سیگنال‌ها و سیستم‌ها

درس ۲۹

تبدیل Z (۲)

The z-Transform (2)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

خصوصیات تبدیل z
Properties of the z-Transform
توابع سیستم سیستم‌های LTI گسسته–زمان؛ علی بودن، پایداری
System Functions of DT LTI Systems; Causality, Stability
ارزیابی هندسی تبدیل‌های z و پاسخ‌های فرکانسی گسسته–زمان
Geometric Evaluation of z-Transforms and DT Frequency Responses
سیستم‌های مرتبه–اول و مرتبه–دوم
First- and Second-Order Systems
جبر تابع سیستم و نمودارهای بلوکی
System Function Algebra and Block Diagrams
تبدیل‌های z یک‌طرفه
Unilateral z-Transforms

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خصوصیات تبدیل z

Properties of z-Transforms

(1) Time Shifting $x[n-n_0] \longleftrightarrow z^{-n_0}X(z),$

The rationality of $X(z)$ unchanged, *different* from LT. ROC unchanged except for the possible addition or deletion of the origin or infinity

$n_o > 0 \Rightarrow$ ROC $z \neq 0$ (maybe)

$n_o < 0 \Rightarrow$ ROC $z \neq \infty$ (maybe)

(2) z-Domain Differentiation $nx[n] \longleftrightarrow -z\dfrac{dX(z)}{dz},$ same ROC

Derivation:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$$

$$-z\frac{dX(z)}{dx} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

تبدیل z

خصوصیات

THE Z-TRANSFORM

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Delay	$x[n-1]$	$z^{-1}X(z)$	R
Multiply by n	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
Convolve in n	$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

تبدیل z

مثال

THE Z-TRANSFORM

$$Y(z) = \left(\frac{z}{z-1}\right)^2 \quad \leftrightarrow \quad y[n] = ?$$

$$\frac{z}{z-1} \quad \leftrightarrow \quad u[n]$$

$$-z\frac{d}{dz}\left(\frac{z}{z-1}\right) = z\left(\frac{1}{z-1}\right)^2 \quad \leftrightarrow \quad nu[n]$$

$$z \times \left(-z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right) = \left(\frac{z}{z-1}\right)^2 \quad \leftrightarrow \quad (n+1)u[n+1] = (n+1)u[n]$$

تبدیل z

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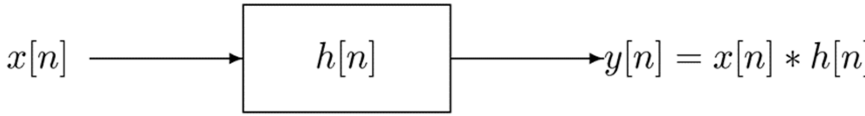
توابع سیستم سیستم‌های LTI گسسته–زمان: علی بودن، پایداری

Convolution Property and System Functions



$Y(z) = H(z)X(z)$, ROC at least the intersection of the ROCs of $H(z)$ and $X(z)$, can be bigger if there is pole/zero cancellation. *e.g.*

$$
\begin{aligned}
H(z) &= \frac{1}{z-a}, \quad |z| > a \\
X(z) &= z-a, \quad z \neq \infty \\
Y(z) &= 1 \quad \text{ROC all } z
\end{aligned}
$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$ — The System Function

$H(z)$ + ROC tells us everything about system

CAUSALITY

(1) $h[n]$ right-sided ⇒ ROC is the exterior of a circle *possibly* including $z = \infty$:

$$H(z) = \sum_{n=N_1}^{\infty} h[n] z^{-n}$$

If $N_1 < 0$, then the term $h[N_1]z^{-N_1} \rightarrow \infty$ at $z = \infty$
$\Rightarrow$ ROC outside a circle, but does *not* include ∞.

$\text{Causal} \Leftrightarrow N_1 \geq 0$

$\Downarrow$

No z^m terms with $m > 0$
$\Rightarrow z = \infty \in \text{ROC}$

A DT LTI system with system function $H(z)$ is causal ⇔ the ROC of $H(z)$ is the exterior of a circle *including* $z = \infty$

Causality for Systems with Rational System Functions

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$$\Downarrow \text{ No poles at } \infty, \text{ if } M \leq N$$

A DT LTI system with rational system function $H(z)$ is causal

$\Leftrightarrow$ (a) the ROC is the exterior of a circle outside the outermost pole;

and (b) if we write $H(z)$ as a ratio of polynomials

$$H(z) = \frac{N(z)}{D(z)}$$

then

$$\text{degree } N(z) \leq \text{ degree } D(z)$$

Stability

- LTI System Stable $\Leftrightarrow$ $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ $\Leftrightarrow$ ROC of $H(z)$ includes the unit circle $|z| = 1$

  $\Rightarrow$ Frequency Response $H(e^{j\omega})$ (DTFT of $h[n]$) exists.

- A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1

تبدیل z

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ارزیابی هندسی تبدیل‌های z و پاسخ‌های فرکانسی گسسته–زمان

Geometric Evaluation of a Rational z-Transform

Example #1:

$$X_1(z) = z - a \text{ - A first-order zero}$$

Example #2:

$$X_2(z) = \frac{1}{z-a} \text{ - A first-order pole}$$

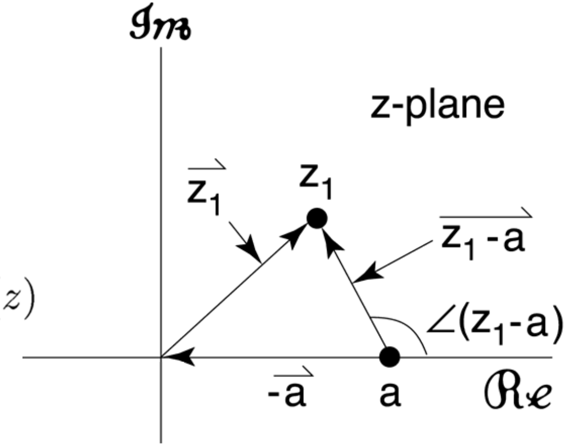
$$|X_2(z)| = \frac{1}{|X_1(z)|}, \quad \angle X_2(z) = -\angle X_1(z)$$

Example #3:

$$X(z) = M\frac{\Pi_{i=1}^{R}(z-\beta_i)}{\Pi_{j=1}^{P}(z-\alpha_j)}$$

$$|X(z)| = |M|\frac{\Pi_{i=1}^{R}|z-\beta_i|}{\Pi_{j=1}^{P}|z-\alpha_j|}$$

$$\angle X(z) = \angle M + \sum_{i=1}^{R}\angle(z-\beta_i) - \sum_{j=1}^{P}\angle(z-\alpha_j)$$



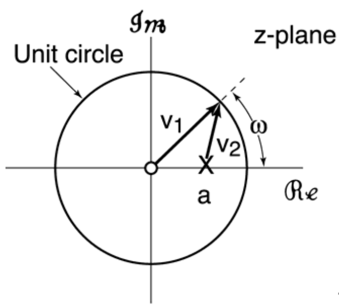
All same as in s-plane

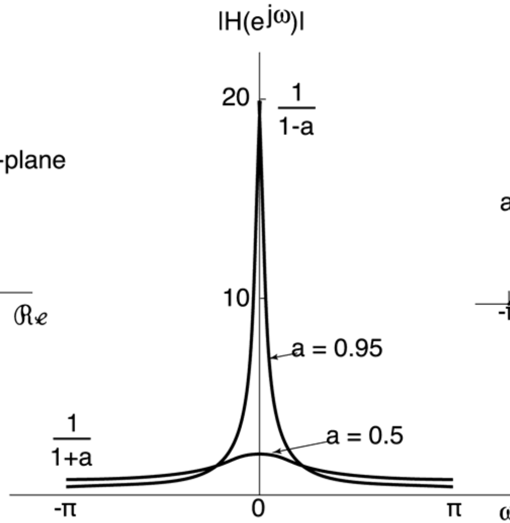
Geometric Evaluation of DT Frequency Responses

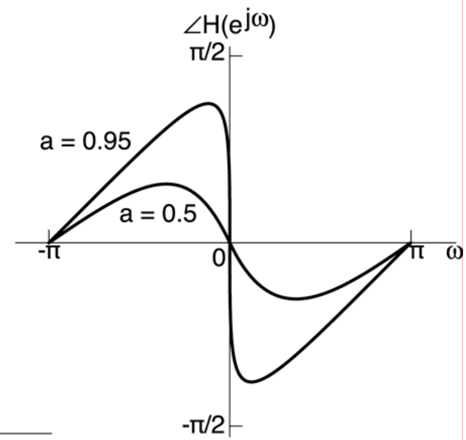
First-Order System
— one *real* pole

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad , \quad |z| > |a|$$

$$h[n] = a^n u[n] \quad , \quad |a| < 1$$







$$H(e^{j\omega}) = \frac{v_1}{v_2}, \; |H(e^{j\omega})| = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}, \quad \angle H(e^{j\omega}) = \angle v_1 - \angle v_2 = \omega - \angle v_2$$

تبدیل Z

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سیستم‌های مرتبه–اول و مرتبه–دوم

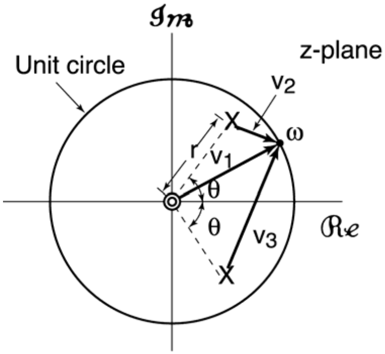
Second-Order System

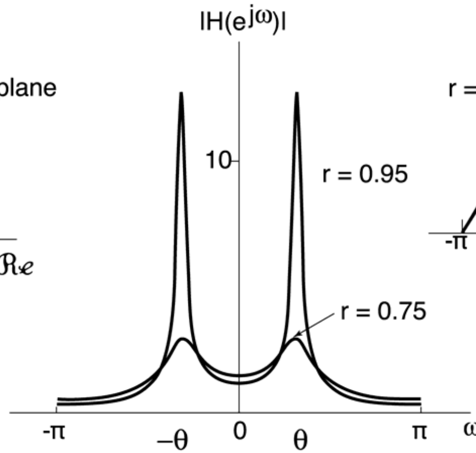
Two poles that are a complex conjugate pair ($z_1 = re^{j\theta} = z_2^*$)

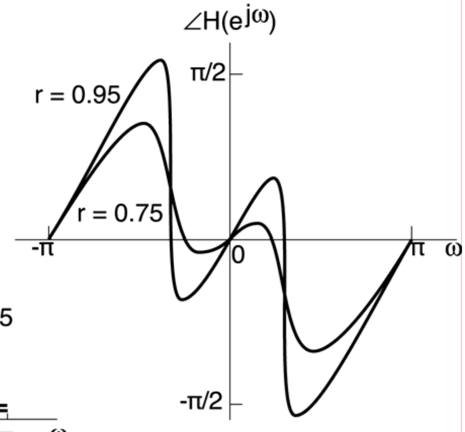
$$H(z) = \frac{z^2}{(z-z_1)(z-z_2)} = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}, \quad 0<r<1, \quad 0\le\theta\le\pi$$

$$|H(e^{j\omega})| = \frac{1}{|(e^{j\omega}-re^{j\theta})(e^{j\omega}-re^{-j\theta})|}, \quad h[n] = r^n\frac{\sin[(n+1)\theta]}{\sin\theta}u[n]$$

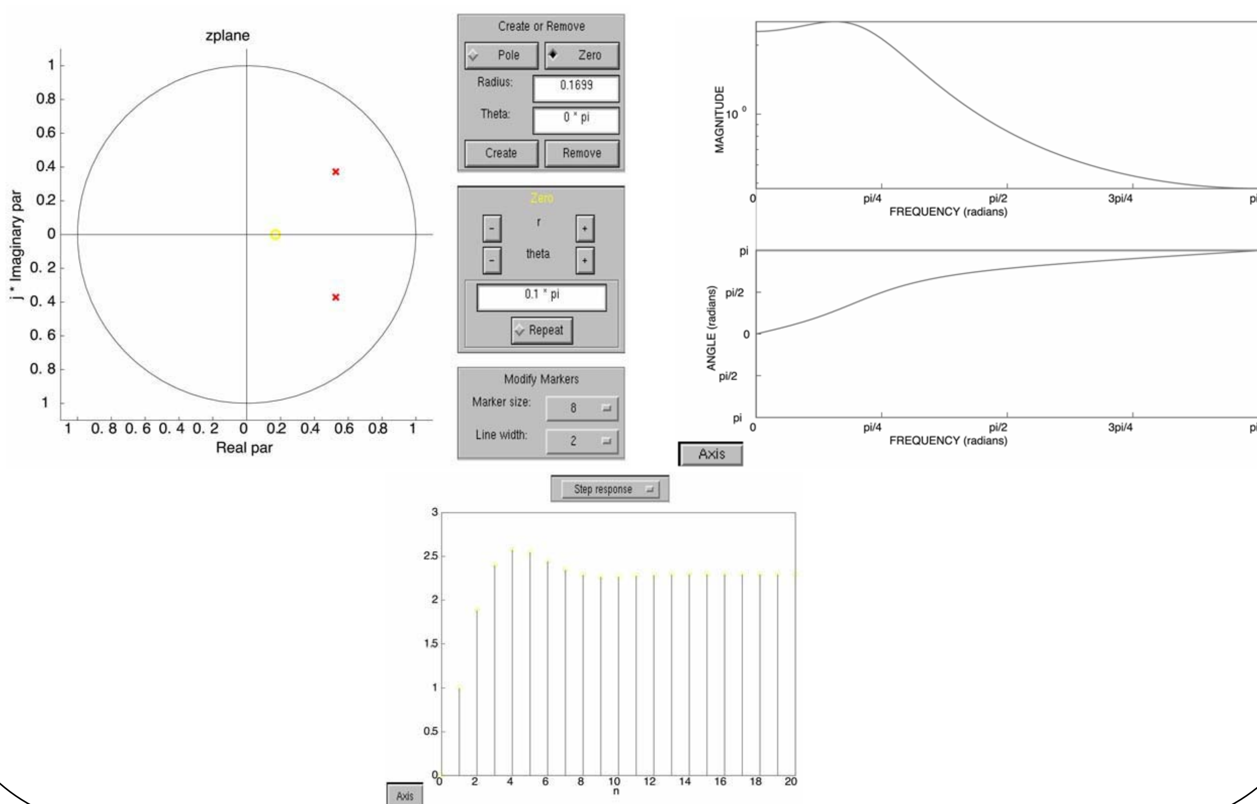
Clearly, $|H|$ peaks near $\omega = \pm\,\theta$







Demo: DT pole-zero diagrams, frequency response, vector diagrams, and impulse- & step-responses



DT LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Use the time-shift property

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Downarrow$$

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$ — Rational

ROC: Depends on Boundary Conditions, left-, right-, or two-sided.

For Causal Systems $\Rightarrow$ ROC is outside the outermost pole

تبدیل Z

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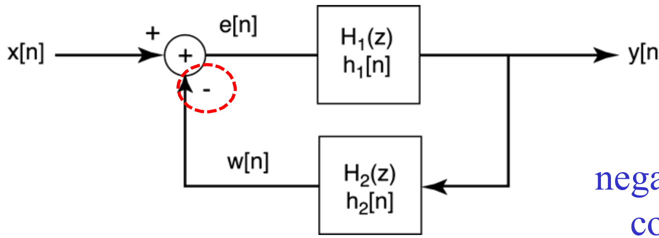
جبر تابع سیستم و نمودارهای بلوکی

System Function Algebra and Block Diagrams

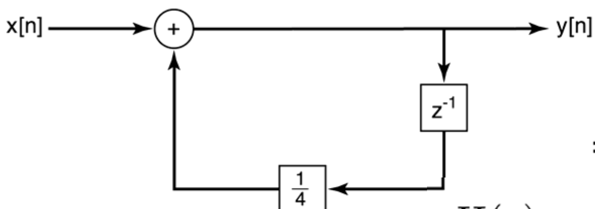
Feedback System
(causal systems)



negative feedback configuration

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

Example #1:



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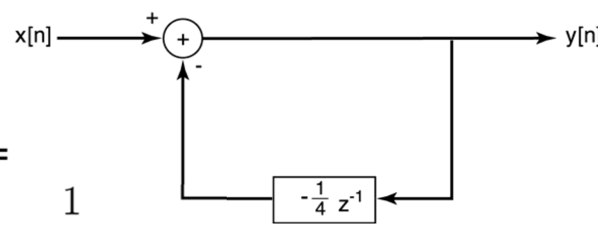


$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$\Downarrow$$

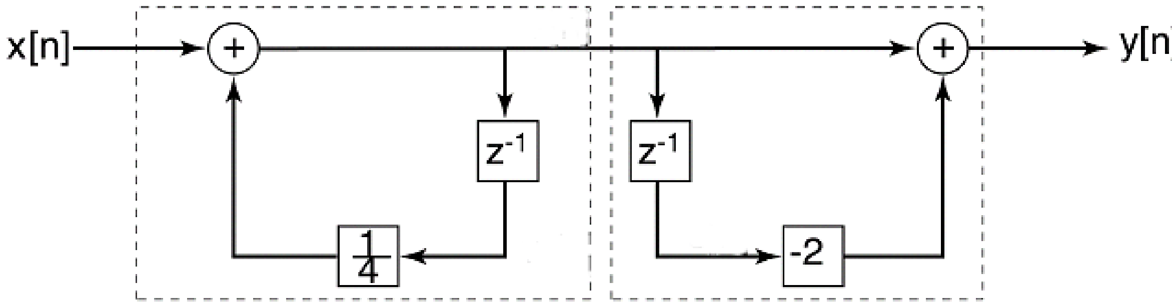
$$y[n] = \frac{1}{4}y[n-1] + x[n]$$

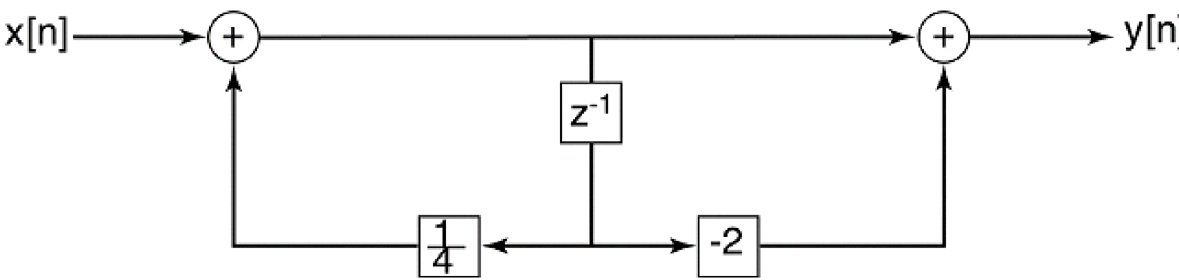
z^{-1} ⇔ D

Delay

Example #2:

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)(1 - 2z^{-1})$$ — Cascade of two systems





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تبدیل‌های z یک‌طرفه

Unilateral z-Transform

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Note:

(1) If $x[n] = 0$ for $n < 0$, then $\mathcal{X}(z) = X(z)$

(2) UZT of $x[n]$ = BZT of $x[n]u[n]$ $\Rightarrow$ ROC *always* outside a circle and *includes* $z = \infty$

(3) For causal LTI systems, $\mathcal{H}(z) = H(z)$

Properties of Unilateral z-Transform

Many properties are analogous to properties of the BZT e.g.

- Convolution property (for $x_1[n < 0] = x_2[n < 0] = 0$)

$$x_1[n] * x_2[n] \stackrel{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}_1(z)\mathcal{X}_2(z)$$

- But there are important differences. For example, *time-shift*

$$y[n] = x[n-1] \longleftrightarrow \mathcal{Y}(z) = x[-1] + z^{-1}\mathcal{X}(z)$$

Initial condition

Derivation:

$$\mathcal{Y}(z) = \sum_{n=0}^{\infty} y[n]z^{-n} = \sum_{n=0}^{\infty} x[n-1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$= x[-1] + z^{-1} \underbrace{\sum_{m=0}^{\infty} x[m]z^{-m}}_{\mathcal{X}(z)}$$

Use of UZTs in Solving Difference Equations with Initial Conditions

$$y[n] + 2y[n-1] = x[n]$$

$$y[-1] = \beta, \quad x[n] = \alpha u[n] \longleftrightarrow \frac{\alpha}{1 - z^{-1}}$$

UZT of Difference Equation

$$\mathcal{Y}(z) + 2\overbrace{\left[\beta + z^{-1}\mathcal{Y}(z)\right]}^{\mathcal{UZ}\{y[n-1]\}} = \frac{\alpha}{1 - z^{-1}}$$

$$\Downarrow$$

$$\mathcal{Y}(z) = -\underbrace{\frac{2\beta}{1 + 2z^{-1}}}_{ZIR} + \underbrace{\frac{\alpha}{(1 + 2z^{-1})(1 - z^{-1})}}_{ZSR}$$

ZIR — Output purely due to the initial conditions,

ZSR — Output purely due to the input.

Example (continued)

$\beta = 0$ $\Rightarrow$ System is initially at rest:

ZSR

$$\mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) = \underbrace{\frac{1}{1+2z^{-1}}}_{\mathcal{H}(z)}\underbrace{\frac{\alpha}{1-z^{-1}}}_{\mathcal{X}(z)}$$

$$\mathcal{H}(z) = H(z) = \frac{1}{1+2z^{-1}}$$

$\alpha = 0$ $\Rightarrow$ Get response to initial conditions

ZIR

$$\mathcal{Y}(z) = -\frac{2\beta}{1+2z^{-1}}$$

$$y[n] = -2\beta(-2)^n u[n]$$

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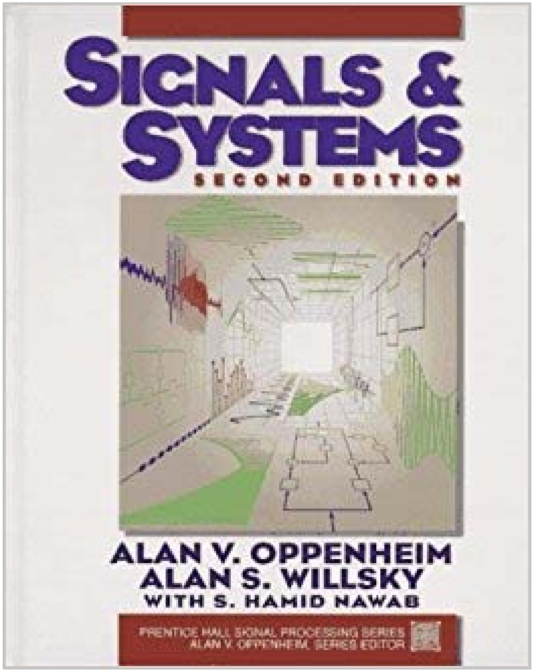
منابع

منبع اصلی



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Chapter 10