سیگنال‌ها و سیستم‌ها

درس ۲۰

نمونه‌برداری (۱)

Sampling (1)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

مفهوم و بازنمایی نمونه‌برداری متناوب یک سیگنال پیوسته–زمان
The Concept and Representation of Periodic Sampling of a CT Signal
تحلیل نمونه‌برداری در حوزه‌ی فرکانس
Analysis of Sampling in the Frequency Domain
قضیه‌ی نمونه‌برداری ــ نرخ نایکوئیست
The Sampling Theorem — the Nyquist Rate
در حوزه‌ی زمان: درون‌یابی
In the Time Domain: Interpolation
زیرنمونه‌برداری و آلیاسینگ
Undersampling and Aliasing

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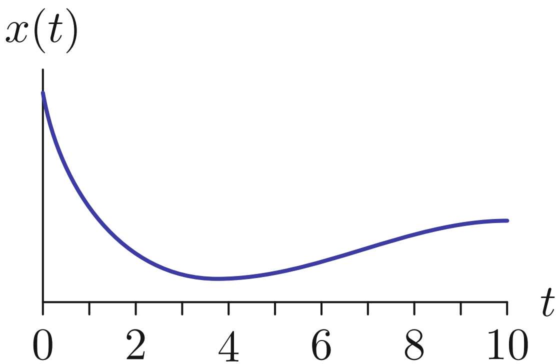
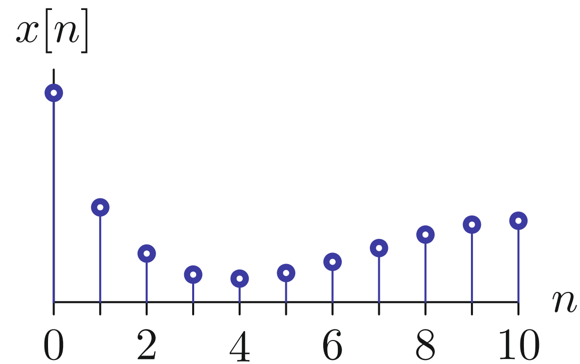
مفهوم و بازنمایی نمونه‌برداری متناوب یک سیگنال پیوسته-زمان

نمونه‌برداری

Sampling

نمونه‌برداری: تبدیل یک سیگنال پیوسته–زمان به سیگنال گسسته–زمان





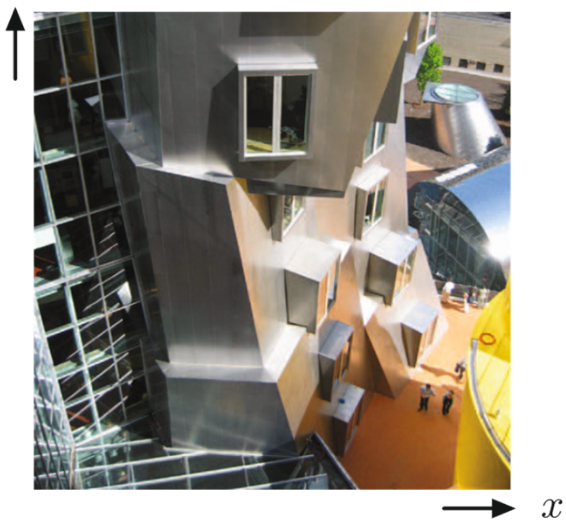
نمونه‌برداری

مثال: دوربین دیجیتال

SAMPLING

دوربین‌های دیجیتال تصاویر نمونه‌برداری شده را ثبت می‌کنند.

$I(x,y)$



$I[m,n]$

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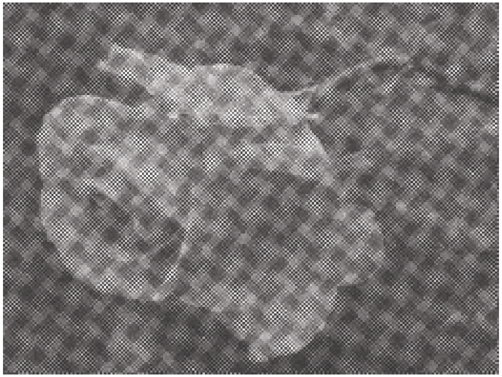
نمونه‌برداری

مثال: تصاویر خاکستری در روزنامه‌ها (۱ از ۲)

SAMPLING

Photographs in newsprint are “half-tone” images.
Each point is black or white and the average conveys brightness.

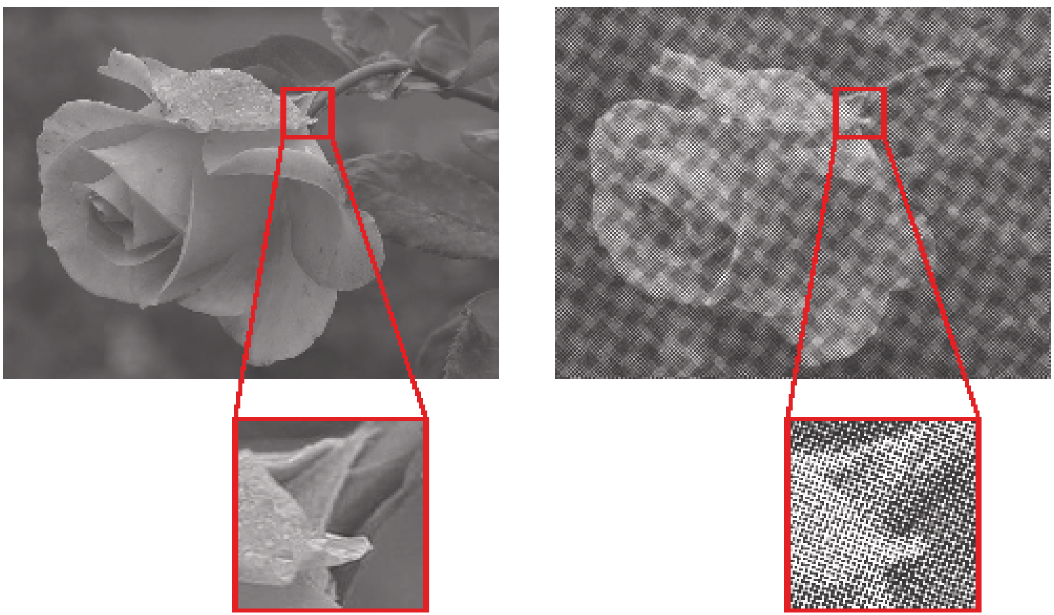


نمونه‌برداری

مثال: تصاویر خاکستری در روزنامه‌ها (۲ از ۲)

SAMPLING

Photographs in newsprint are “half-tone” images.
Each point is black or white and the average conveys brightness.
Zoom in to see the binary pattern.

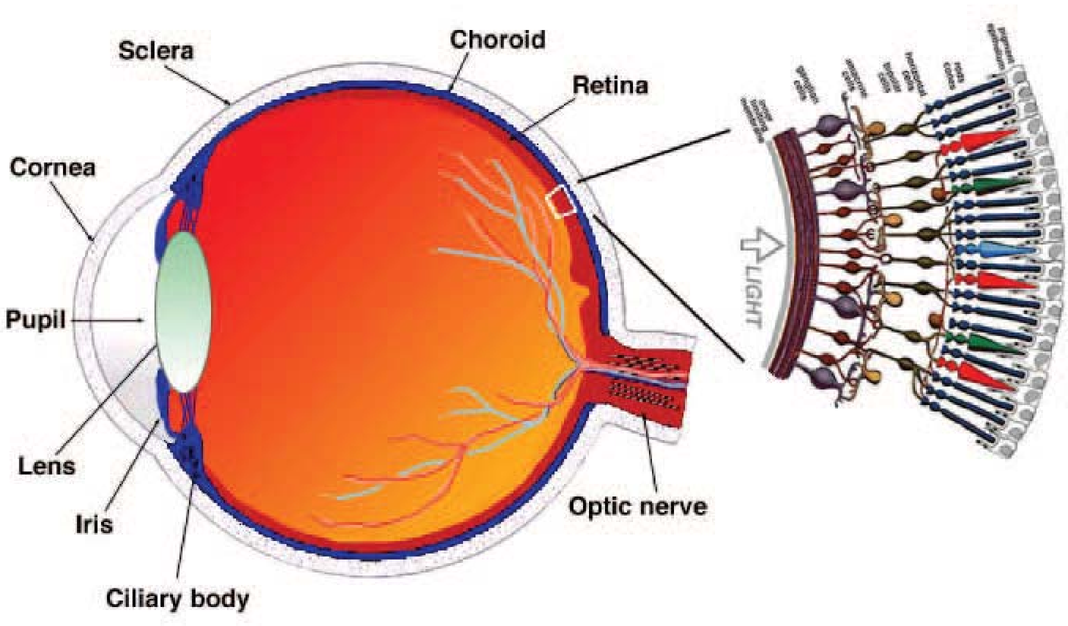


نمونه‌برداری

مثال: نمونه‌برداری در چشم انسان

SAMPLING

Every image that we see is sampled by the retina,
which contains ≈ 100 million rods and 6 million cones (average spacing ≈ 3μm)
which act as discrete sensors.



SAMPLING

We live in a continuous-time world: most of the signals we encounter are CT signals, e.g. $x(t)$. How do we convert them into DT signals $x[n]$?

— Sampling, taking snap shots of $x(t)$ every T seconds.

T – sampling period
$x[n] \equiv x(nT)$, $n = \ldots, -1, 0, 1, 2, \ldots$ — regularly spaced samples

Applications and Examples

- Digital Processing of Signals
- Strobe
- Images in Newspapers
- Sampling Oscilloscope
- …

How do we perform sampling?

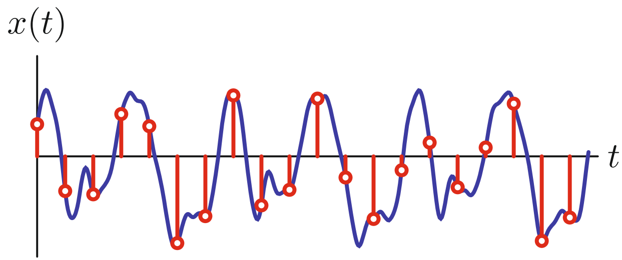
نمونه‌برداری

حفظ اطلاعات در نمونه‌برداری

SAMPLING

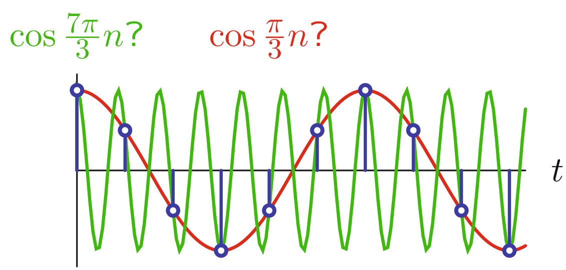
نمونه‌برداری چه اثری بر اطلاعات موجود در سیگنال می‌گذارد؟

ما دوست داریم نمونه‌برداری را به‌گونه‌ای انجام دهیم که اطلاعات را حفظ کند (که به‌نظر ممکن نمی‌رسد).



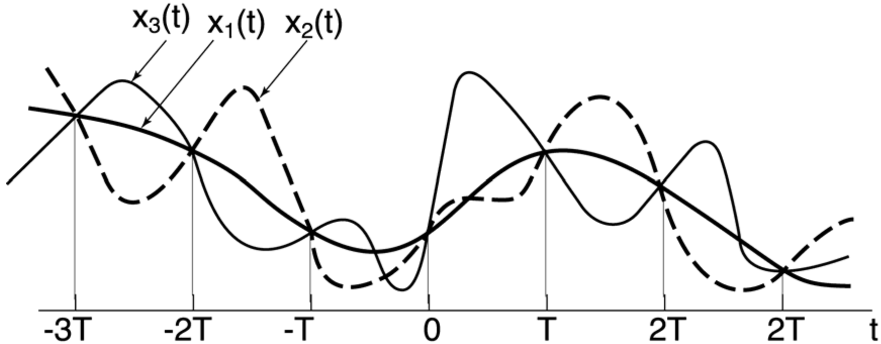
اطلاعات بین نمونه‌ها از دست می‌رود.

بنابراین، همان نمونه‌ها می‌توانند سیگنال‌های متعددی را بازنمایی کنند.



Why/When Would a Set of Samples Be Adequate?

- Observation: *Lots* of signals have the same samples



- By sampling we throw out lots of information
  – all values of $x(t)$ between sampling points are lost.

- **Key Question for Sampling**:
  Under what conditions can we reconstruct the original CT signal $x(t)$ from its samples?

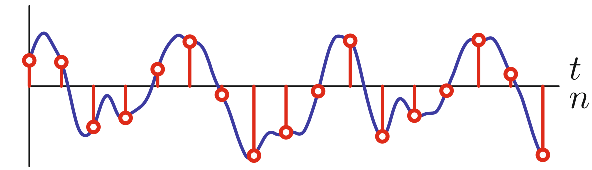
نمونه‌برداری و بازسازی

Sampling and Reconstruction

برای تعیین اثر نمونه‌برداری، سیگنال اصلی $x(t)$ را با سیگنال $x_p(t)$ مقایسه می‌کنیم که از روی نمونه‌های $x[n]$ **بازسازی** شده است.

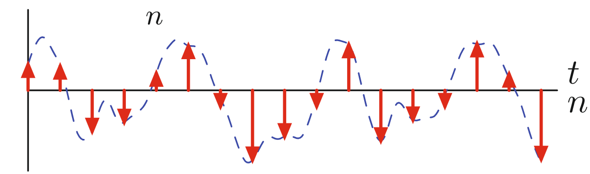
نمونه‌برداری یکنواخت: Uniform sampling (sampling interval T).

$$x[n] = x(nT)$$



بازسازی با ضربه: Impulse reconstruction.

$$x_p(t) = \sum_n x[n]\delta(t - nT)$$



بازسازی

RECONSTRUCTION

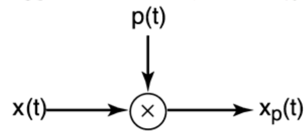
بازسازی ضربه‌ای یک سیگنال $x_p(t)$ را تولید می‌کند که مساوی با سیگنال اصلی $x(t)$ ضرب در قطار ضربه است.

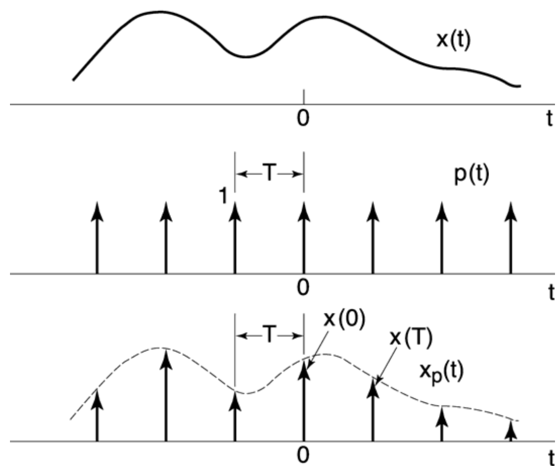
$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) \\ &= x(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\equiv\ p(t)} \end{aligned}$$

Impulse Sampling — Multiplying *x*(*t*) by the sampling function

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$





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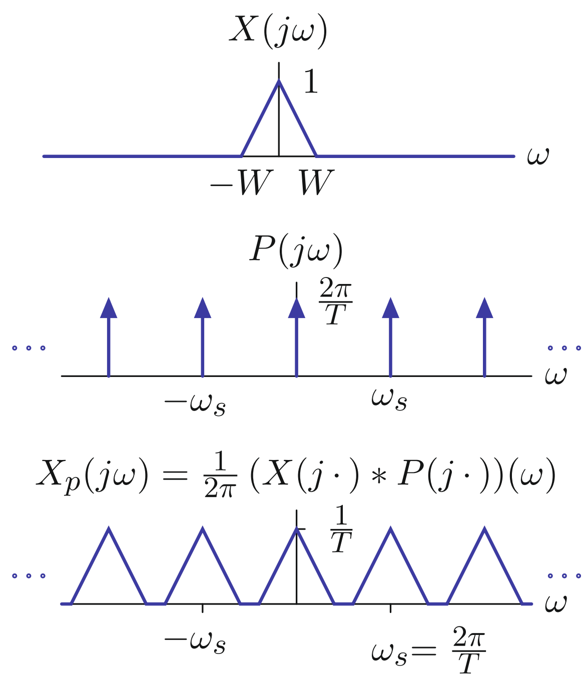
تحلیل نمونه‌برداری در حوزه‌ی فرکانس

نمونه‌برداری

Sampling

ضرب در ضربه، در حوزه‌ی زمان معادل است با کانوولوشن با قطار ضربه در حوزه‌ی فرکانس
$\Leftarrow$ کپی‌های متعددی از محتوای فرکانسی اصلی تولید می‌کند.



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Analysis of Sampling in the Frequency Domain

$$x_p(t) = x(t) \cdot p(t)$$

$$\text{Multiplication Property} \Rightarrow X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T} = \text{Sampling Frequency}$$

Important to note: $\omega_s \propto 1/T$

$$\Downarrow$$

$$\begin{aligned} X_p(j\omega) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) * \delta(\omega - k\omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$

نمونه‌برداری

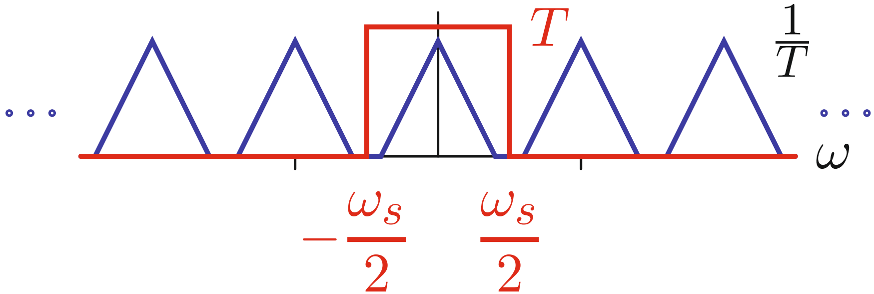
بازسازی

SAMPLING

کپی‌های فرکانس بالا می‌توانند با یک فیلتر پایین‌گذر حذف شوند.

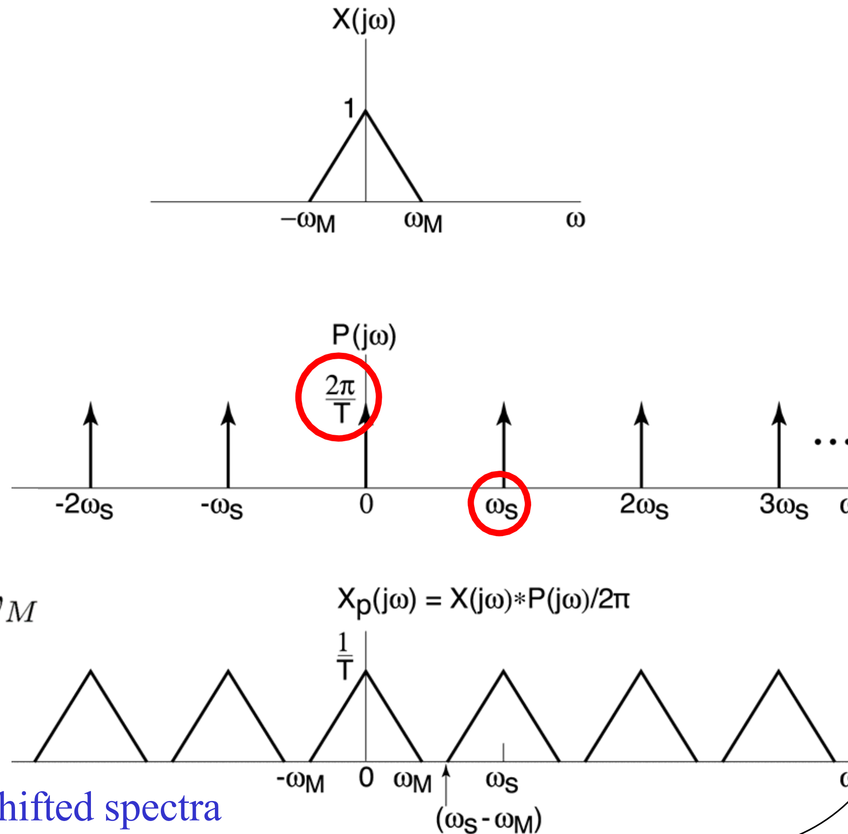
(همچنین با ضرب در T برای لغو اثر تغییر مقیاس دامنه)

$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



بازسازی ضربه‌ای که به‌دنبال آن یک فیلتر پایین‌گذر بیاید، **بازسازی باند محدود** نام دارد.

Illustration of sampling in the frequency-domain for a band-limited ($X(j\omega) = 0$ for $|\omega| > \omega_M$) signal



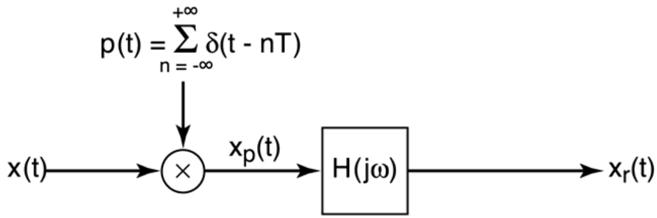
$X_p(j\omega)$ drawn assuming

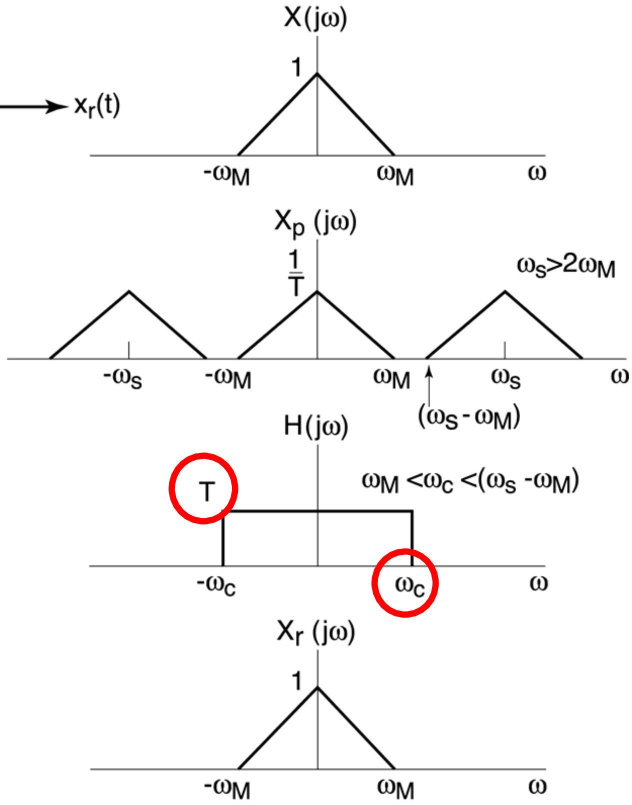
$$\omega_s - \omega_M > \omega_M$$

$$i.e. \quad \omega_s > 2\omega_M$$

No overlap between shifted spectra

Reconstruction of $x(t)$ from sampled signals





If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$

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قضیه‌ی نمونه‌برداری
—
نرخ نایکوئیست

قضیه‌ی نمونه‌برداری

The Sampling Theorem

اگر یک سیگنال دارای باند محدود باشد،
می‌توان نمونه‌برداری را بدون از دست رفتن اطلاعات انجام داد.

اگر $x(t)$ یک سیگنال باند محدود باشد به‌گونه‌ای که

$$X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_m$$

آن‌گاه x(t) می‌تواند به‌طور یکتا توسط نمونه‌هایش $x(nT)$ تعیین شود اگر

$$\omega_s = \frac{2\pi}{T} > 2\omega_m.$$

کوچکترین فرکانس نمونه‌برداری، $2\omega_m$، «نرخ نایکوئیست» نام دارد.

The Sampling Theorem

Suppose $x(t)$ is bandlimited, so that

$$X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $\{x(nT)\}$ if

$$\boxed{\omega_s > 2\omega_M = \text{The Nyquist rate}}$$

where $\omega_s = 2\pi/T$

نمونه‌برداری

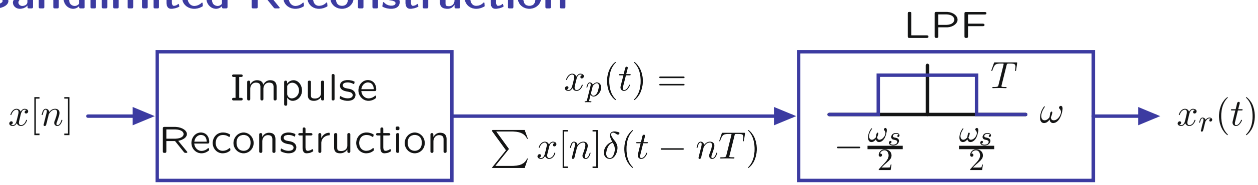
خلاصه

SAMPLING

Sampling

$$x(t) \rightarrow x[n] = x(nT)$$

Bandlimited Reconstruction



Sampling Theorem: If $X(j\omega) = 0 \ \forall \ |\omega| > \frac{\omega_s}{2}$ then $x_r(t) = x(t)$.

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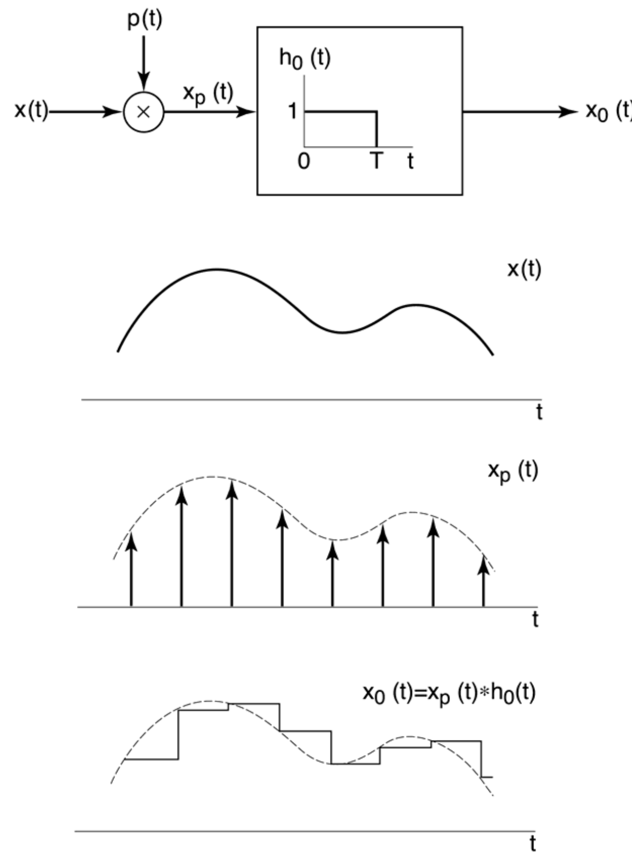
Observations on Sampling

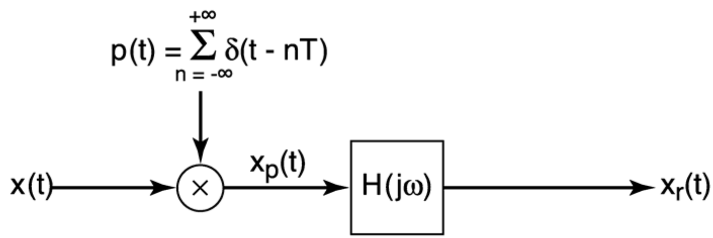
(1) In practice, we obviously don't sample with impulses or implement ideal lowpass filters.

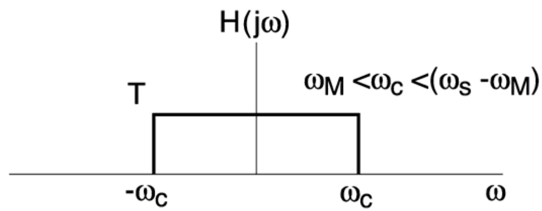
— One practical example: The Zero-Order Hold



Observations (Continued)

(2) Sampling is fundamentally a *time-varying* operation, since we multiply $x(t)$ with a time-varying function $p(t)$. However,





is the identity system (which is *TI*) for bandlimited $x(t)$ satisfying the sampling theorem ($\omega_s > 2\omega_M$).

(3) What if $\omega_s \leq 2\omega_M$? Something different: more later.

نمونه‌برداری (۱)

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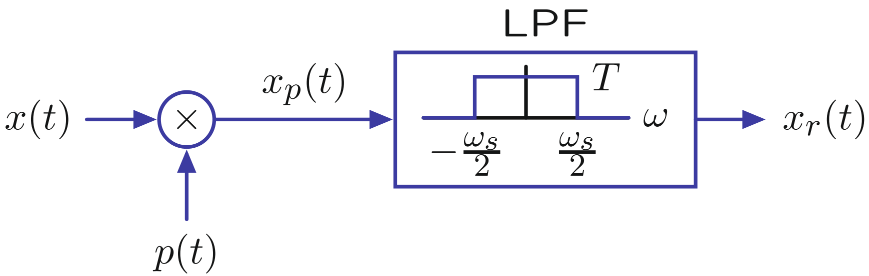
در حوزه‌ی زمان: درون‌یابی

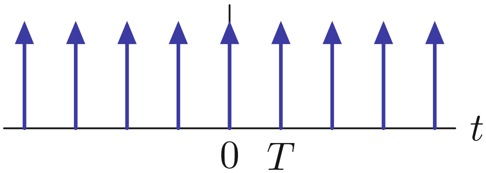
مدل پیوسته-زمان نمونه‌برداری و بازسازی

CT Model of Sampling and Reconstruction

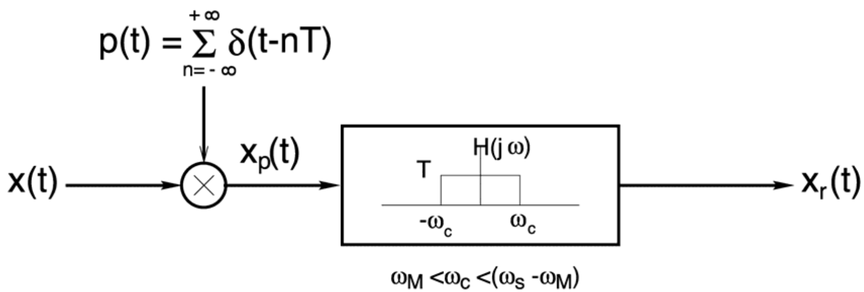
نمونه‌برداری که به‌دنبال آن بازسازی باندمحدود انجام شود،
معادل است با ضرب در قطار ضربه و به‌دنبال آن فیلتر کردن پایین‌گذر



$p(t) =$ "sampling function"



Time-Domain Interpretation of Reconstruction of Sampled Signals — Band-Limited Interpolation



$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \quad , \quad \text{where } h(t) = \frac{T \sin \omega_c t}{\pi t} \\
&= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \right) * h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin[\omega_c(t-nT)]}{\pi(t-nT)}
\end{aligned}
$$

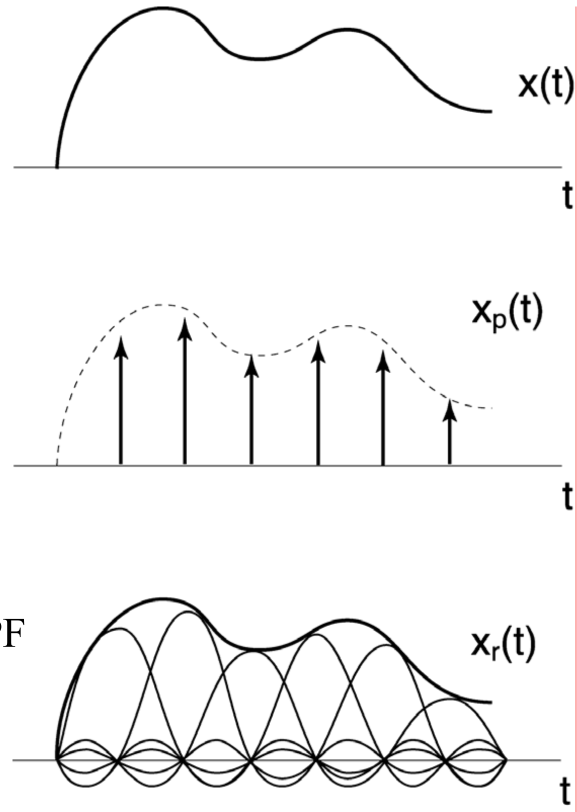
The lowpass filter interpolates the samples *assuming* $x(t)$ contains no energy at frequencies $\geq \omega_c$

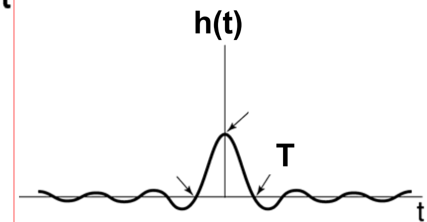
Graphic Illustration of Time-Domain Interpolation

Original
CT signal

After sampling

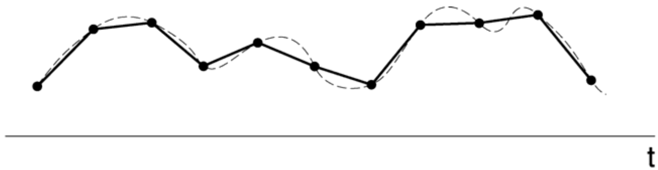
After passing the LPF

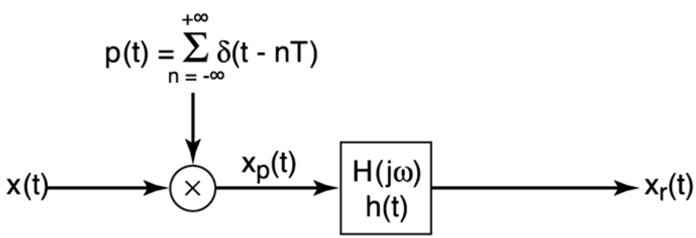


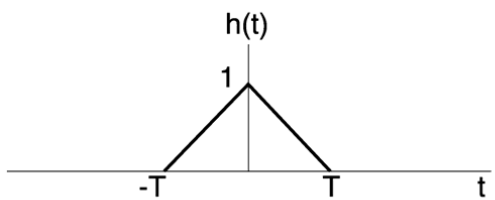


Interpolation Methods

- Bandlimited Interpolation
- Zero-Order Hold
- First-Order Hold — Linear interpolation







نمونه‌برداری (۱)

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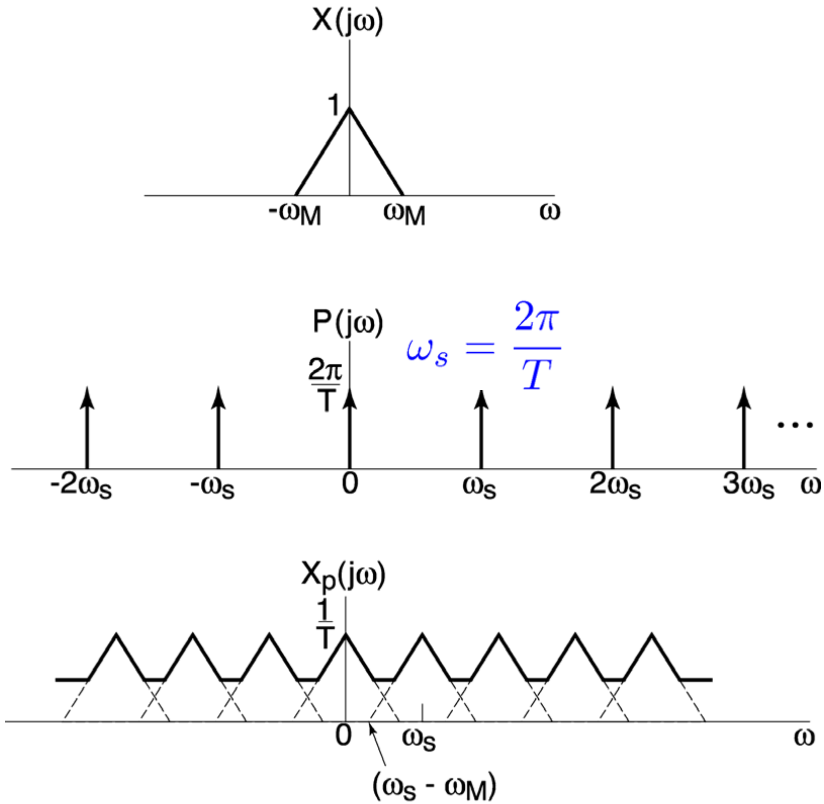
زیرنمونه‌برداری و آلیاسینگ

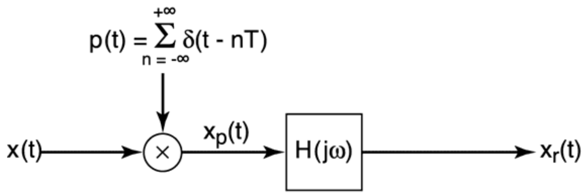
Undersampling and Aliasing

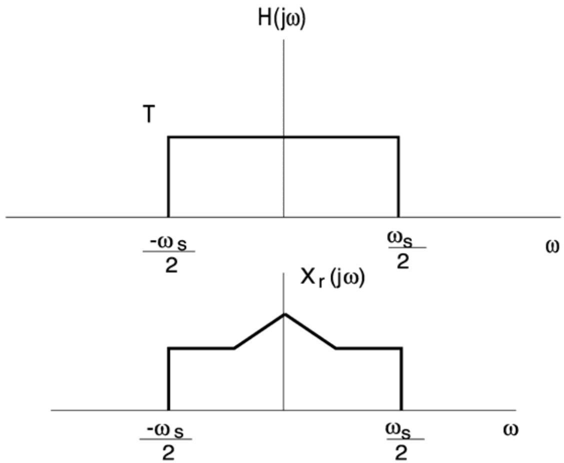
When $\omega_s \leq 2\omega_M \Rightarrow$ Undersampling



$$\omega_s = \frac{2\pi}{T}$$

Undersampling and Aliasing (continued)





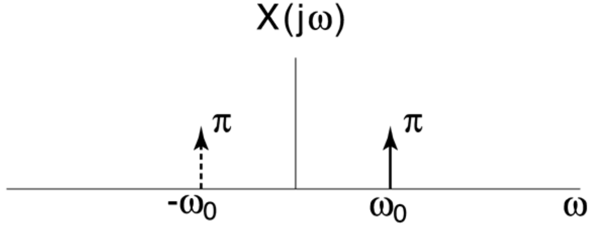
$X_r(j\omega) \neq X(j\omega)$
Distortion because of *aliasing*

— Higher frequencies of $x(t)$ are "folded back" and take on the "aliases" of lower frequencies

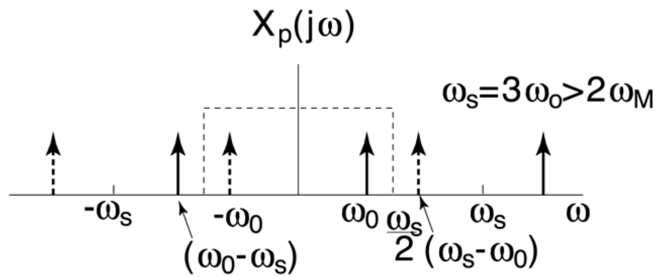
— Note that at the sample times, $x_r(nT) = x(nT)$

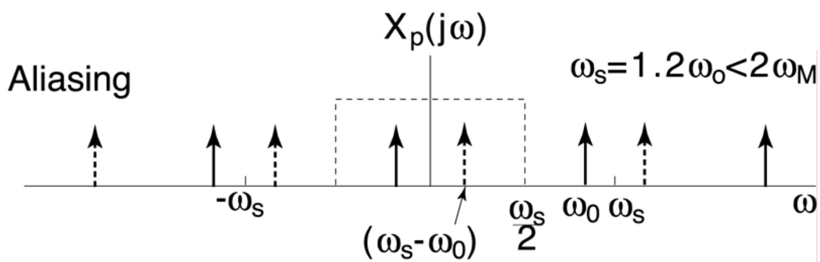
A Simple Example

$$x(t) = \cos(\omega_0 t + \phi)$$



Picture would be Modified…



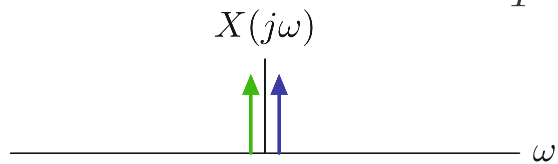


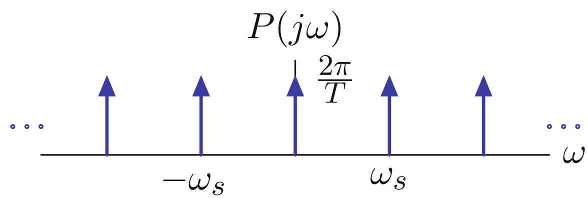
Demo: Sampling and reconstruction of $\cos\omega_0 t$

آلیاسینگ

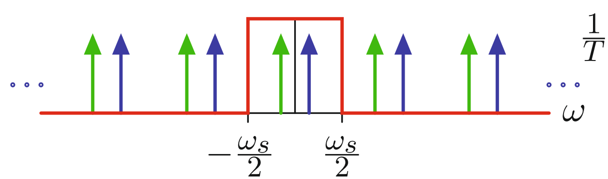
[ALIASING]

What happens if X contains frequencies $|\omega| > \dfrac{\pi}{T}$?





$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$

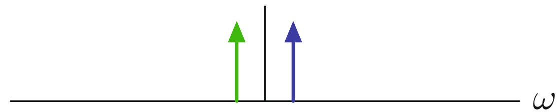


آلیاسینگ

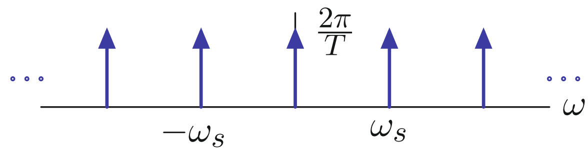
ALIASING

What happens if X contains frequencies $|\omega| > \frac{\pi}{T}$?

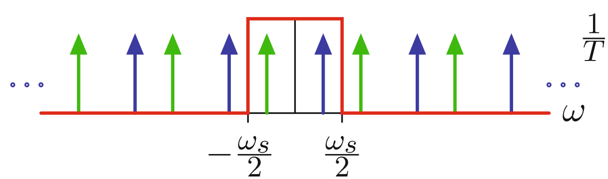
$X(j\omega)$



$P(j\omega)$



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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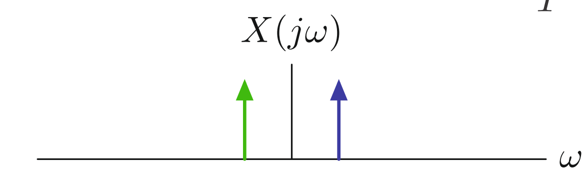
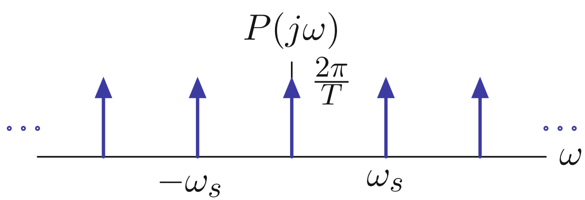
ALIASING

What happens if X contains frequencies $|\omega| > \dfrac{\pi}{T}$?





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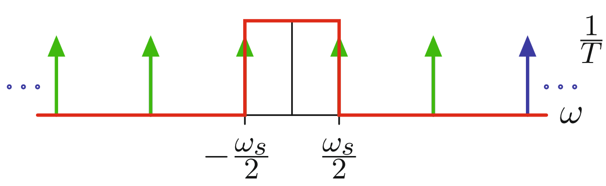

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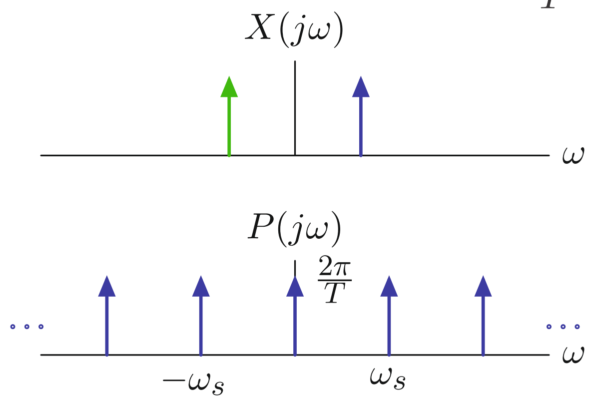
آلیاسینگ

Aliasing

What happens if X contains frequencies $|\omega| > \frac{\pi}{T}$?



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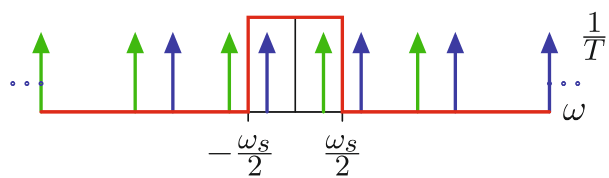
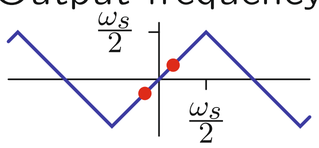
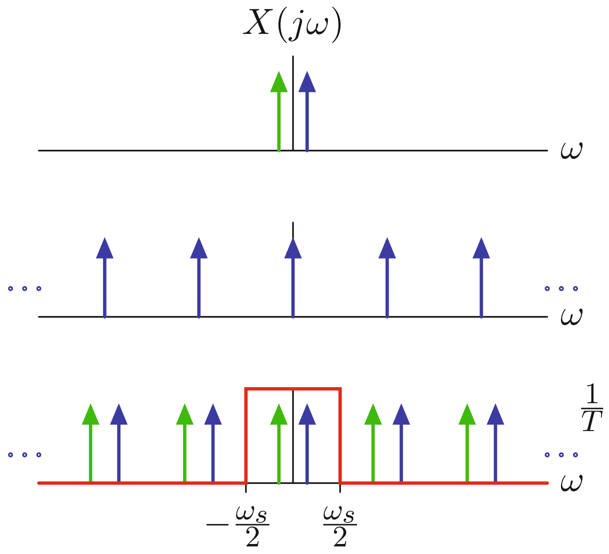


آلیاسینگ

ALIASING

The effect of aliasing is to wrap frequencies.

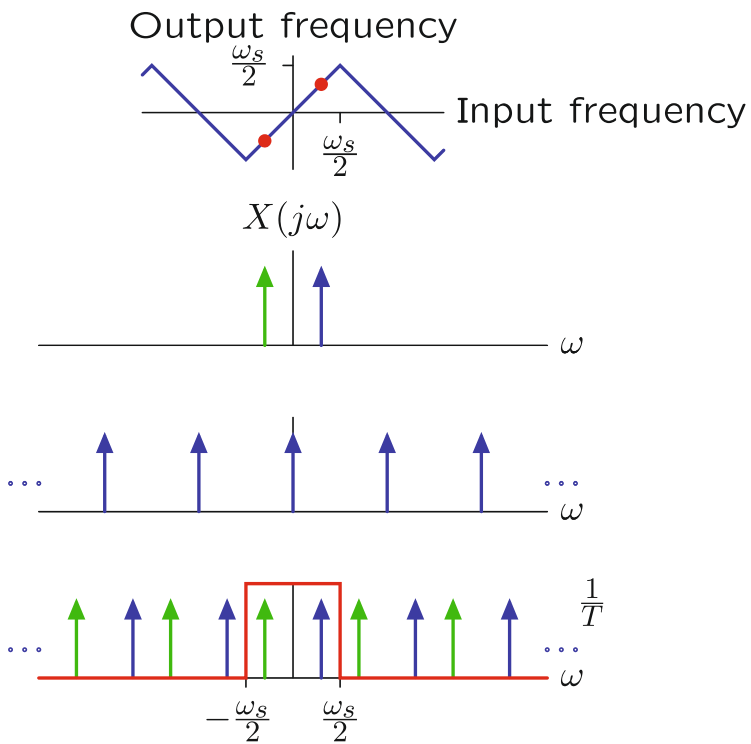




آلیاسینگ

ALIASING

The effect of aliasing is to wrap frequencies.



آلیاسینگ

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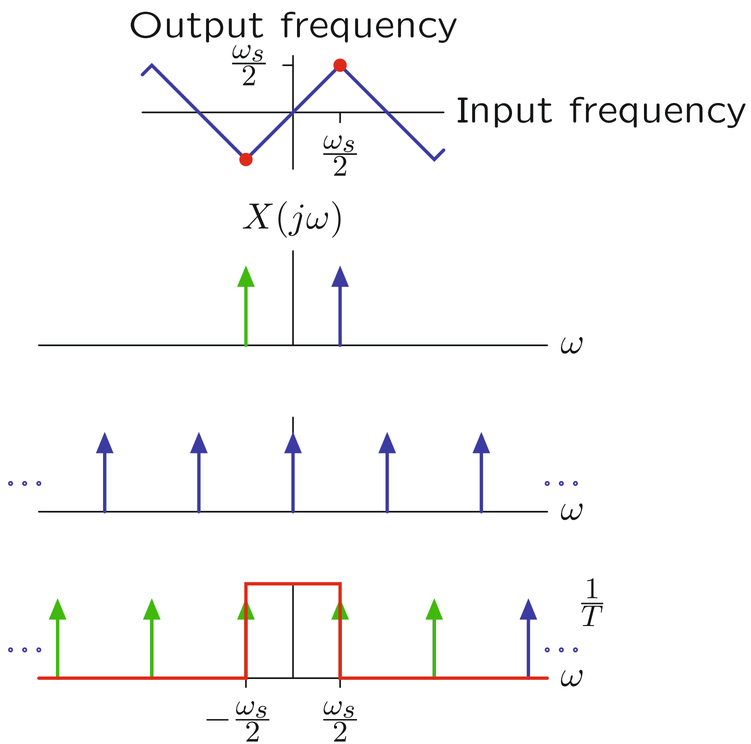

آلیاسینگ

ALIASING

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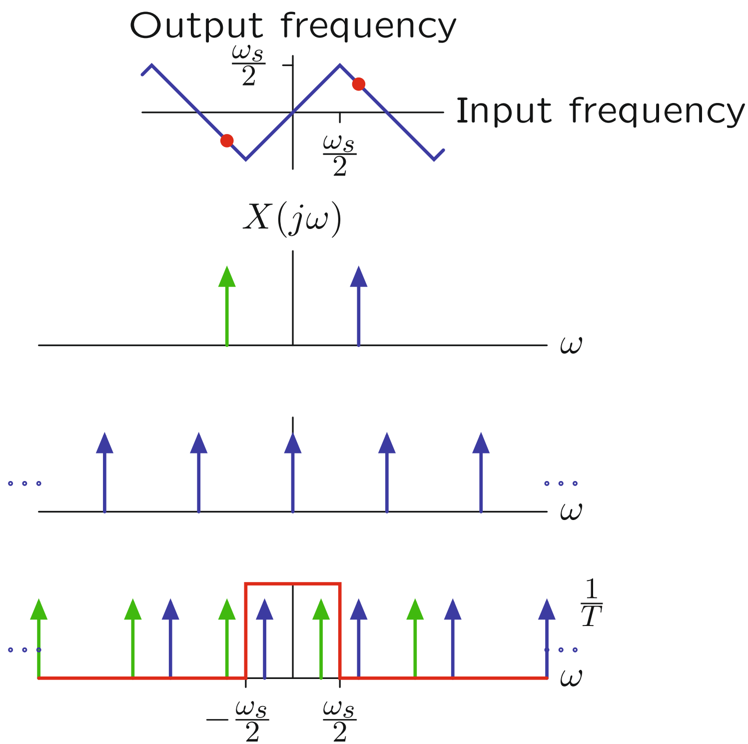


آلیاسینگ

مثال (۱ از ۴)

ALIASING

A periodic signal with a period of 0.1 ms is sampled at 44 kHz.

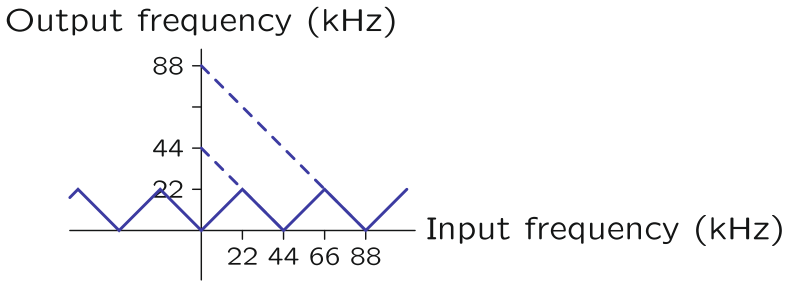
To what frequency does the eighth harmonic alias?

1. 18 kHz
2. 16 kHz
3. 14 kHz
4. 8 kHz
5. 6 kHz
6. none of the above

آلیاسینگ

مثال (۲ از ۴)

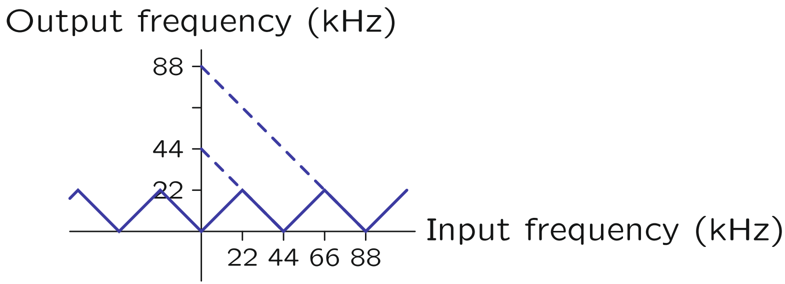
ALIASING



آلیاسینگ

مثال (۳ از ۴)

ALIASING



Harmonic	Alias
10 kHz	10 kHz
20 kHz	20 kHz
30 kHz	44 kHz-30 kHz =14 kHz
40 kHz	44 kHz-40 kHz = 4 kHz
50 kHz	50 kHz-44 kHz = 6 kHz
60 kHz	60 kHz-44 kHz =16 kHz
70 kHz	88 kHz-70 kHz =18 kHz
80 kHz	88 kHz-80 kHz = 8 kHz

آلیاسینگ

مثال (۴ از ۴)

ALIASING

A periodic signal with a period of 0.1 ms is sampled at 44 kHz.

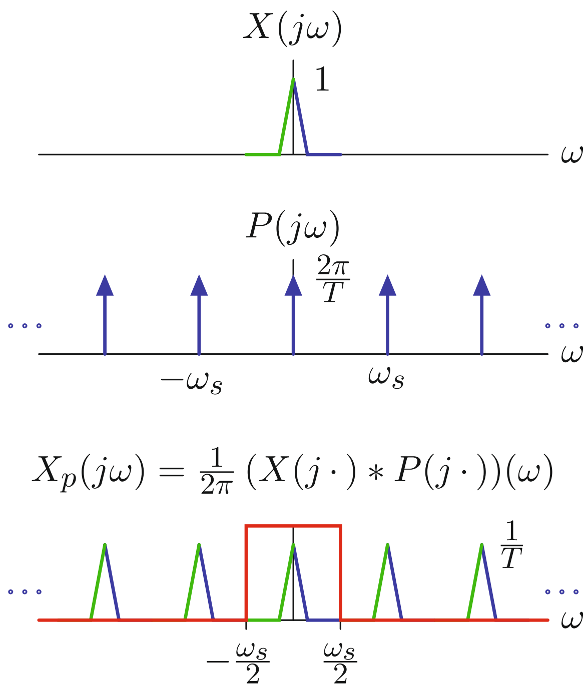
To what frequency does the eighth harmonic alias?

1. 18 kHz
2. 16 kHz
3. 14 kHz
4. 8 kHz
5. 6 kHz
6. none of the above

آلیاسینگ

ALIASING

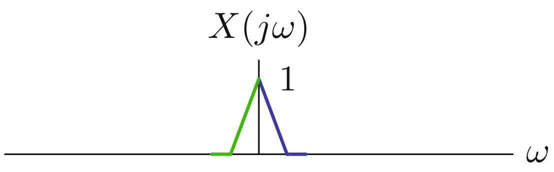
High frequency components of complex signals also wrap.

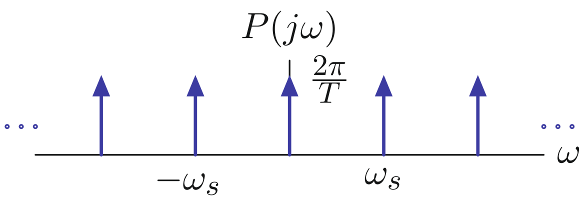


آلیاسینگ

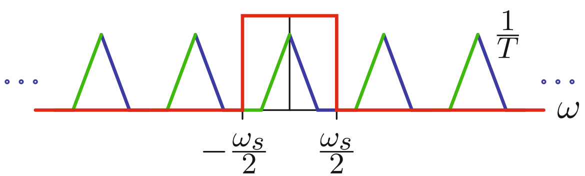
Aliasing

High frequency components of complex signals also wrap.





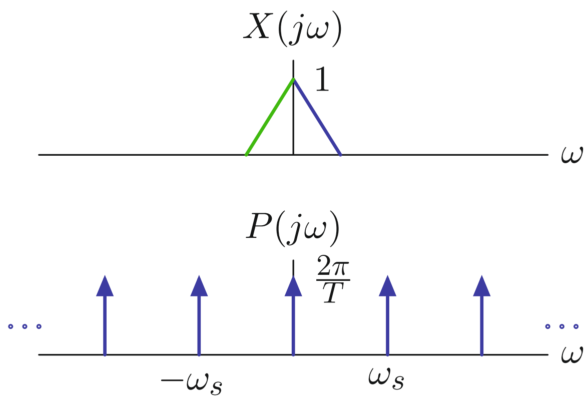
$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



آلیاسینگ

ALIASING

High frequency components of complex signals also wrap.



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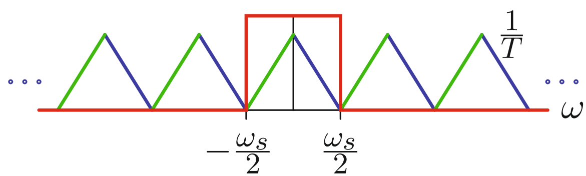

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آلیاسینگ

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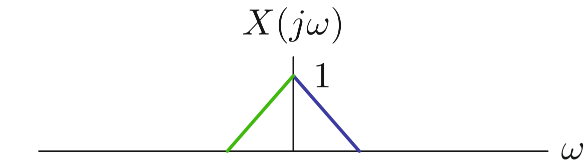
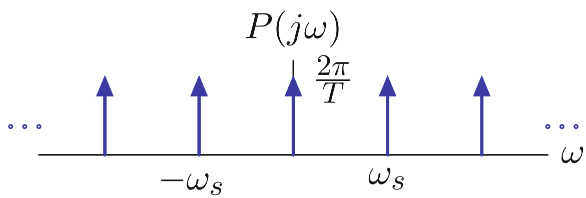
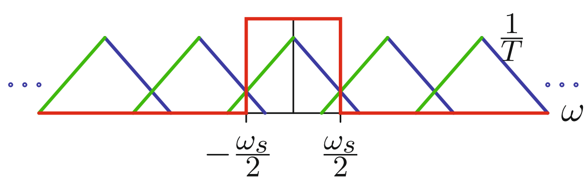
High frequency components of complex signals also wrap.





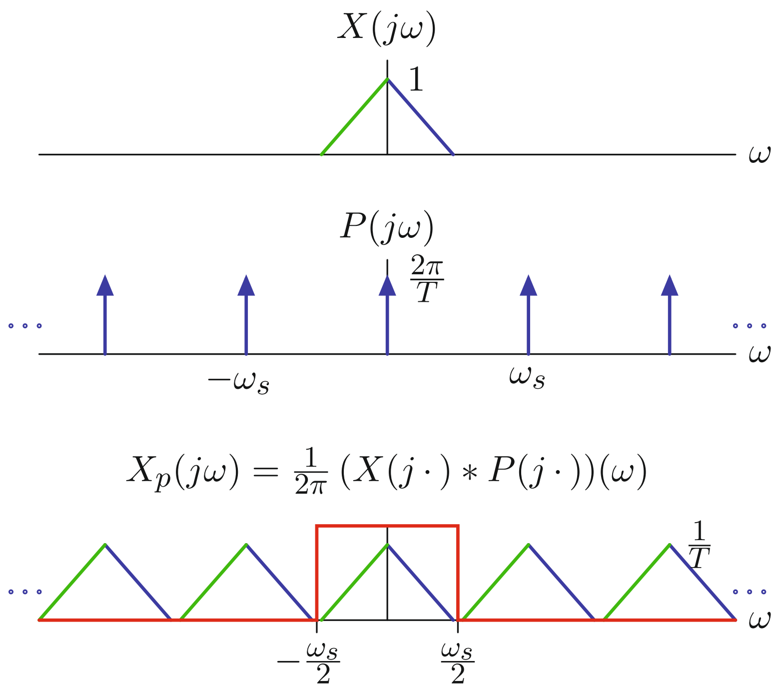
$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



آلیاسینگ

Aliasing

Aliasing increases as the sampling rate decreases.



آلیاسینگ

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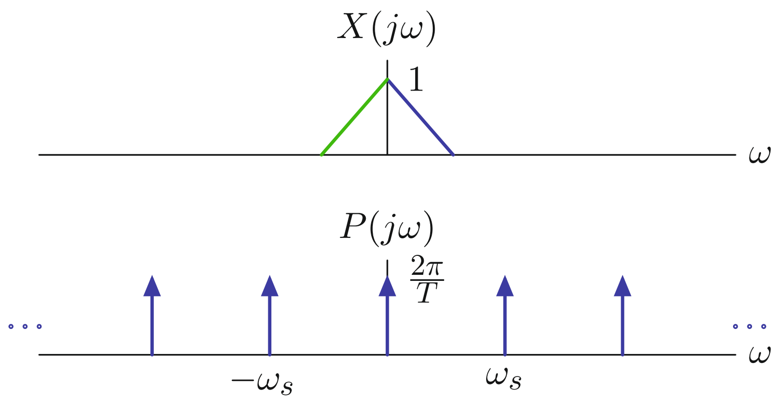
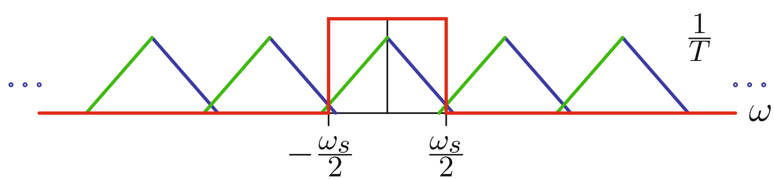


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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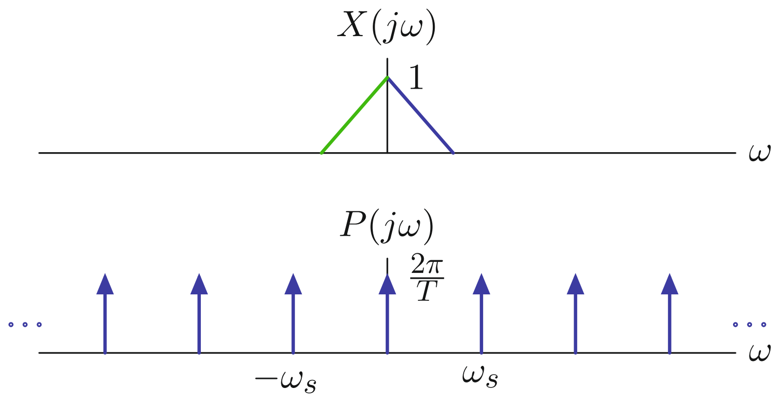
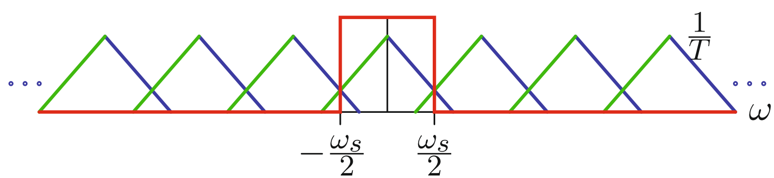


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



آلیاسینگ

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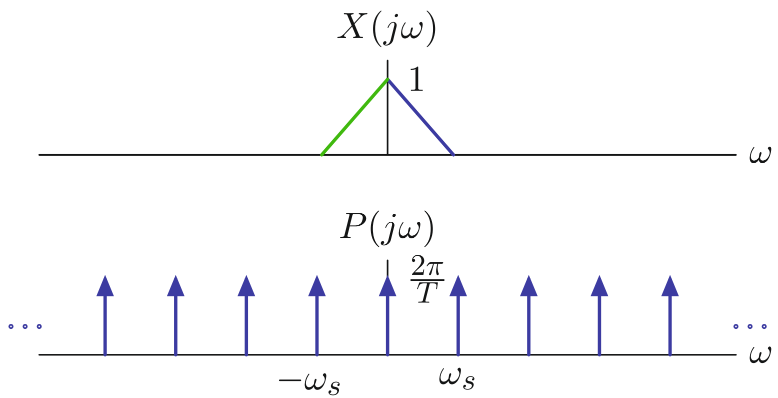
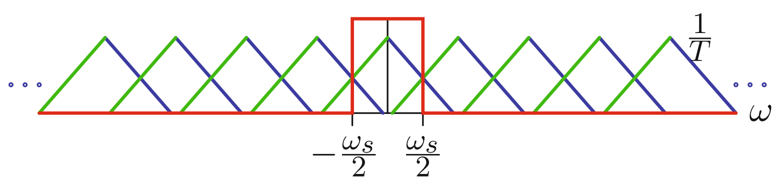


$$X_p(j\omega) = \frac{1}{2\pi}\,(X(j\,\cdot\,) * P(j\,\cdot\,))(\omega)$$



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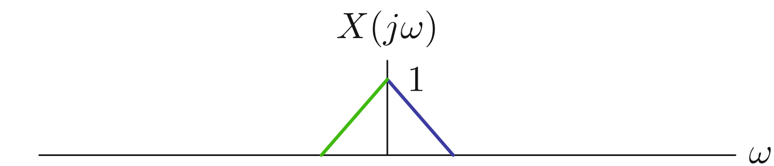
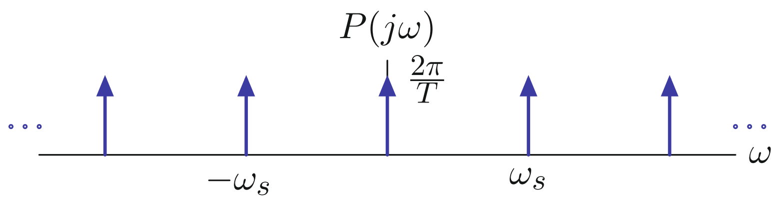
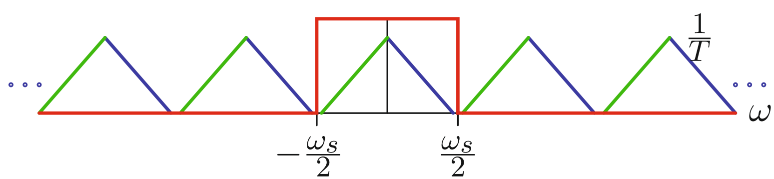




$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot) * P(j\,\cdot)\right)(\omega)$$



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آلیاسینگ

ALIASING

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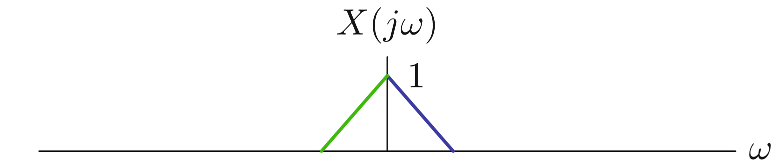
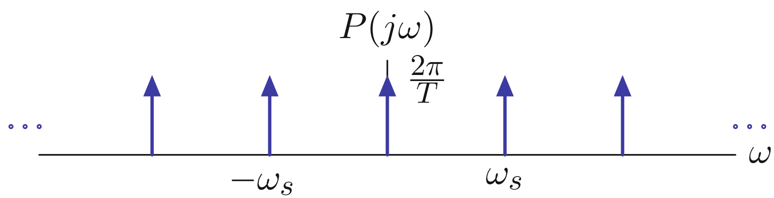
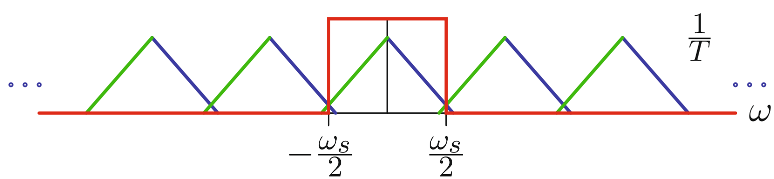




$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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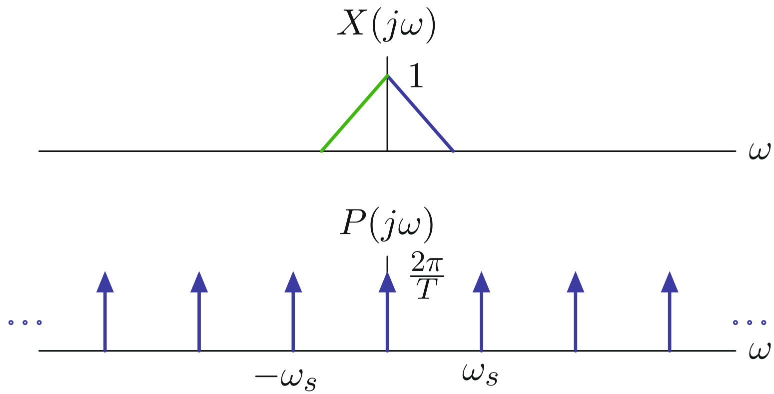

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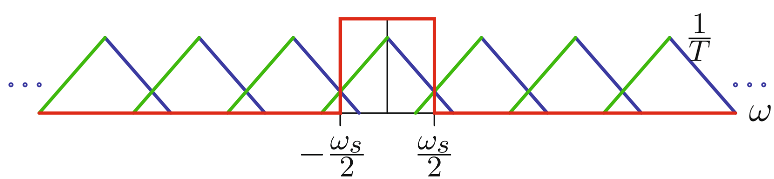

آلیاسینگ

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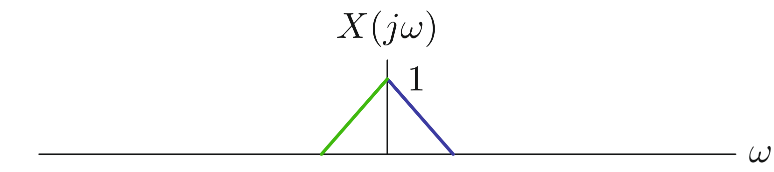
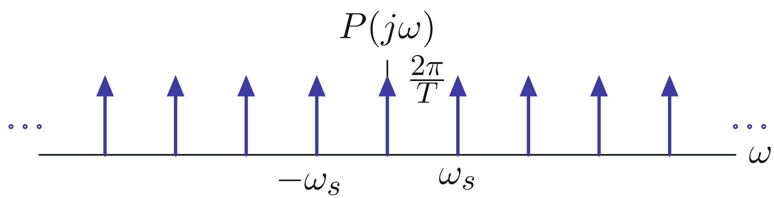
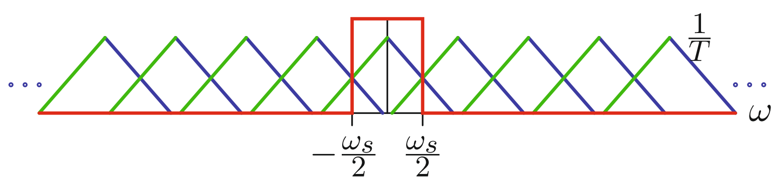




$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$

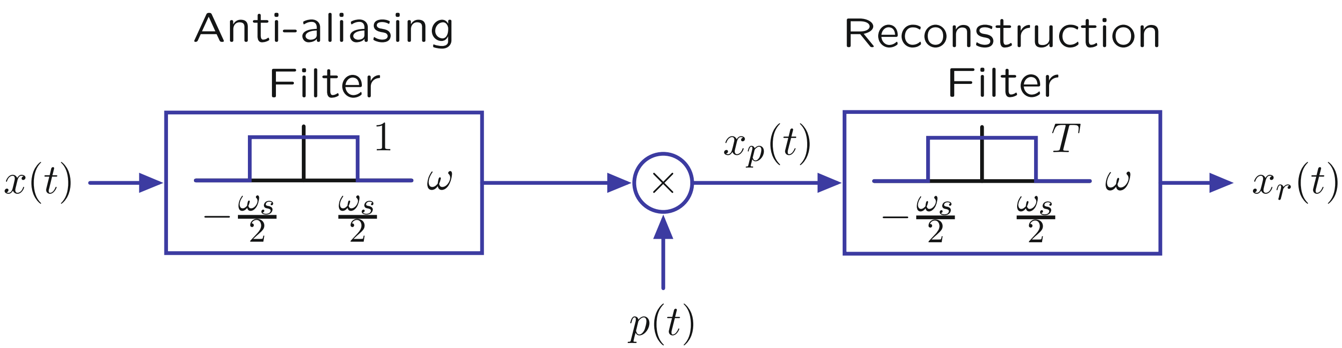


آلیاسینگ

فیلتر ضد آلیاسینگ

ANTI-ALIASING FILTER

برای اجتناب از آلیاسینگ،
مؤلفه‌های فرکانسی که آلیاس می‌شوند را پیش از نمونه‌برداری حذف می‌کنیم.



آلیاسینگ

ALIASING

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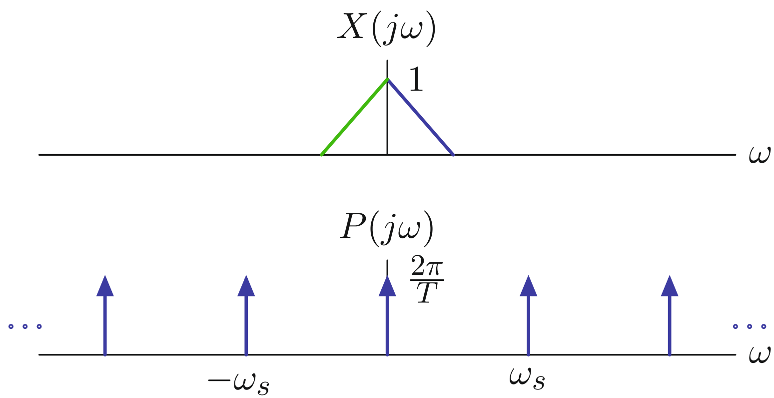
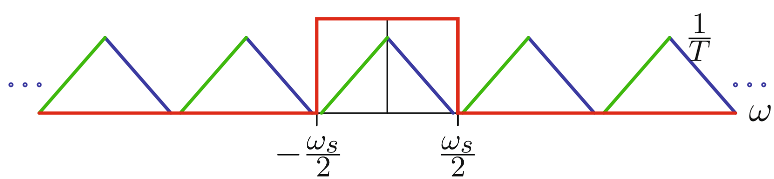


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$

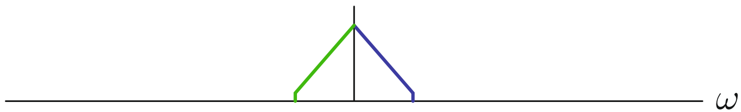


آلیاسینگ

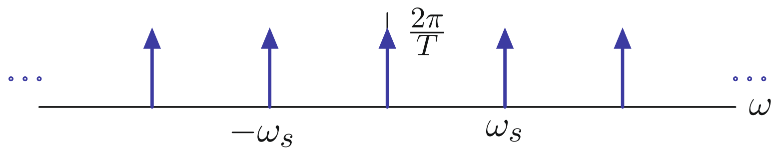
ALIASING

Aliasing increases as the sampling rate decreases.

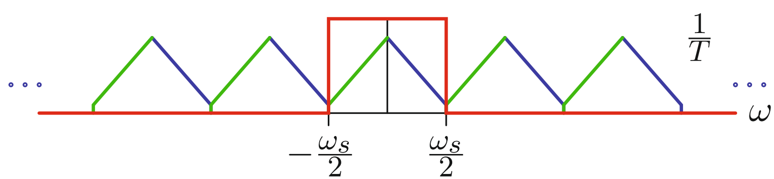
Anti-aliased $X(j\omega)$



$P(j\omega)$



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



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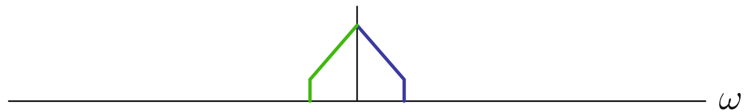
آلیاسینگ

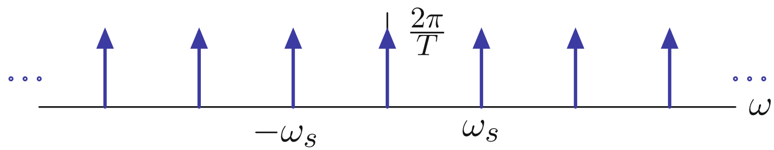
ALIASING

Aliasing increases as the sampling rate decreases.

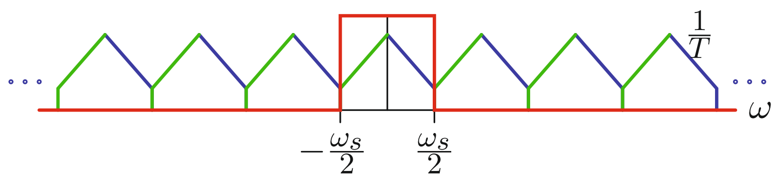
Anti-aliased $X(j\omega)$



$P(j\omega)$



$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$

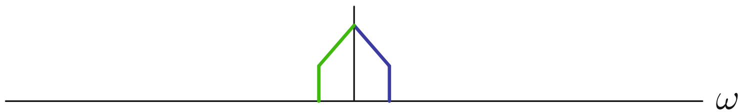


آلیاسینگ

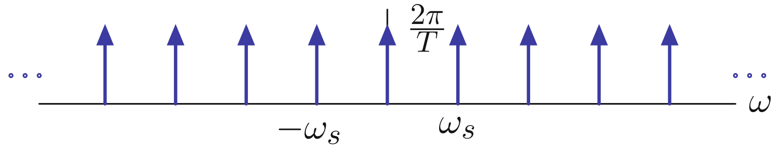
ALIASING

Aliasing increases as the sampling rate decreases.

Anti-aliased $X(j\omega)$



$P(j\omega)$



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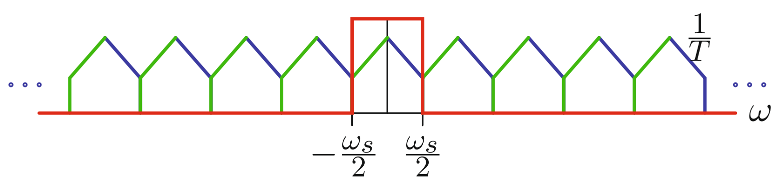

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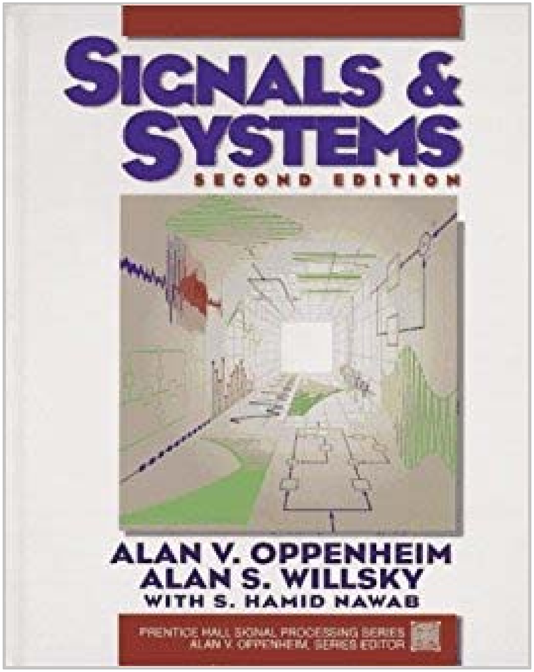
منابع

منبع اصلی



A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
Signals and Systems,
Second Edition, Prentice Hall, 1997.

Chapter 7