# سیگنال‌ها و سیستم‌ها

درس ۲۶

# تبدیل لاپلاس (۳)

## The Laplace Transform (3)

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# طرح درس

## COURSE OUTLINE

| |
|---|
| **خصوصیات تابع سیستم پیوسته–زمان** |
| CT System Function Properties |
| **جبر تابع سیستم و نمودارهای بلوکی** |
| System Function Algebra and Block Diagrams |
| **تبدیل لاپلاس یک‌طرفه و کاربردهای آن** |
| Unilateral Laplace Transform and Applications |

تبدیل لاپلاس (۳)

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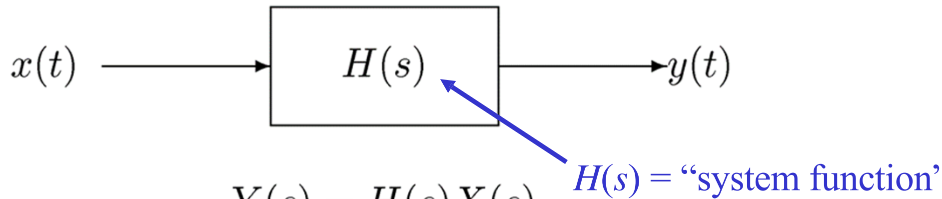
# خصوصیات تابع سیستم پیوسته-زمان

# CT System Function Properties




$$Y(s) = H(s)X(s)$$

1) System is stable $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \Leftrightarrow$ ROC of $H(s)$ includes $j\omega$ axis

2) Causality $\Rightarrow$ $h(t)$ right-sided signal $\Rightarrow$ ROC of $H(s)$ is a right-half plane

**Question:**

If the ROC of $H(s)$ is a right-half plane, is the system causal?

**Ex.**

$$\begin{aligned} H(s) &= \frac{e^{sT}}{s+1}, \quad \Re e\{s\} > -1 \Rightarrow h(t) \text{ right-sided} \\ h(t) &= \mathcal{L}^{-1}\left\{\frac{e^{sT}}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}_{t\to t+T} = e^{-t}u(t)|_{t\to t+T} \\ &= e^{-(t+T)}u(t+T) \neq 0 \quad \text{at} \quad t<0 \end{aligned}$$

Non-causal

# Properties of CT Rational System Functions

a) However, if $H(s)$ is ***rational,*** then

The system is causal $\Leftrightarrow$ The ROC of $H(s)$ is to the right of the rightmost pole

b) If $H(s)$ is rational and is the system function of a causal system, then

The system is stable $\Leftrightarrow$ $j\omega$-axis is in ROC

$\Leftrightarrow$ all poles are in LHP

## Checking if All Poles Are In the Left-Half Plane

$$H(s) = \frac{N(s)}{D(s)}$$

Poles are the roots of $D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

*Method #1:* Calculate all the roots and see!

*Method #2:* Routh-Hurwitz – Without having to solve for roots.

| | Polynomial | Condition so that all roots are in the LHP |
|---|---|---|
| First-order | $s + a_0$ | $a_0 > 0$ |
| Second-order | $s^2 + a_1 s + a_0$ | $a_1 > 0, a_0 > 0$ |
| Third-order | $s^3 + a_2 s^2 + a_1 s + a_0$ | $a_2 > 0, a_1 > 0, a_0 > 0$ <u>and</u> $a_0 < a_1 a_2$ |
| | $\vdots$ | $\vdots$ |

## Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and there are no impulses or higher order discontinuities at the origin, then

$$\boxed{x(0^+) = \lim_{s\to\infty} sX(s)}$$

**Initial value**

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\boxed{x(\infty) = \lim_{s\to 0} sX(s)}$$

**Final value**

# Applications of the Initial- and Final-Value Theorem

For $$X(s) = \frac{N(s)}{D(s)}$$

$n$ - order of polynomial $N(s)$, $d$ - order of polynomial $D(s)$

- Initial value:

$$x(0^+) = \lim_{s \to \infty} sX(s) = \begin{cases} 0 & d > n+1 \\ \text{finite } \neq 0 & d = n+1 \\ \infty & d < n+1 \end{cases}$$

$$\text{E.g.} \quad X(s) = \frac{1}{s+1} \qquad x(0^+) = ?$$

- Final value

$$\text{If } x(\infty) = \lim_{s \to 0} sX(s) = 0 \Rightarrow \lim_{s \to 0} X(s) < \infty$$

$$\Rightarrow \text{No poles at } s = 0$$

# LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Repeated use of differentiation property: $\dfrac{d}{dt} \leftrightarrow s, \quad \dfrac{d^k}{dt^k} \leftrightarrow s^k$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Downarrow$$

$$Y(s) = H(s)X(s)$$

where $H(s) = \underbrace{\dfrac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}_{\text{Rational}}$ $\longleftarrow$ roots of numerator $\Rightarrow$ *zeros*
$\longleftarrow$ roots of denominator $\Rightarrow$ *poles*

ROC =? Depends on: 1) Locations of *all* poles.
2) Boundary conditions, *i.e.* right-, left-, two-sided signals.

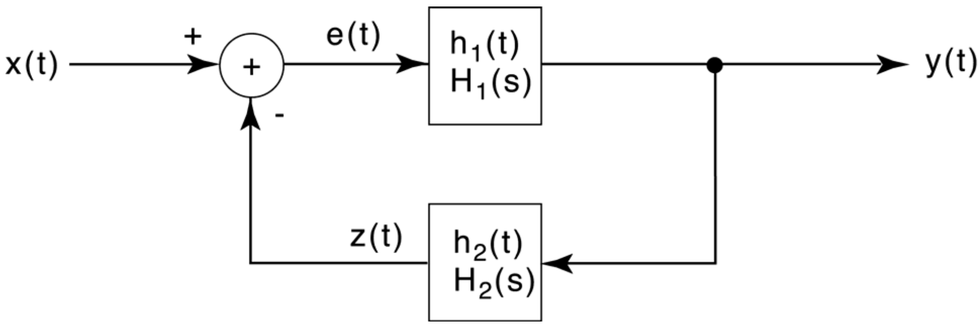
تبدیل لاپلاس (۳)

# ۲

# جبر
# تابع سیستم
# و
# نمودارهای
# بلوکی

# System Function Algebra

**Example:** A basic feedback system consisting of ***causal*** blocks




$$E(s) = X(s) - Z(s) = X(s) - H_2(s)Y(s)$$
$$Y(s) = H_1(s)E(s) = H_1(s)[X(s) - H_2(s)Y(s)]$$

$$\Downarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

More on this later in feedback

ROC: Determined by the roots of $1+H_1(s)H_2(s)$, instead of $H_1(s)$

# Block Diagram for Causal LTI Systems with Rational System Functions

**Example:**

$$Y(s) = H(s)X(s)$$

$$H(s) = \frac{2s^2+4s-6}{s^2+3s+2} = \left(\frac{1}{s^2+3s+2}\right)(2s^2+4s-6)$$

— Can be viewed as cascade of two systems.

Define:

$$W(s) = \frac{1}{s^2+3s+2}X(s)$$

$$\frac{d^2w(t)}{dt^2} + 3\frac{dw(t)}{dt} + 2w(t) = x(t), \qquad \text{initially at rest}$$

$$\text{or} \qquad \frac{d^2w(t)}{dt^2} = x(t) - 3\frac{dw(t)}{dt} - 2w(t)$$

Similarly

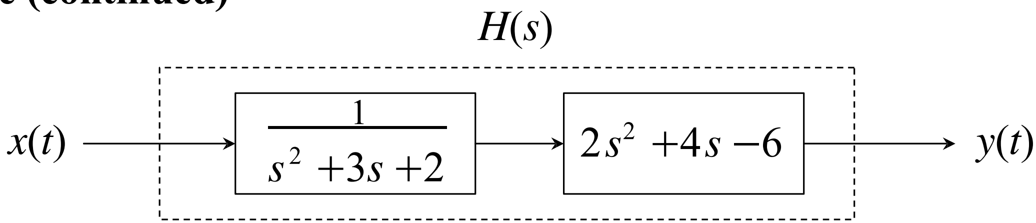
$$Y(s) = (2s^2+4s-6)W(s)$$

$$\Downarrow$$

$$y(t) = 2\frac{d^2w(t)}{dt^2} + 4\frac{dw(t)}{dt} - 6w(t)$$

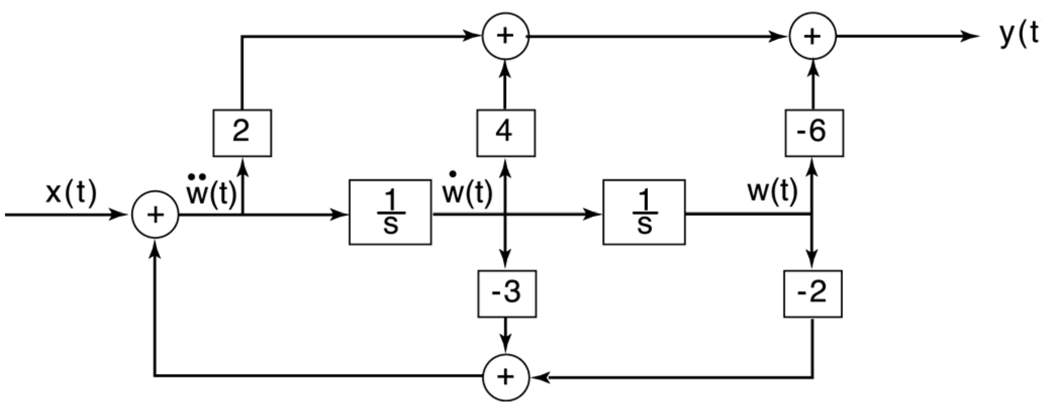
**Example (continued)**

Instead of




We can construct $H(s)$ using:

$$
\begin{aligned}
\frac{d^2w(t)}{dt^2} &= x(t) - 3\frac{dw(t)}{dt} - 2w(t) \\
y(t) &= 2\frac{d^2w(t)}{dt^2} + 4\frac{dw(t)}{dt} - 6w(t)
\end{aligned}
$$

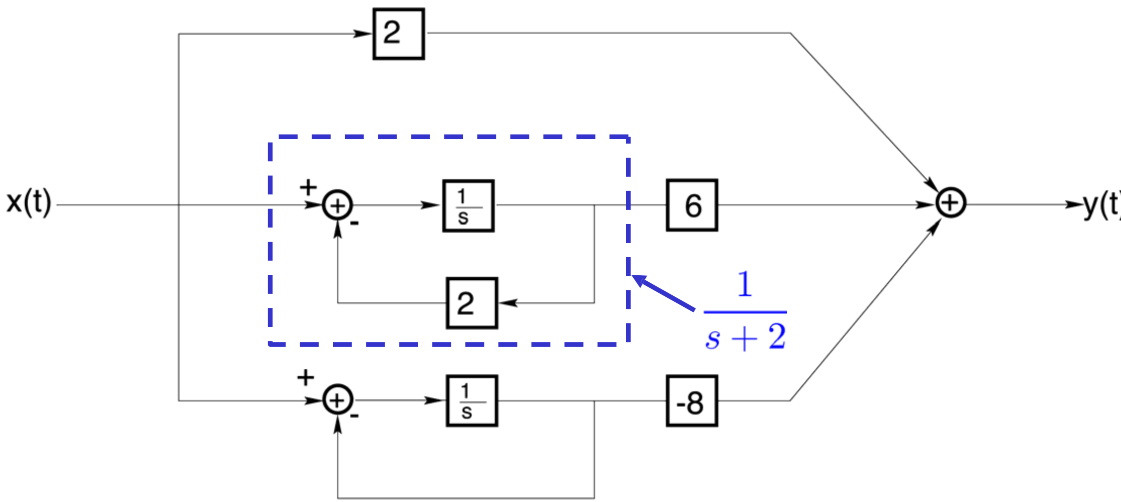



Notation: $1/s$ — an integrator

**Note also that**

$$H(s) = \left[\frac{2(s-1)}{s+2}\right]\left[\frac{s+3}{s+1}\right] = \left[\frac{s+3}{s+2}\right]\left[\frac{2(s-1)}{s+1}\right] \quad - \text{ Cascade}$$

$$\stackrel{PFE}{=} \quad 2 + \frac{6}{s+2} - \frac{8}{s+1} \quad - \text{ parallel connection}$$




Lesson to be learned: There are many *different* ways to construct a system that performs a certain function.

# ۳

# تبدیل لاپلاس یک‌طرفه و کاربردهای آن

# The Unilateral Laplace Transform

(The preferred tool to analyze causal CT systems described by LCCDEs with **initial conditions**)

$$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt = \mathcal{UL}\{x(t)\}$$

Note:

1) If $x(t) = 0$ for $t < 0$, then $X(s) = \mathcal{X}(s)$

2) Unilateral LT of $x(t)$ = Bilateral LT of $x(t)u(t^-)$

3) For example, if $h(t)$ is the impulse response of a causal LTI system, then

$$H(s) = \mathcal{H}(s)$$

4) Convolution property: If $x_1(t) = x_2(t) = 0$ for $t < 0$, then

$$\boxed{\mathcal{UL}\{x_1(t) * x_2(t)\} = \mathcal{X}_1(x)\mathcal{X}_2(s)}$$

Same as Bilateral Laplace transform

## Differentiation Property for Unilateral Laplace Transform

$$x(t) \longleftrightarrow \mathcal{X}(s)$$

$$\Downarrow$$

$$\boxed{\frac{dx(t)}{dt} \longleftrightarrow s\mathcal{X}(s) - x(0^-)}$$

Initial condition!

Derivation:

integration by parts

$\int f \cdot dg = fg - \int g \cdot df$

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = s \underbrace{\int_{0^-}^{\infty} x(t) e^{-st} dt}_{\mathcal{X}(s)} + x(t)e^{-st}\Big|_{0-}^{\infty}$$

$$= s\mathcal{X}(s) - x(0^-)$$

Note:

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left\{\frac{dx(t)}{dt}\right\} \longleftrightarrow s\overbrace{(s\mathcal{X}(s) - x(0^-))}^{\mathcal{UL}\frac{dx(t)}{dt}} - x'(0^-)$$

$$\longleftrightarrow s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

# Use of ULTs to Solve Differentiation Equations with Initial Conditions

**Example:**

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$y(0^-) = \beta\,,\; y'(0^-) = \gamma\,,\; x(t) = \alpha u(t)$$

Take ULT:

$$\underbrace{s^2\mathcal{Y}(s) - \beta s - \gamma}_{\mathcal{UL}\left\{\frac{d^2y}{dt^2}\right\}} + 3\underbrace{\left(\mathcal{Y}(s) - \beta\right)}_{\mathcal{UL}\left\{\frac{dy}{dt}\right\}} + 2\mathcal{Y}(s) = \frac{\alpha}{s}$$

$$\Downarrow$$

$$\mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{ZIR} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{ZSR}$$

***ZIR*** — Response for zero input $x(t)=0$

***ZSR*** — Response for zero state, $\beta=\gamma=0$, initially at rest

**Example (continued)**

- Response for LTI system initially at rest ($\beta = \gamma = 0$)

$$\Downarrow$$

$$\mathcal{H}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)} = \frac{1}{(s+1)(s+2)} = H(s)$$

- Response to initial conditions alone ($\alpha = 0$).
  For example:

$$x(t) = 0\,(\text{no input}), \quad y(0^-) = 1, \quad y'(0^-) = 0 \quad (\beta = 1, \gamma = 0)$$

$$\Downarrow$$

$$\mathcal{Y}(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\Downarrow$$

$$y(t) = 2e^{-t} - e^{-2t}, \quad t \geq 0$$

# حل معادلات دیفرانسیل با تبدیل لاپلاس

مثال

Start with differential equation:

$$\dot{y}(t) + y(t) = \delta(t)$$

Take the Laplace transform of this equation:

$$sY(s) + Y(s) = 1$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s+1}$$

Take inverse Laplace transform (by recognizing form of transform):

$$y(t) = e^{-t}u(t)$$

## حل معادلات دیفرانسیل با تبدیل لاپلاس

مثال

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \delta(t)$$

Laplace transform:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1$$

Solve:

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Inverse Laplace transform:

$$y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$

These forward and inverse Laplace transforms are easy if

- differential equation is linear with constant coefficients, and
- the input signal is an impulse function.

تبدیل لاپلاس (۳)

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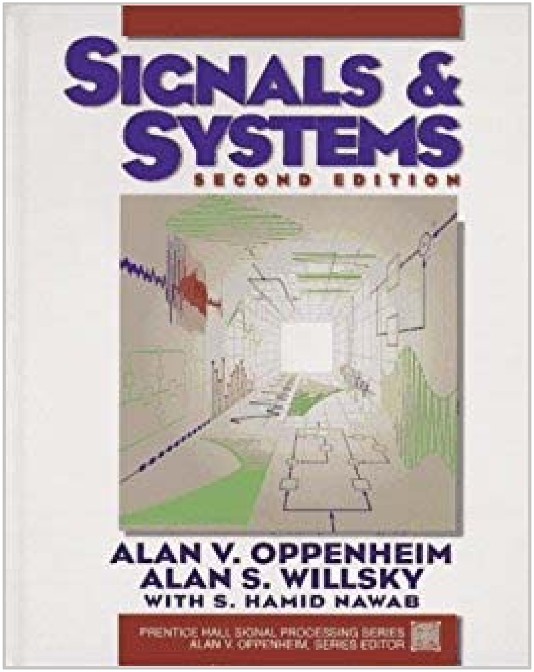
# منابع

# منبع اصلی

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**Chapter 9**