سیگنال‌ها و سیستم‌ها

درس ۱۳

تبدیل فوریه‌ی پیوسته–زمان (۱)

The Continuous-Time Fourier Transform (1)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

استخراج جفت تبدیل فوریه‌ی پیوسته–زمان
Derivation of the CT Fourier Transform pair
مثال‌هایی از تبدیل‌های فوریه
Examples of Fourier Transforms
تبدیل‌های فوریه‌ی سیگنال‌های متناوب
Fourier Transforms of Periodic Signals
خصوصیات تبدیل فوریه‌ی پیوسته–زمان
Properties of the CT Fourier Transform

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استخراج جفت تبدیل فوریه‌ی پیوسته-زمان

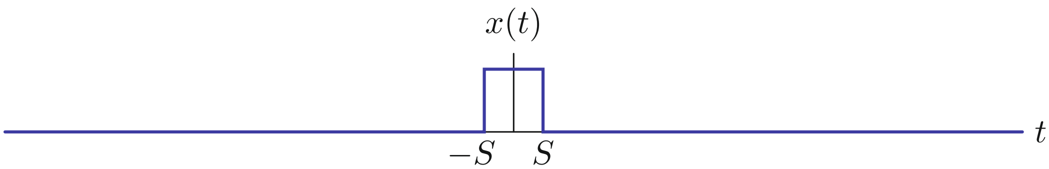
تبدیل فوریه

سیگنال نامتناوب

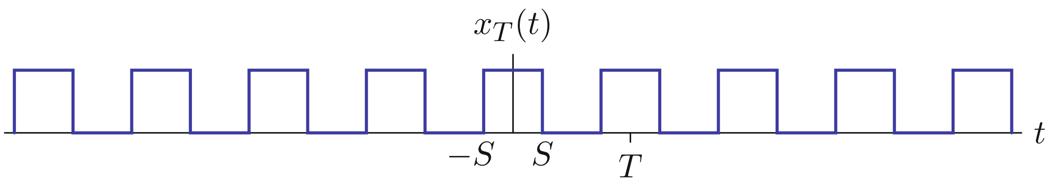
FOURIER TRANSFORM

یک سیگنال نامتناوب می‌تواند به‌عنوان یک سیگنال متناوب با دوره‌ی تناوب نامتناهی دیده شود.

Let $x(t)$ represent an aperiodic signal.



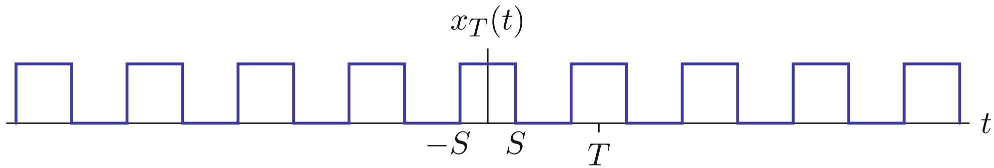
"Periodic extension": $x_T(t) = \sum_{k=-\infty}^{\infty} x(t+kT)$



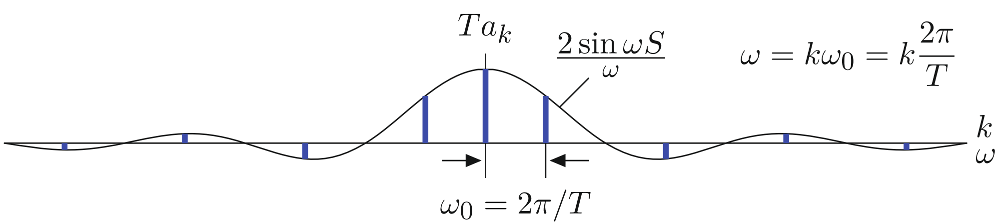
Then $x(t) = \lim_{T\to\infty} x_T(t)$.

Fourier Transform

Represent $x_T(t)$ by its Fourier series.

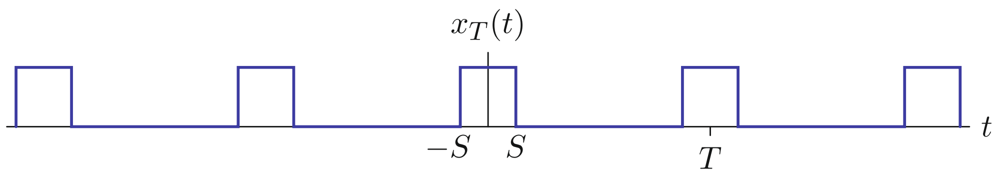


$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$

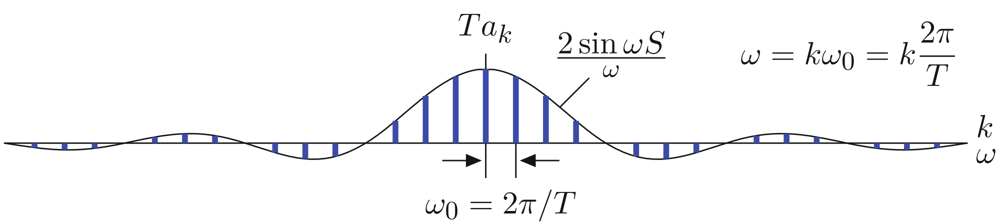


FOURIER TRANSFORM

Doubling period doubles # of harmonics in given frequency interval.



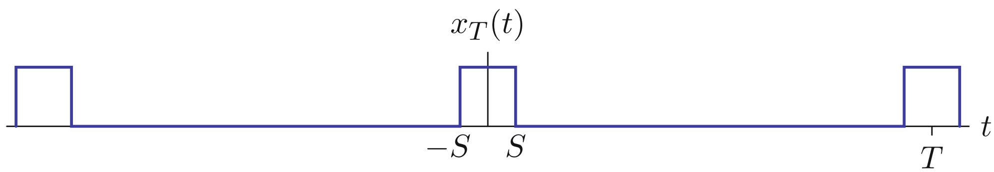
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$



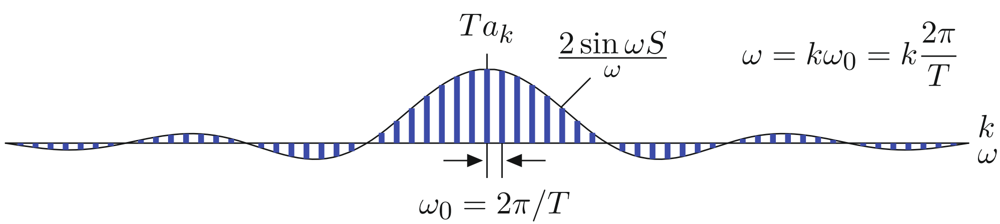
تبدیل فوریه

FOURIER TRANSFORM

As $T \rightarrow \infty$, discrete harmonic amplitudes $\rightarrow$ a continuum $E(\omega)$.



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt} dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin \omega S}{\omega}$$

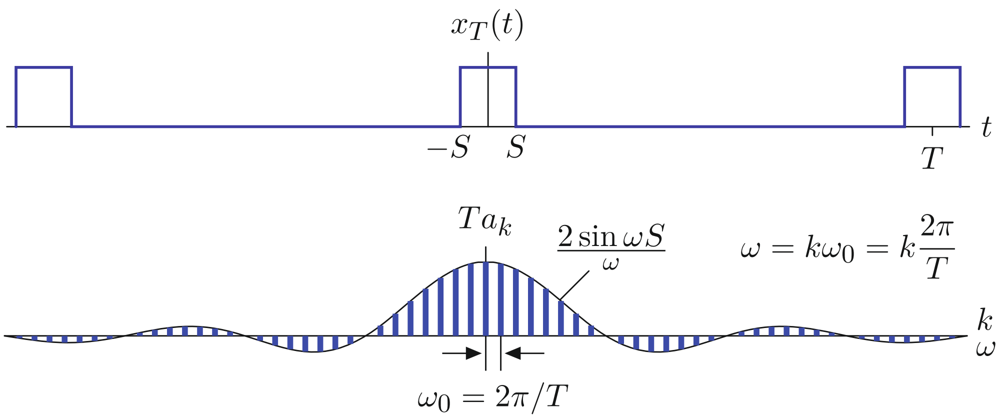


$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

تبدیل فوریه

Fourier Transform

As $T \to \infty$, synthesis sum $\to$ integral.



$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}E(\omega)}_{a_k} e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi}E(\omega)e^{j\omega t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\omega)e^{j\omega t}d\omega$$

Fourier's Derivation of the CT Fourier Transform

- $x(t)$ - an aperiodic signal
  - view it as the limit of a periodic signal as $T \to \infty$

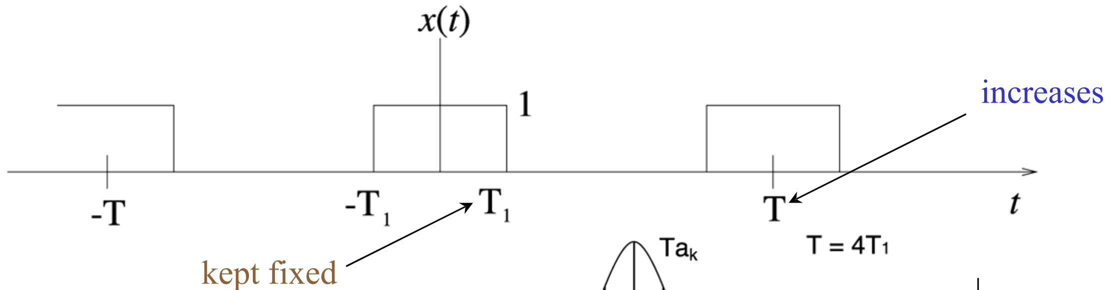
- For a periodic signal, the harmonic components are spaced $\omega_0 = 2\pi/T$ apart ...

- As $T \to \infty$, $\omega_0 \to 0$, and harmonic components are spaced closer and closer in frequency

$$\Downarrow$$

$$\text{Fourier series} \longrightarrow \text{Fourier integral}$$

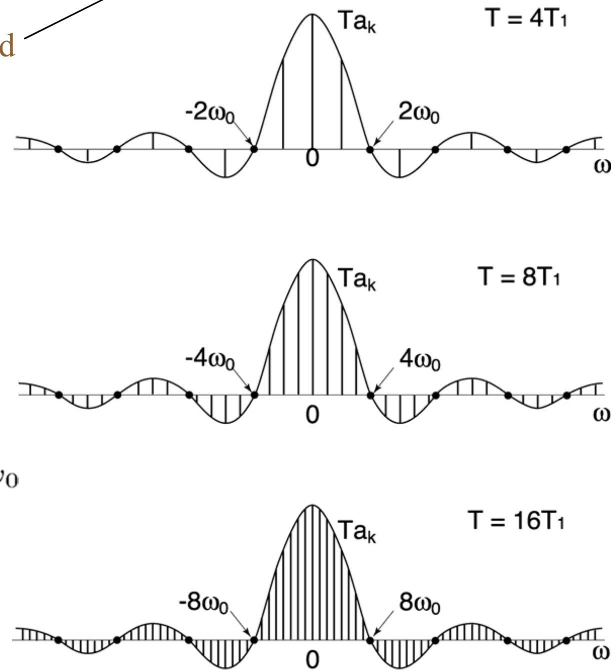
Motivating Example: Square wave



$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

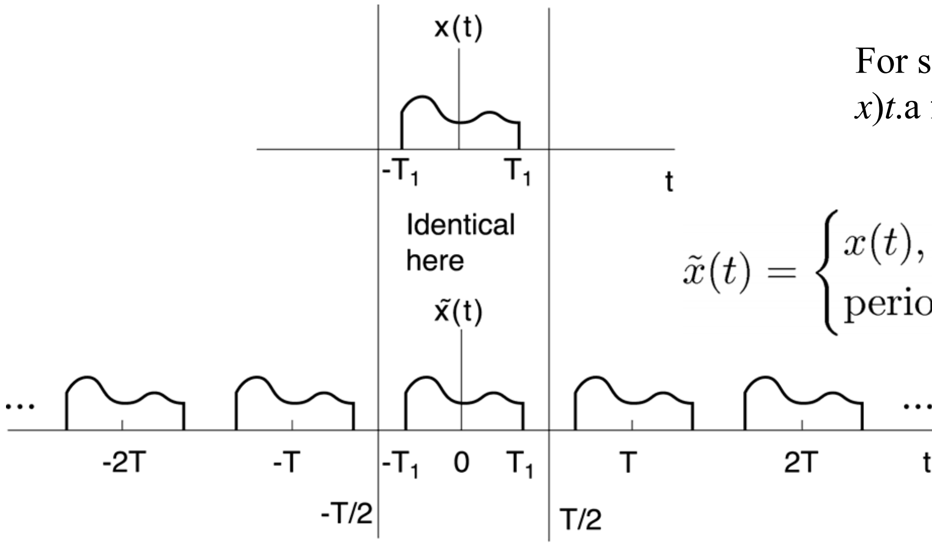
$$\Downarrow$$

$$Ta_k = \left.\frac{2\sin \omega T_1}{\omega}\right|_{\omega = k\omega_0}$$



Discrete frequency points become denser in ω as T increases

So, on with the derivation ...



For simplicity, assume
x)*t*.a finite duration has (

$$\tilde{x}(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ \text{periodic}, & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \to \infty,\ \tilde{x}(t) = x(t) \text{ for all } t$$

Derivation (continued)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

$\uparrow$

$\tilde{x}(t) = x(t)$ in this interval

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt \qquad (1)$$

If we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

then Eq.(1) $\Rightarrow$

$$a_k = \frac{X(jk\omega_0)}{T}$$

Derivation (continued)

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$

$$\begin{aligned} x(t) = \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t} \end{aligned}$$

$$\Downarrow$$

As $T \to \infty$, $\sum \omega_0 \to \int d\omega$, we get the CT Fourier Transform pair

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Synthesis equation} \\ X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Analysis equation} \end{aligned}}$$

For what kinds of signals can we do this?

(1) It works also even if $x(t)$ is infinite duration, but satisfies:

a) Finite energy $$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

In this case, there is *zero* energy in the error

$$e(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \text{Then} \int_{-\infty}^{\infty} |e(t)|^2 dt = 0$$

b) Dirichlet conditions

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = x(t)$ at points of continuity

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega =$ midpoint at discontinuity

(iii) Gibb's phenomenon

c) By allowing impulses in $x(t)$ or in $X(j\omega)$, we can represent even *more* signals

E.g. It allows us to consider *FT* for *periodic* signals

تبدیل فوریه‌ی پیوسته–زمان (۱)

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مثال‌هایی از تبدیل‌های فوریه

Example #1

(a) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

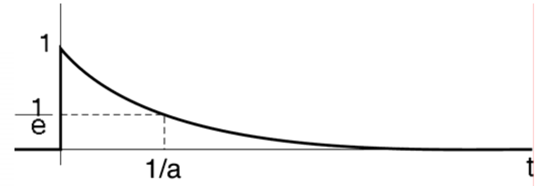
— Synthesis equation for $\delta(t)$

(b) $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_0}$$

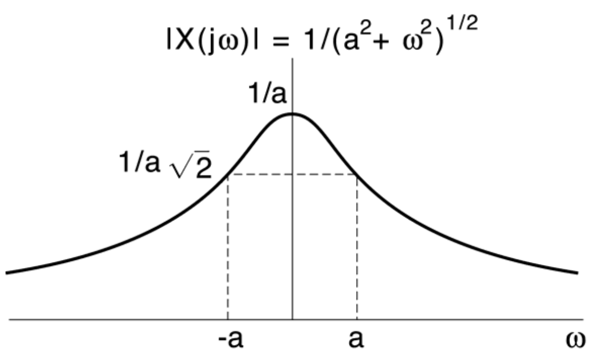
Example #2: Exponential function



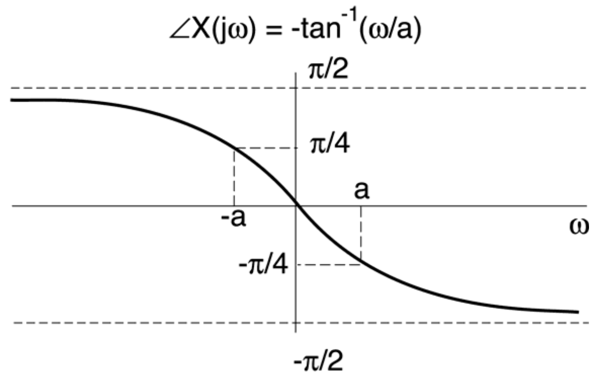
$$x(t) = e^{-at}u(t), a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_0^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt$$

$$= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$



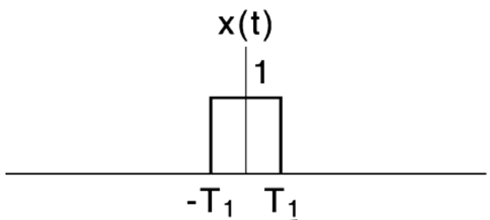
Even symmetry

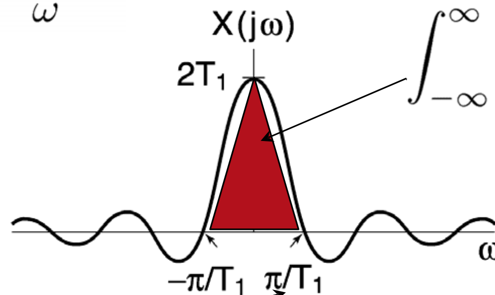


Odd symmetry

Example #3: A square pulse in the time-domain

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin\omega T_1}{\omega}$$





$$\int_{-\infty}^{\infty} X(j\omega) d\omega$$

Note the inverse relation between the two widths $\Rightarrow$ Uncertainty principle

Useful facts about CTFT's

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

Example above: $\int_{-\infty}^{\infty} x(t)dt = 2T_1 = X(0)$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) d\omega$$

Ex. above: $x(0) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) d\omega$

$$= \frac{1}{2\pi} \times (\text{Area of the triangle})$$

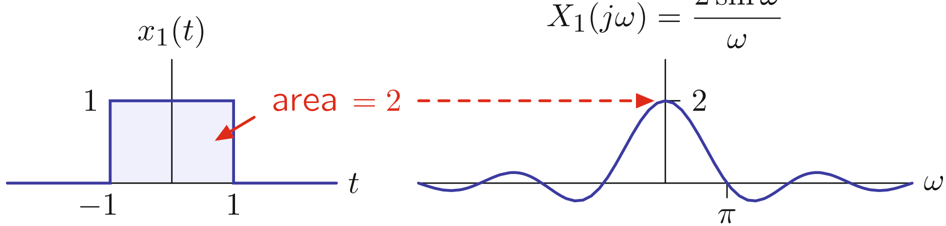
خصوصیات تبدیل فوریه‌ی پیوسته–زمان

رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۱)

The value of $X(j\omega)$ at $\omega = 0$ is the integral of $x(t)$ over time t.

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j0t}dt = \int_{-\infty}^{\infty} x(t)\,dt$$

$$X_1(j\omega) = \frac{2\sin\omega}{\omega}$$

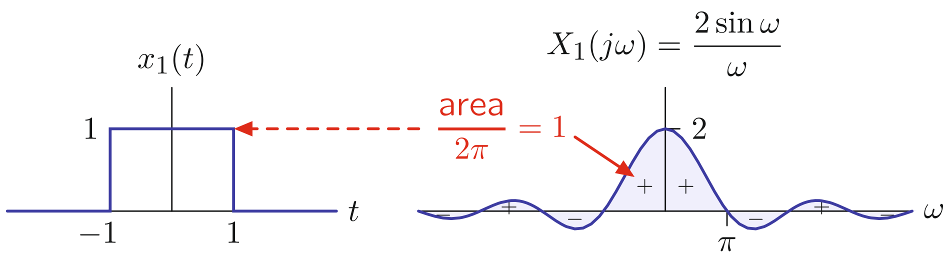


خصوصیات تبدیل فوریه‌ی پیوسته-زمان

رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۲)

The value of $x(0)$ is the integral of $X(j\omega)$ divided by 2π.

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$



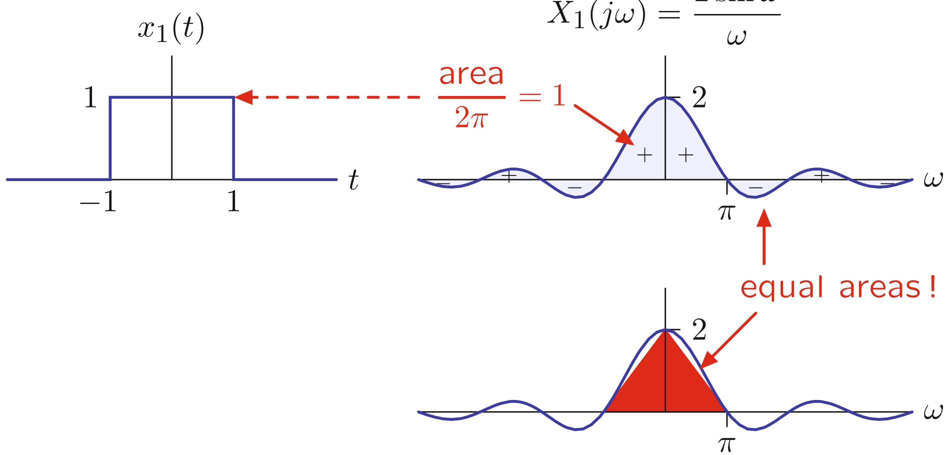
خصوصیات تبدیل فوریه‌ی پیوسته-زمان

رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۳)

The value of $x(0)$ is the integral of $X(j\omega)$ divided by 2π.

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$

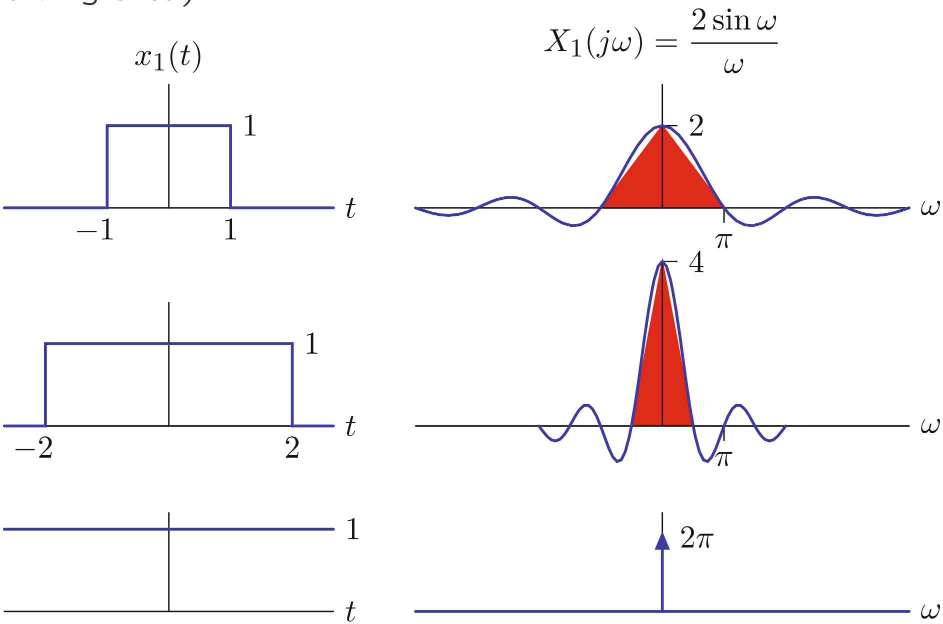
$$X_1(j\omega) = \frac{2\sin\omega}{\omega}$$



خصوصیات تبدیل فوریه‌ی پیوسته-زمان

رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۴)

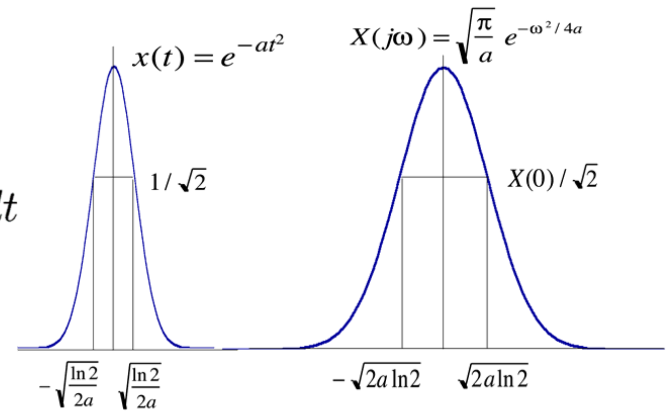
Stretching time compresses frequency and increases amplitude (preserving area).



New way to think about an impulse!

Example #4: $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2} dt \\
&= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t+\frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi}/a} \cdot e^{-\frac{\omega^2}{4a}} \\
&= \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}
\end{aligned}
$$



Also a Gaussian!

(Pulse width in t)•(Pulse width in ω)

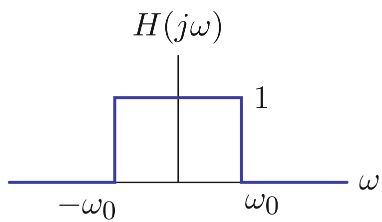
$\Rightarrow \Delta t \bullet \Delta\omega \sim (1/a^{1/2}) \bullet (a^{1/2}) = 1$

Uncertainty Principle! Cannot make *both* Δt and $\Delta\omega$ arbitrarily small.

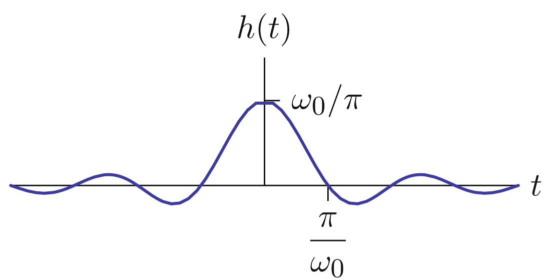
تبدیل وارون فوریه

Inverse Fourier Transform

Find the impulse reponse of an "ideal" low pass filter.



$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}d\omega = \frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0 t}{\pi t}$$



This result is not so easily obtained without inverse relation.

خاصیت دوگانی

تبدیل مستقیم و تبدیل وارون فوریه

DUALITY

The Fourier transform and its inverse have very similar forms.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{(Fourier transform)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{(“inverse” Fourier transform)}$$

Convert one to the other by

- $t \rightarrow \omega$
- $\omega \rightarrow -t$
- scale by 2π

خاصیت دوگانی

تبدیل مستقیم و تبدیل وارون فوریه

DUALITY

The Fourier transform and its inverse have very similar forms.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Two differences:

- minus sign: flips time axis (or equivalently, frequency axis)
- divide by 2π (or multiply in the other direction)

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega)$$

$\omega \rightarrow t$ $\qquad$ $t \rightarrow \omega$; flip ; $\times 2\pi$

$$x_2(t) = g(t) \leftrightarrow X_2(j\omega) = 2\pi f(-\omega)$$

خاصیت دوگانی

تبدیل مستقیم و تبدیل وارون فوریه

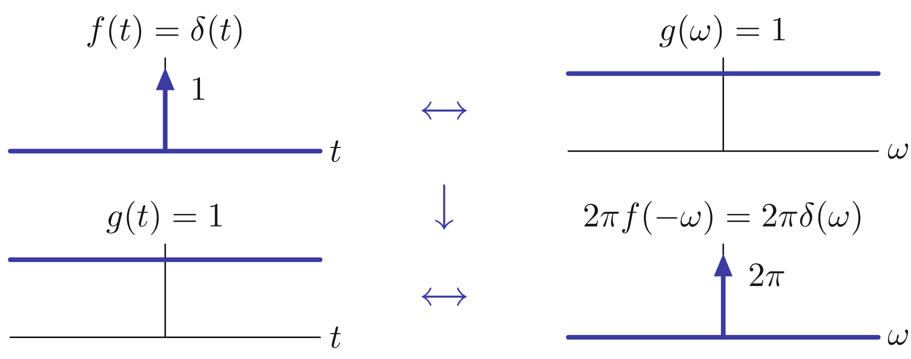
DUALITY

Using duality to find new transform pairs.

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega)$$

$\omega \to t$ $\qquad$ $t \to \omega$; flip ; $\times 2\pi$

$$x_2(t) = g(t) \leftrightarrow X_2(j\omega) = 2\pi f(-\omega)$$



The function $g(t) = 1$ does not have a Laplace transform!

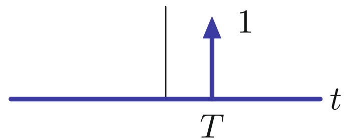
خاصیت دوگانی

تبدیل مستقیم و تبدیل وارون فوریه

DUALITY

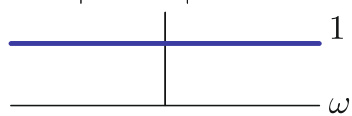
Fourier transform of delayed impulse: $\delta(t-T) \leftrightarrow e^{-j\omega T}$.

$$x(t) = \delta(t-T)$$

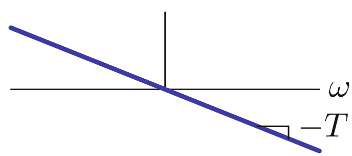


$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t-T)e^{-j\omega t}dt = e^{-j\omega T}$$

$$\left|X(j\omega)\right| = 1$$



$$\angle X(j\omega) = -\omega T$$



تبدیل فوریه‌ی پیوسته–زمان (۱)

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تبدیل‌های فوریه‌ی سیگنال‌های متناوب

تبدیل فوریه‌ی سیگنال‌های متناوب

تبدیل فوریه‌ی سینوس

Using duality to find the Fourier transform of an eternal sinusoid.

$$\delta(t-T) \quad \leftrightarrow \quad e^{-j\omega T}$$

$$\omega \rightarrow t \qquad\qquad t \rightarrow \omega \;;\; \text{flip} \;;\; \times 2\pi$$

$$e^{-jtT} \quad \leftrightarrow \quad 2\pi\delta(\omega+T)$$

$$T \rightarrow \omega_0 :$$

$$e^{-j\omega_0 t} \quad \leftrightarrow \quad 2\pi\delta(\omega+\omega_0)$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad \{a_k\}$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi}{T}k\right)$$

CT Fourier Transforms of Periodic Signals

Suppose

$$X(j\omega) = \delta(\omega - \omega_0)$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

— periodic in t with frequency ω_o

That is

$$\boxed{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

— All the energy is concentrated in one frequency — ω_o

More generally

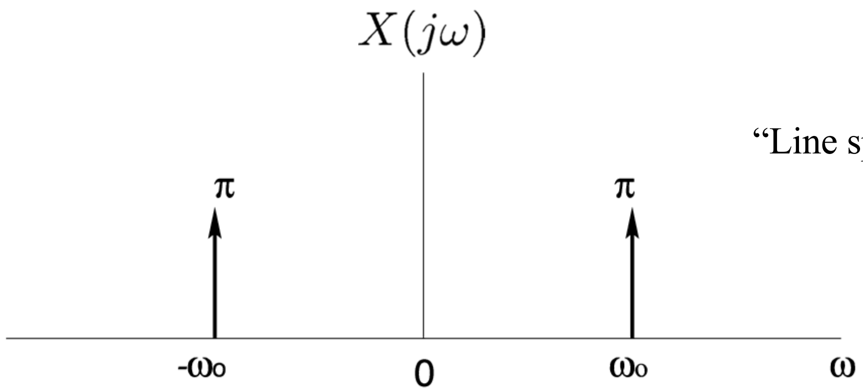
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example #4:

$$x(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$\updownarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



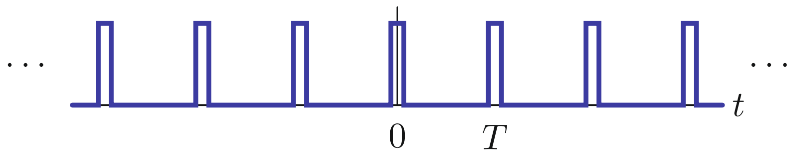
"Line spectrum"

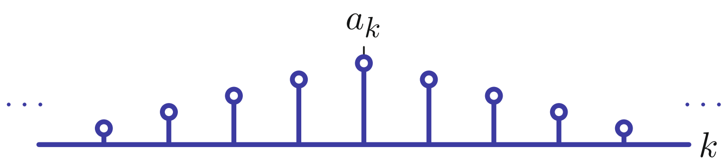
تبدیل فوریه‌ی سیگنال‌های متناوب

رابطه‌ی کلی

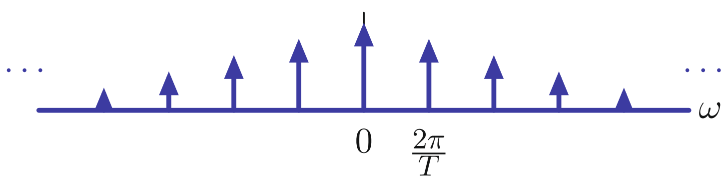
Each term in the Fourier series is replaced by an impulse.

$$x(t) = \sum_{k=-\infty}^{\infty} x_p(t - kT)$$

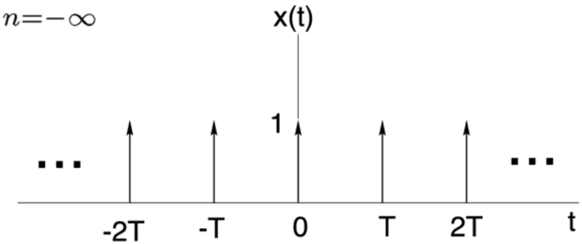




$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \, a_k \, \delta(\omega - k\frac{2\pi}{T})$$



Example #5: $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ — Sampling function

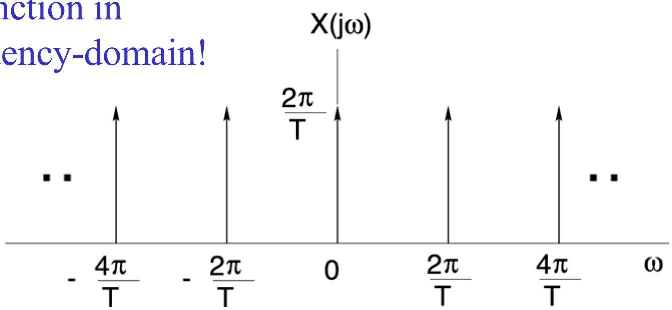


$$x(t) \leftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$$

$$\Downarrow$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{2\pi}{T}}_{2\pi a_k} \delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_0})$$

Same function in
the frequency-domain!



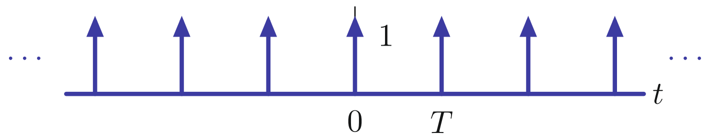
Note: (period in t) T
$\Leftrightarrow$ (period in ω) $2\pi/T$
Inverse relationship again!

تبدیل فوریه‌ی سیگنال‌های متناوب

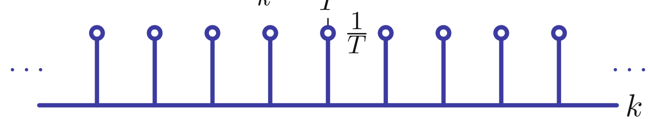
تبدیل فوریه‌ی قطار ضربه

The Fourier transform of an impulse train is an impulse train.

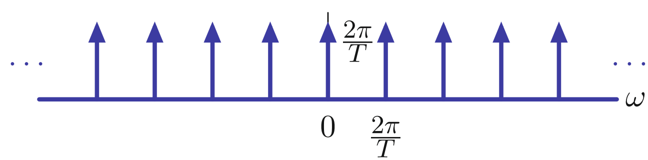
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$a_k = \frac{1}{T} \quad \forall \ k$$



$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\frac{2\pi}{T})$$



تبدیل فوریه‌ی پیوسته–زمان (۱)

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خصوصیات تبدیل فوریه‌ی پیوسته–زمان

Properties of the CT Fourier Transform

1) Linearity $\qquad ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$

2) Time Shifting $\qquad x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$

Proof: $\displaystyle \int_{-\infty}^{\infty} x(\underbrace{t - t_0}_{t'})e^{-j\omega t} dt = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\omega t'} dt'}_{X(j\omega)}$

FT magnitude unchanged

$$|e^{-j\omega t_0} X(j\omega)| = |X(j\omega)|$$

Linear change in *FT* phase

$$\angle(e^{-j\omega t_0} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

Properties (continued)

3) Conjugate Symmetry

$$x(t) \text{ real} \leftrightarrow X(-j\omega) = X^*(j\omega)$$

$$\Downarrow$$

$$|X(-j\omega)| = |X(j\omega)|$$ *Even*

$$\angle X(-j\omega) = -\angle X(j\omega)$$ *Odd*

$$Re\{X(-j\omega)\} = Re\{X(j\omega)\}$$ *Even*

$$Im\{X(-j\omega)\} = -Im\{X(j\omega)\}$$ *Odd*

The Properties Keep on Coming ...

4) Time-Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|}X\left(j\frac{\omega}{a}\right)$$

E.g. $a > 1 \rightarrow at > t$
compressed in time $\Leftrightarrow$
stretched in frequency

$$\Downarrow \; a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$$\Downarrow$$

a) $x(t)$ real and even

$$x(t) = x(-t)$$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) - \text{ Real \& even}$$

b) $x(t)$ real and odd

$$x(t) = -x(-t)$$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) - \text{ Purely imaginary \& odd}$$

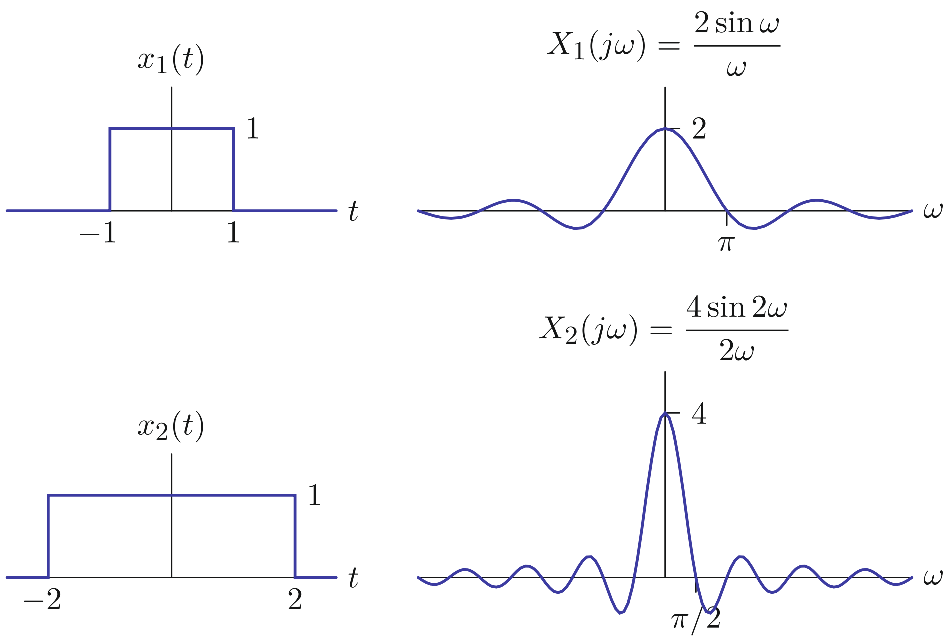
c)

$$\begin{array}{ccccc} X(j\omega) & = & Re\{X(j\omega)\} & + & jIm\{X(j\omega)\} \\ & & \uparrow & & \uparrow \\ \textbf{For real } x(t) & = & Ev\{x(t)\} & + & Od\{x(t)\} \end{array}$$

خصوصیات تبدیل فوریه‌ی پیوسته-زمان

کشش زمانی موجب فشردگی فرکانسی می‌شود

Stretching time compresses frequency.



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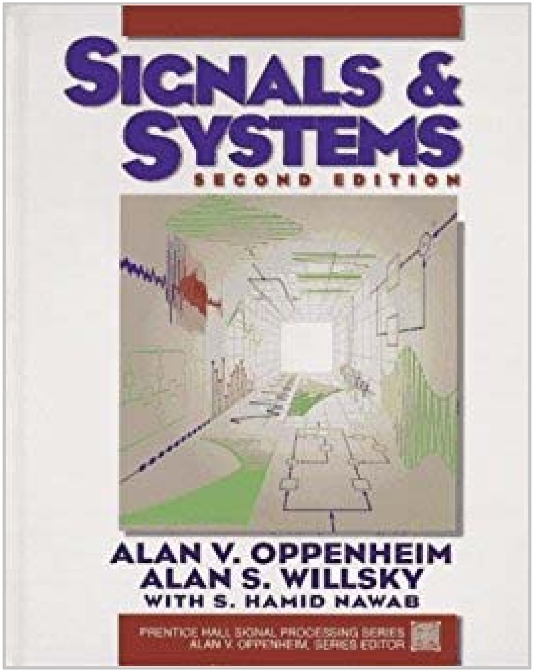
منابع

منبع اصلی



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Chapter 4

Prepared by Kazim Fouladi | Spring 2018 | 3rd Edition
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