b) $x(t) = e^{j\omega_0 t}$ for all t

FT: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \int_{-\infty}^{\infty} e^{-\sigma t} dt = \infty \text{ for all } \sigma$$

$X(s)$ is defined only in ROC; we don't allow impulses in LTs

تبدیل لاپلاس (۱)

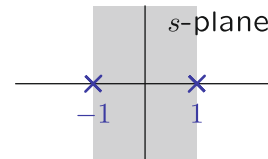
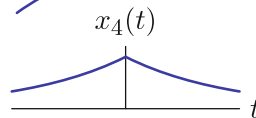
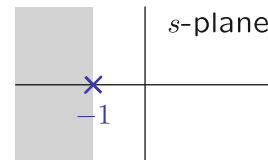
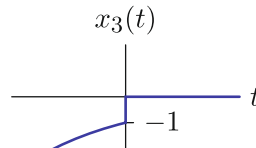
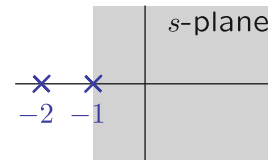
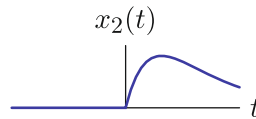
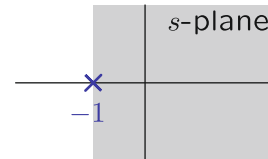
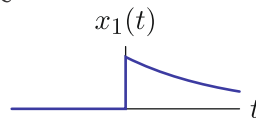
۴

خصوصیات
نواحی
همگرایی

تبدیل لاپلاس

تعبیر حوزه‌ی زمان ناحیه‌ی همگرایی

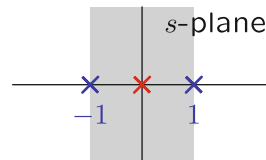
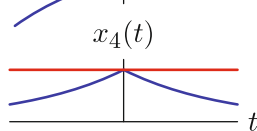
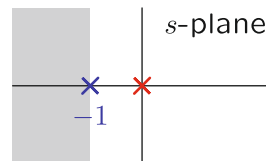
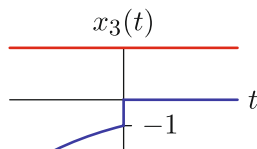
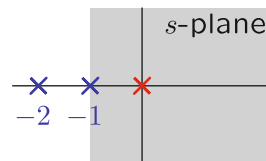
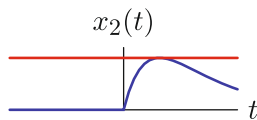
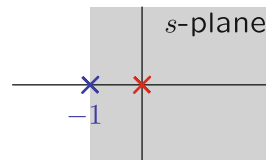
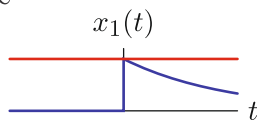
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



تبدیل لاپلاس

تعبیر حوزه‌ی زمان ناحیه‌ی همگرایی

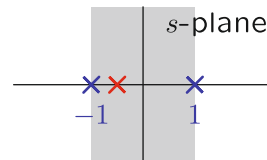
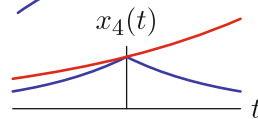
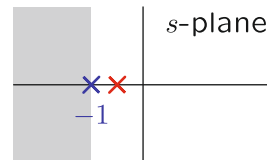
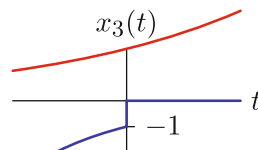
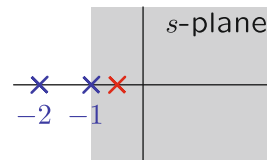
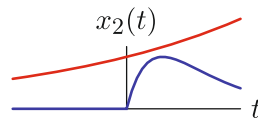
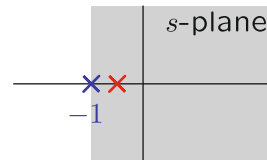
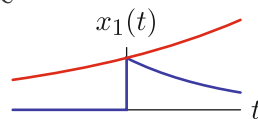
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



تبدیل لاپلاس

تعبیر حوزه‌ی زمان ناحیه‌ی همگرایی

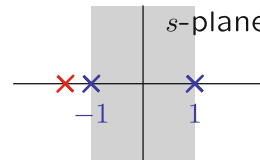
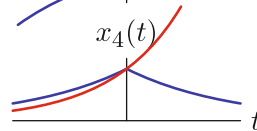
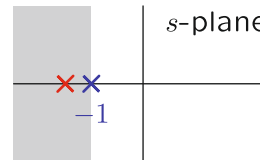
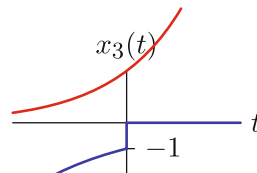
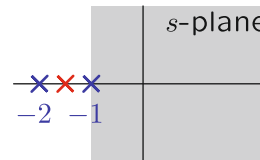
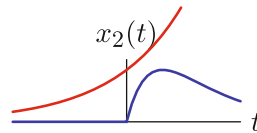
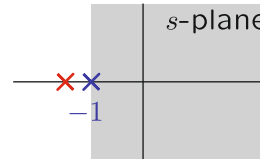
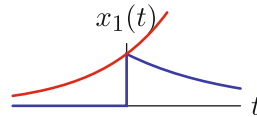
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



تبدیل لاپلاس

تعبیر حوزه‌ی زمان ناحیه‌ی همگرایی

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



Properties of the ROC

- The ROC can take on only a small number of different forms
 - 1) The ROC consists of a collection of lines parallel to the $j\omega$ -axis in the s -plane (i.e. the ROC only depends on σ).
Why?

$$\int_{-\infty}^{\infty} |x(t)e^{-st}| dt = \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty \text{ depends only on } \sigma = \Re\{s\}$$

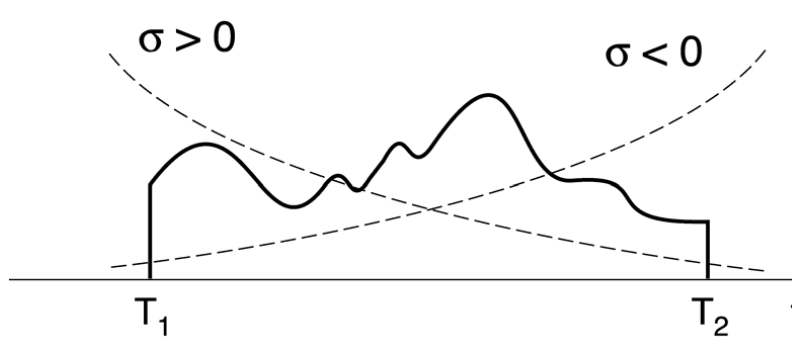
- 2) If $X(s)$ is rational, then the ROC does not contain any poles.
Why?

Poles are places where $D(s) = 0$

$$X(s) = \frac{N(s)}{D(s)} = \infty \quad \text{Not convergent.}$$

More Properties

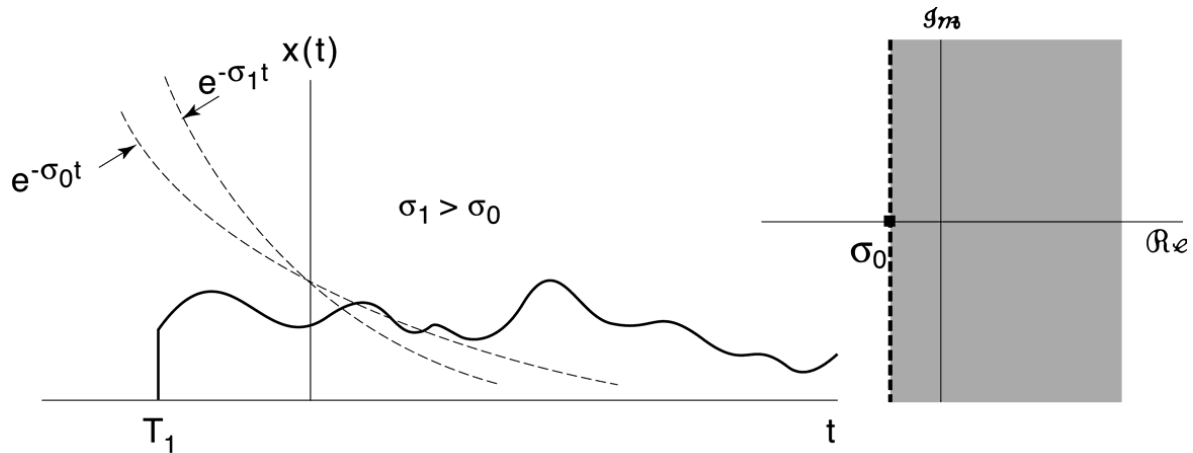
- 3) If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s -plane.



$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \underbrace{\int_{T_1}^{T_2} x(t)e^{-st} dt}_{\text{A finite integration interval}}$$
$$< \infty \quad \text{if} \quad \int_{T_1}^{T_2} |x(t)| dt < \infty$$

ROC Properties that Depend on Which Side You Are On - I

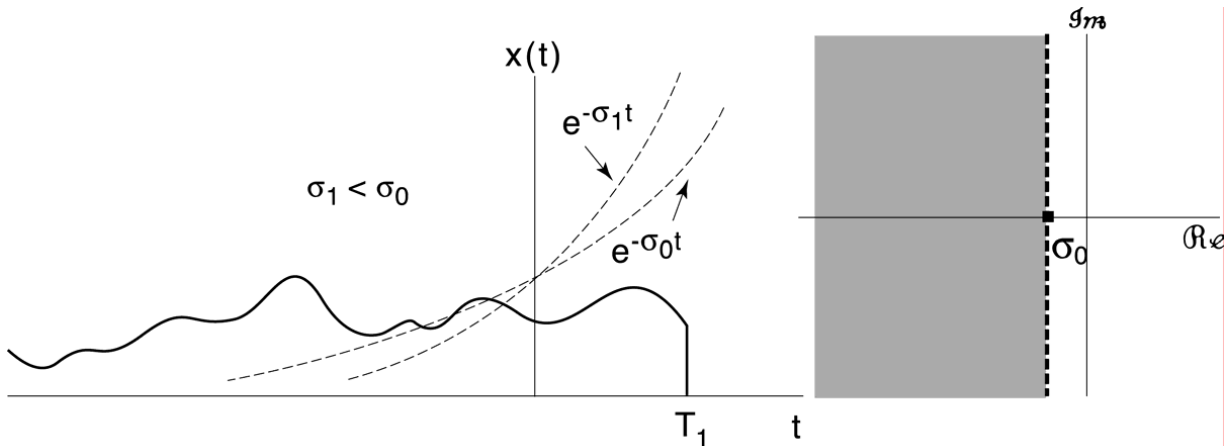
- 4) If $x(t)$ is right-sided (i.e. if it is zero *before* some time), and if $\text{Re}(s) = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}(s) > \sigma_0$ are also in the ROC.



ROC is a right half plane (RHP)

ROC Properties that Depend on Which Side You Are On - II

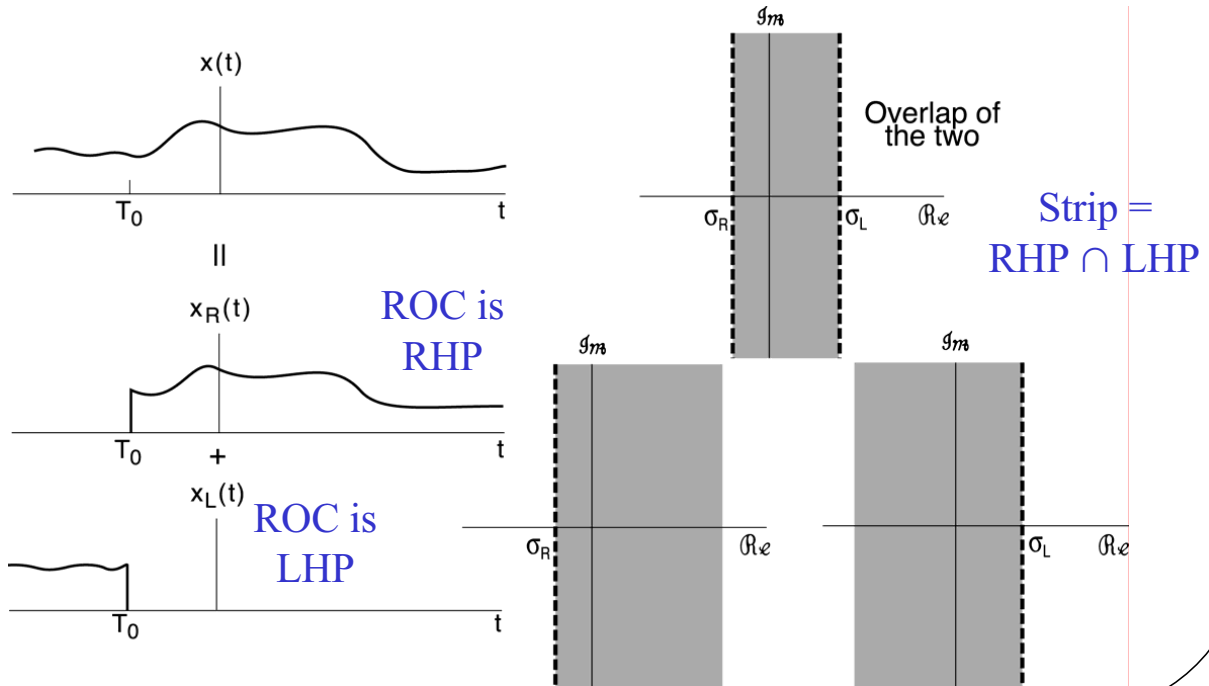
- 4) If $x(t)$ is left-sided (i.e. if it is zero *after* some time), and if $\text{Re}(s) = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}(s) < \sigma_0$ are also in the ROC.



ROC is a left half plane (LHP)

Still More ROC Properties

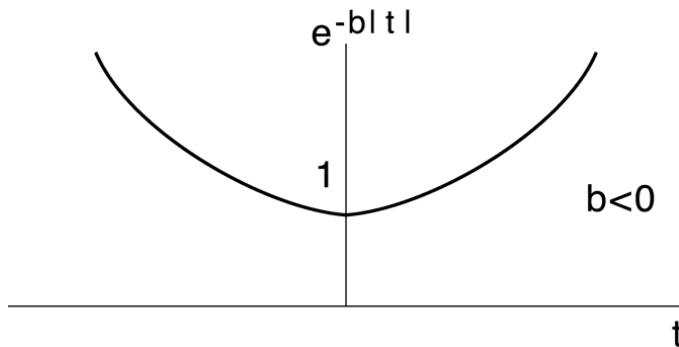
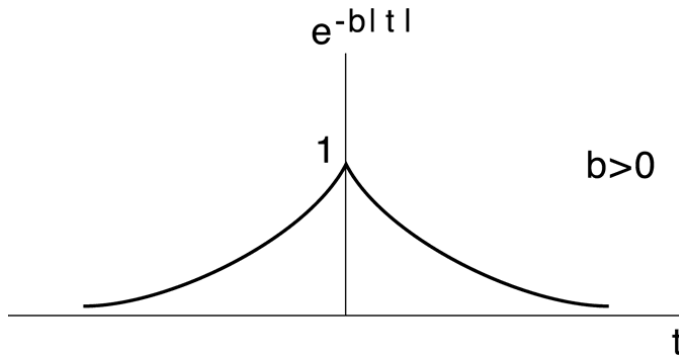
- 6) If $x(t)$ is two-sided and if the line $\text{Re}(s) = \sigma_0$ is in the ROC, then the ROC consists of a strip in the s -plane that includes the line $\text{Re}(s) = \sigma_0$.



Example: $x(t) = e^{-b|t|}$

Intuition?

Okay: multiply by
constant (e^{0t}) and
will be integrable

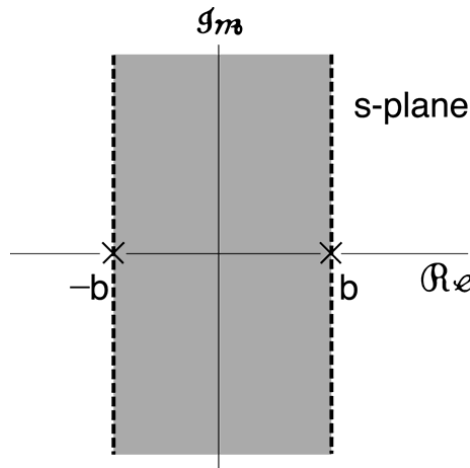


Looks bad: no $e^{\sigma t}$
will dampen both
sides

Example (continued):

$$\begin{array}{ccc} x(t) & = & e^{bt}u(-t) \quad + \quad e^{-bt}u(t) \\ & & \Downarrow \quad \quad \quad \Downarrow \\ & & -\frac{1}{s-b}, \Re\{s\} < b \quad \quad \quad \frac{1}{s+b}, \Re\{s\} > -b \end{array}$$

Overlap if $b > 0 \Rightarrow X(s) = \frac{-2b}{s^2 - b^2}$, with ROC:



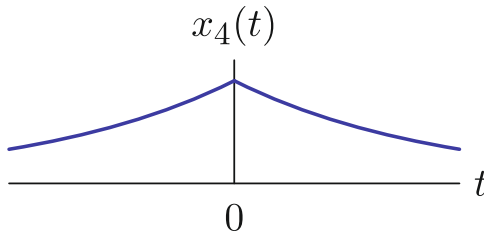
What if $b < 0$? $\Rightarrow$ No overlap $\Rightarrow$ No Laplace Transform

تبدیل لاپلاس

مثال

LAPLACE TRANSFORM

$$x_4(t) = e^{-|t|}$$



$$\begin{aligned}
 X_4(s) &= \int_{-\infty}^{\infty} e^{-|t|} e^{-st} dt \\
 &= \int_{-\infty}^0 e^{(1-s)t} dt + \int_0^{\infty} e^{-(1+s)t} dt \\
 &= \left. \frac{e^{(1-s)t}}{(1-s)} \right|_{-\infty}^0 + \left. \frac{e^{-(1+s)t}}{-(1+s)} \right|_0^{\infty} \\
 &= \underbrace{\frac{1}{1-s}}_{\text{Re}(s) < 1} + \underbrace{\frac{1}{1+s}}_{\text{Re}(s) > -1} \\
 &= \frac{1+s+1-s}{(1-s)(1+s)} = \frac{2}{1-s^2} \\
 &\quad ; \quad -1 < \text{Re}(s) < 1
 \end{aligned}$$

The ROC is the intersection of

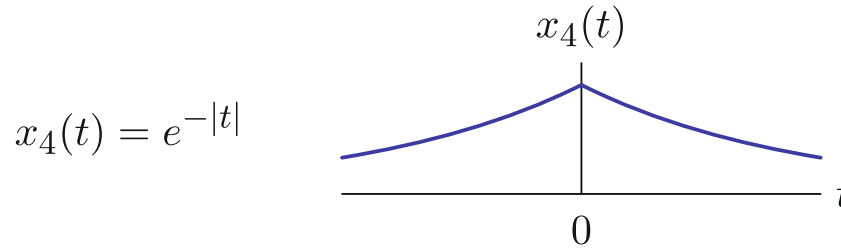
$$\text{Re}(s) < 1 \text{ and } \text{Re}(s) > -1.$$

تبدیل لاپلاس

مثال

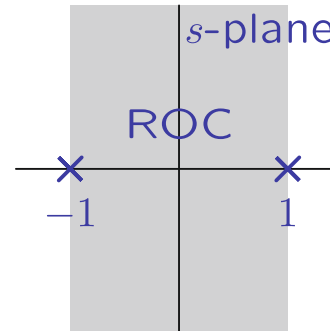
LAPLACE TRANSFORM

ناحیه‌ی همگرایی سیگنالی که دوستی باشد، یک نوار عمودی است.



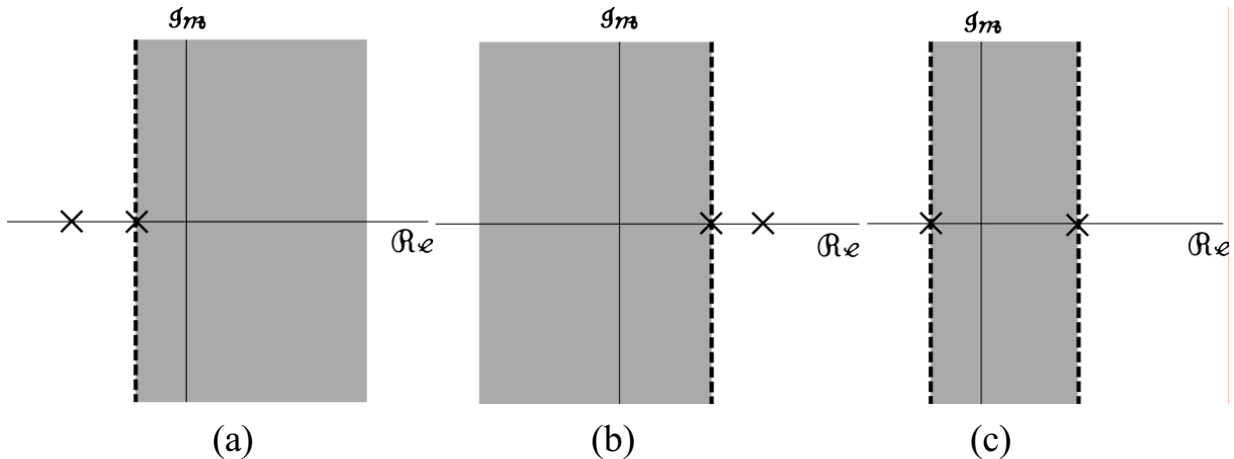
$$X_4(s) = \frac{2}{1 - s^2}$$

$$-1 < \text{Re}(s) < 1$$



Properties, Properties

- 7) If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.
- 8) Suppose $X(s)$ is rational, then
 - (a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.
 - (b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.



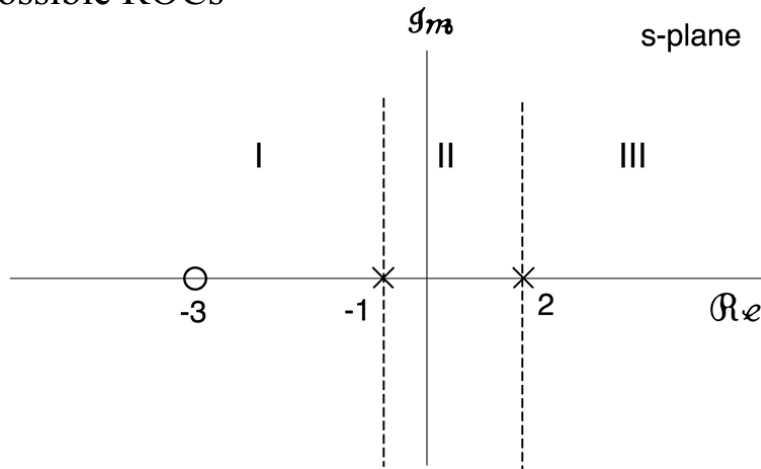
- 9) If ROC of $X(s)$ includes the $j\omega$ -axis, then *FT* of $x(t)$ exists.

9) If ROC of $X(s)$ includes the $j\omega$ -axis, then *FT* of $x(t)$ exists.

Example:

$$X(s) = \frac{(s + 3)}{(s + 1)(s - 2)}$$

Three possible ROCs



$x(t)$ is right-sided

ROC: III

Fourier
Transform
exists?

No

$x(t)$ is left-sided

ROC: I

No

$x(t)$ extends for all time

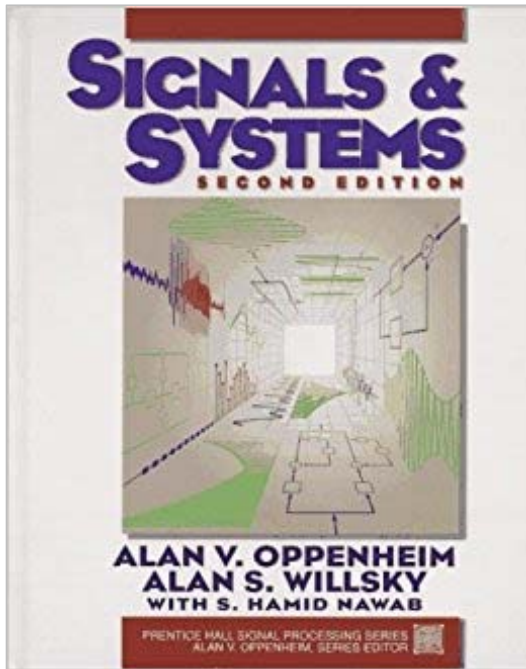
ROC: II

Yes

تبدیل لاپلاس (۱)

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منابع



A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
Signals and Systems,
Second Edition, Prentice Hall, 1997.

Chapter 9