سیگنال‌ها و سیستم‌ها

درس ۲۵

تبدیل لاپلاس (۲)

The Laplace Transform (2)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

تبدیل لاپلاس معکوس
Inverse Laplace Transforms
خصوصیات تبدیل لاپلاس
Laplace Transform Properties
تابع سیستم برای یک سیستم خطی تغییرناپذیر با زمان
The System Function of an LTI System
ارزیابی هندسی تبدیل‌های لاپلاس و پاسخ‌های فرکانسی
Geometric Evaluation of Laplace Transforms and Frequency Responses

سیگنال‌ها و سیستم‌ها

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تبدیل لاپلاس معکوس

Inverse Laplace Transform

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega \in \text{ ROC} \\ &= \mathcal{F}\{x(t)e^{-\sigma t}\} \end{aligned}$$

Fix $\sigma \in$ ROC and apply the inverse Fourier transform

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

But $s = \sigma + j\omega$ (σ fixed) $\Rightarrow ds = j d\omega$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}ds$$

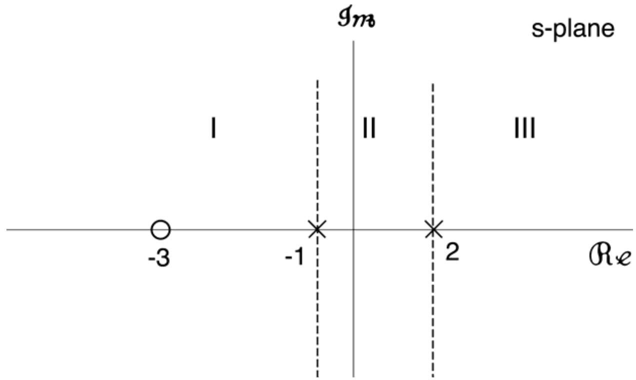
Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

Example:

$$X(s) = \frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = -\frac{2}{3}, \quad B = \frac{5}{3}$$

Three possible ROC's — corresponding to three *different* signals



Recall
$$\frac{1}{s+a}, \quad \Re e\{s\} < -a \longleftrightarrow -e^{-at}u(-t) \text{ left-sided}$$
$$\frac{1}{s+a}, \quad \Re e\{s\} > -a \longleftrightarrow e^{-at}u(t) \text{ right-sided}$$

ROC I: — Left-sided signal.

$$\begin{aligned} x(t) &= -Ae^{-t}u(-t) - Be^{2t}u(-t) \\ &= \left[\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t}\right]u(-t) \qquad \text{Diverges as } t \to -\infty \end{aligned}$$

ROC II: — Two-sided signal, has Fourier Transform.

$$\begin{aligned} x(t) &= Ae^{-t}u(t) - Be^{2t}u(-t) \\ &= -\left[\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t)\right] \quad \to 0 \text{ as } t \to \pm\infty \end{aligned}$$

ROC III:— Right-sided signal.

$$\begin{aligned} x(t) &= Ae^{-t}u(t) + Be^{2t}u(t) \\ &= \left[-\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t}\right]u(t) \qquad \text{Diverges as } t \to +\infty \end{aligned}$$

تبدیل لاپلاس معکوس

Expand with partial fractions:

$$\frac{2s}{s^2-4} = \underbrace{\frac{1}{s+2}}_{\text{pole at } -2} + \underbrace{\frac{1}{s-2}}_{\text{pole at } 2}$$

pole	function	right-sided; ROC	left-sided (ROC)
-2	e^{-2t}	$e^{-2t}u(t)$; $\mathrm{Re}(s) > -2$	$-e^{-2t}u(-t)$; $\mathrm{Re}(s) < -2$
2	e^{2t}	$e^{2t}u(t)$; $\mathrm{Re}(s) > 2$	$-e^{2t}u(-t)$; $\mathrm{Re}(s) < 2$

1.	$e^{-2t}u(t) + e^{2t}u(t)$	$\mathrm{Re}(s) > -2 \cap \mathrm{Re}(s) > 2$	$\mathrm{Re}(s) > 2$
2.	$e^{-2t}u(t) - e^{2t}u(-t)$	$\mathrm{Re}(s) > -2 \cap \mathrm{Re}(s) < 2$	$-2 < \mathrm{Re}(s) < 2$
3.	$-e^{-2t}u(-t) + e^{2t}u(t)$	$\mathrm{Re}(s) < -2 \cap \mathrm{Re}(s) > 2$	none
4.	$-e^{-2t}u(-t) - e^{2t}u(-t)$	$\mathrm{Re}(s) < -2 \cap \mathrm{Re}(s) < 2$	$\mathrm{Re}(s) < -2$

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خصوصیات تبدیل لاپلاس

Properties of Laplace Transforms

- Many parallel properties of the CTFT, but for Laplace transforms we need to determine implications for the ROC
- For example:

Linearity

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$$

ROC at least the intersection of ROCs of $X_1(s)$ and $X_2(s)$

ROC can be bigger (due to pole-zero cancellation)

E.g. $\quad x_1(t) = x_2(t)$ and $a = -b$

Then $\quad ax_1(t) + bx_2(t) = 0 \longrightarrow X(s) = 0$

$\Rightarrow$ROC entire $s$plane-

Time Shift

$$x(t-T) \longleftrightarrow e^{-sT}X(s), \text{ same ROC as } X(s)$$

Example:

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad ?$$

$$\frac{e^{-sT}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2t}u(t)|_{t \to t-T}$$

$$\downarrow T = -3$$

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2(t+3)}u(t+3)$$

Time-Domain Differentiation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}ds, \quad \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} sX(s)e^{st}ds$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s), \text{ with ROC containing the ROC of } X(s)$$

ROC could be bigger than the ROC of $X(s)$, if there is pole-zero cancellation. E.g.,

$$\begin{array}{rcllr} x(t) & = & u(t) & \leftrightarrow & \frac{1}{s}, & \Re e\{s\} > 0 \\ \frac{dx(t)}{dt} & = & \delta(t) & \leftrightarrow & 1 = s \cdot \frac{1}{s} & \text{ROC} = \text{entire s-plane} \end{array}$$

s-Domain Differentiation

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds}, \text{ with same ROC as } X(s)$$

(Derivation is similar to $\frac{d}{dt} \leftrightarrow s$)

E.g., $te^{-at}u(t) \leftrightarrow -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \quad \Re e\{s\} > -a$

تبدیل لاپلاس مشتق

Laplace Transform of a Derivative

Assume that $X(s)$ is the Laplace transform of $x(t)$:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Find the Laplace transform of $y(t) = \dot{x}(t)$.

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt = \int_{-\infty}^{\infty} \underbrace{\dot{x}(t)}_{\dot{v}}\underbrace{e^{-st}}_{u}dt$$

$$= \underbrace{x(t)}_{v}\underbrace{e^{-st}}_{u}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{x(t)}_{v}(\underbrace{-se^{-st}}_{\dot{u}})dt$$

The first term must be zero since $X(s)$ converged. Thus

$$Y(s) = s\int_{-\infty}^{\infty} x(t)e^{-st}dt = sX(s)$$

تبدیل لاپلاس تابع ضربه

Laplace Transform of the Impulse Function

Let $x(t) = \delta(t)$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t)\ e^{-st}\Big|_{t=0}\ dt \\ &= \int_{-\infty}^{\infty} \delta(t)\ 1\ dt \\ &= 1 \end{aligned}$$

Sifting property:

$\delta(t)$ **sifts** out the value of e^{-st} at $t = 0$.

Convolution Property

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

For $\quad x(t) \longleftrightarrow X(s), y(t) \longleftrightarrow Y(s), h(t) \longleftrightarrow H(s)$

Then $\quad Y(s) = H(s) \cdot X(s)$

- ROC of $Y(s) = H(s)X(s)$: at least the overlap of the ROCs of $H(s)$ & $X(s)$
- ROC could be empty if there is no overlap between the two ROCs

  E.g.

  $$x(t) = e^{t}u(t) \quad \text{, and} \quad h(t) = -e^{-t}u(-t)$$

- ROC could be larger than the overlap of the two.

  E.g.

  $$x(t) * h(t) = \delta(t)$$

خواص تبدیل لاپلاس

PROPERTIES OF LAPLACE TRANSFORMS

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t-T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-\frac{dX(s)}{ds}$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	shift R by $-\alpha$
Differentiate in t	$\frac{dx(t)}{dt}$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(s)}{s}$	$\supset \left(R \cap (\mathrm{Re}(s) > 0)\right)$
Convolve in t	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau$	$X_1(s)X_2(s)$	$\supset (R_1 \cap R_2)$

Prepared by Kazim Fouladi | Fall 2017 | 2nd Edition
کاظم فولادی

تبدیل لاپلاس (۲)

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تابع سیستم برای یک سیستم خطی تغییرناپذیر با زمان

The System Function of an LTI System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$h(t) \quad \longleftrightarrow \quad H(s) \text{ – the system function}$$

The system function characterizes the system

$$\Downarrow$$

System properties correspond to properties of $H(s)$ and its ROC

A first example:

$$\text{System is stable} \Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \quad \Leftrightarrow \quad \begin{array}{l}\text{ROC of } H(s) \\ \text{includes the } j\omega \text{ axis}\end{array}$$

تابع تبدیل سیستم

محاسبه‌ی تابع تبدیل سیستم به کمک تبدیل لاپلاس

Start with differential equation:

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

Take the Laplace transform of a each term:

$$2s^2Y(s) + 3sY(s) + Y(s) = 2X(s)$$

Solve for system function:

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

تابع تبدیل سیستم

محاسبه‌ی پاسخ ضربه‌ی سیستم به کمک تبدیل لاپلاس

Differential equation:

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

If $x(t) = \delta(t)$ then $y(t)$ is the impulse response $h(t)$.

If $X(s) = 1$ then $Y(s) = H(s)$.

تابع سیستم، تبدیل لاپلاس پاسخ ضربه است.

تبدیل لاپلاس (۲)

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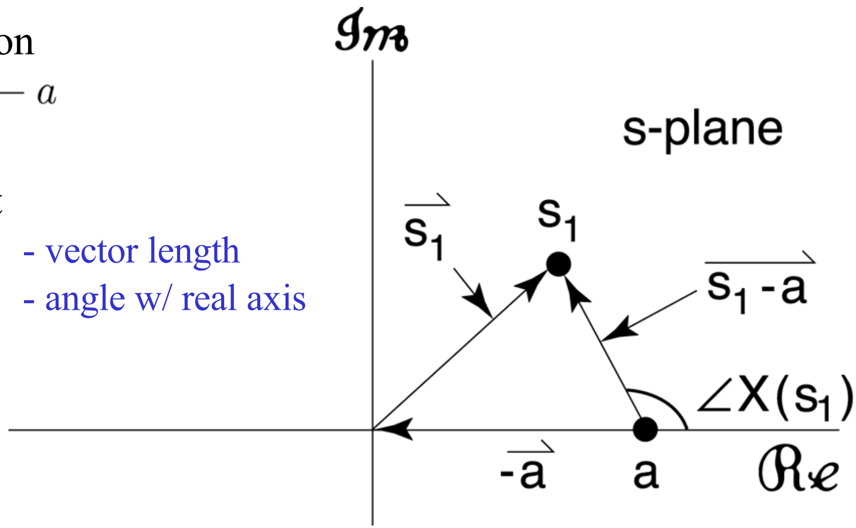
ارزیابی هندسی تبدیل‌های لاپلاس و پاسخ‌های فرکانسی

Geometric Evaluation of Rational Laplace Transforms

Example #1: $X_1(s) = s - a$ A first-order zero

Graphic evaluation
of $X_1(s) = s_1 - a$

Can reason about
$|X_1(s)|$ - vector length
$\angle X_1(s)$ - angle w/ real axis



Example #2: A first-order pole

$$X_2(s) = \frac{1}{s-a} = \frac{1}{X_1(s)}$$

$$\Rightarrow |X_2(s)| = \frac{1}{|X_1(s)|} \quad (\text{or } \log|X_2(s)| = -\log|X_1(s)|)$$

$$\angle X_2(s) = -\angle X_1(s)$$

Still reason with vector, but remember to "invert" for poles

Example #3: A higher-order rational Laplace transform

$$X(s) = M\frac{\Pi_{i=1}^{R}(s-\beta_i)}{\Pi_{j=1}^{P}(s-\alpha_j)}$$

$$|X(s)| = |M|\frac{\Pi_{i=1}^{R}|s-\beta_i|}{\Pi_{j=1}^{P}|s-\alpha_j|}$$

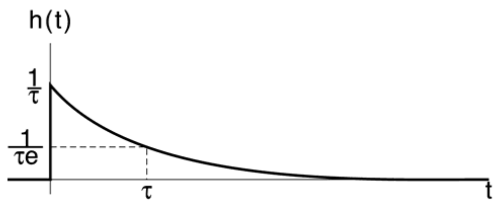
$$\angle X(s) = \angle M + \sum_{i=1}^{R}\angle(s-\beta_i) - \sum_{j=1}^{P}\angle(s-\alpha_j)$$

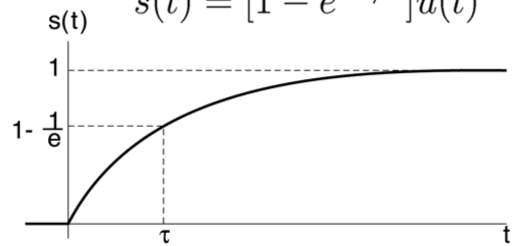
First-Order System

$$H(s) = \frac{1}{s\tau + 1} = \frac{1/\tau}{s + 1/\tau}, \Re e\{s\} > -\frac{1}{\tau}$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

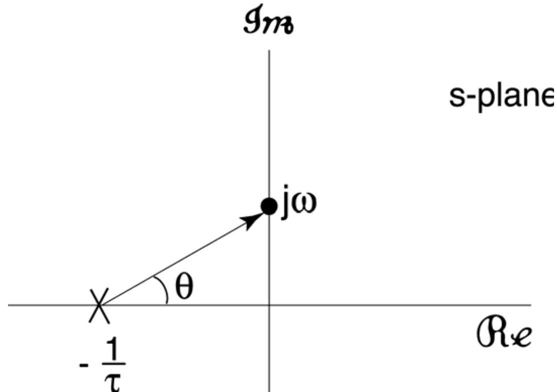
$$s(t) = [1 - e^{-t/\tau}] u(t)$$





Graphical evaluation of $H(j\omega)$:

$$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau} = \frac{1}{\tau} \cdot \frac{1}{j\omega + 1/\tau}$$



Bode Plot of the First-Order System

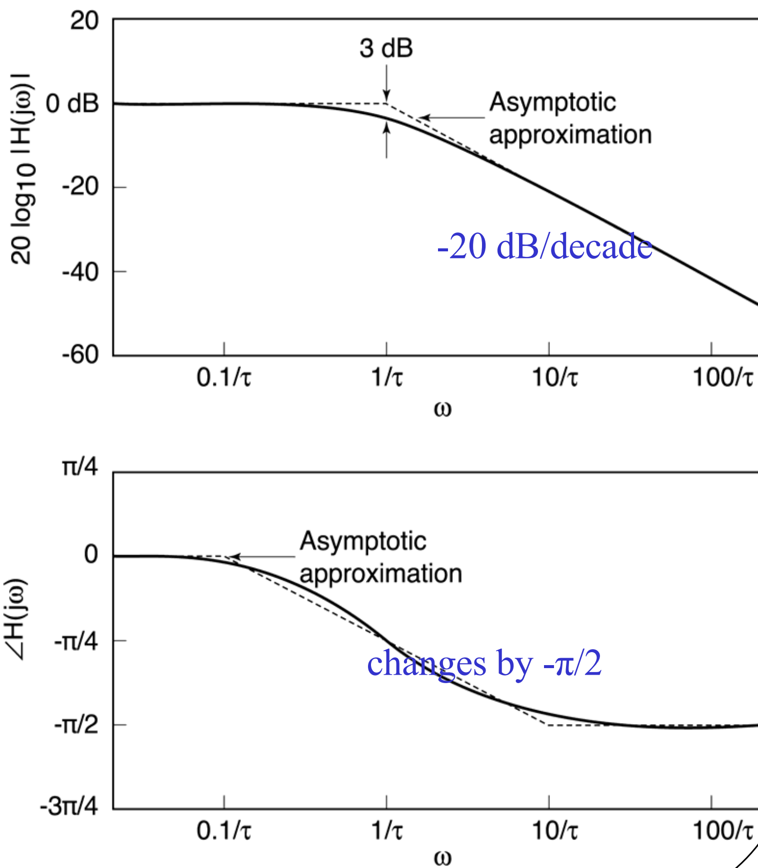
$$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$$

$$|H(j\omega)| = \frac{1/\tau}{\sqrt{\omega^2 + (1/\tau)^2}}$$

$$= \begin{cases} 1 & \omega = 0 \\ 1/\sqrt{2} & \omega = 1/\tau \\ 1/\omega\tau & \omega >> 1/\tau \end{cases}$$

$$\angle H(j\omega) = -\theta = -\tan^{-1}(\omega\tau)$$

$$= \begin{cases} 0 & \omega = 0 \\ -\pi/4 & \omega = 1/\tau \\ -\pi/2 & \omega >> 1/\tau \end{cases}$$



Second-Order System

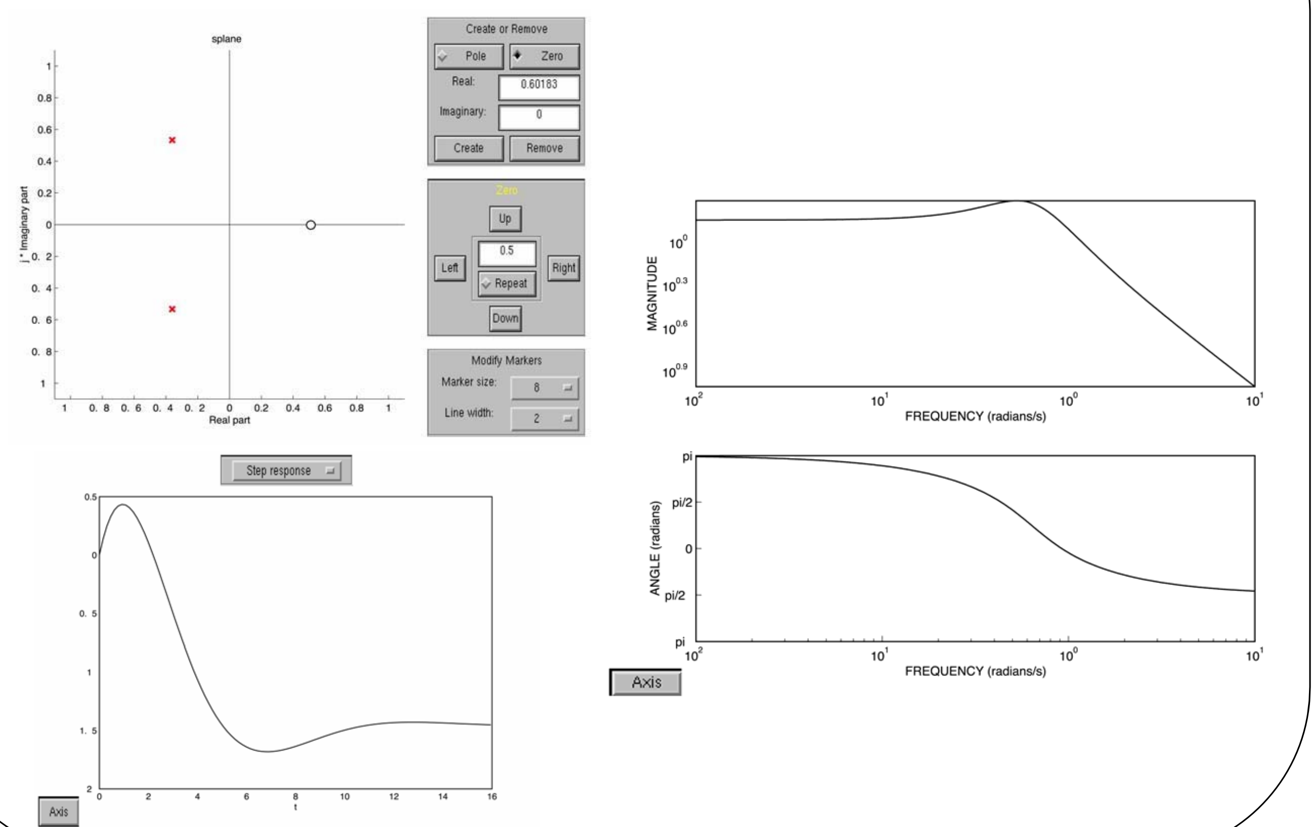
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \text{ROC } \Re e\{s\} > \Re e(\text{pole})$$

$0 < \zeta < 1$ $\Rightarrow$ complex poles
— *Under*damped

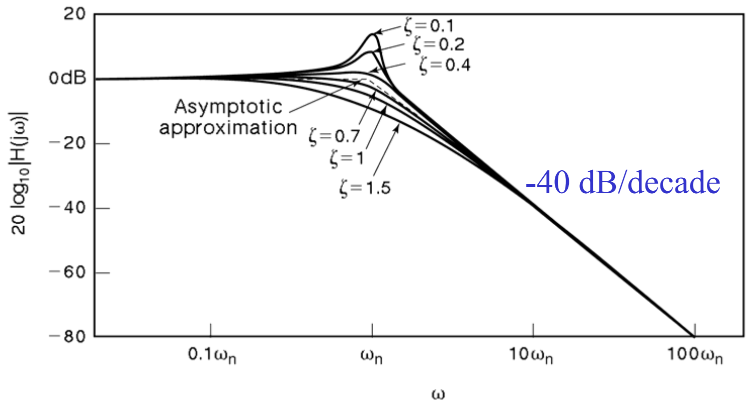
$\zeta = 1$ $\Rightarrow$ double pole at $s = -\omega_n$
— *Critically* damped

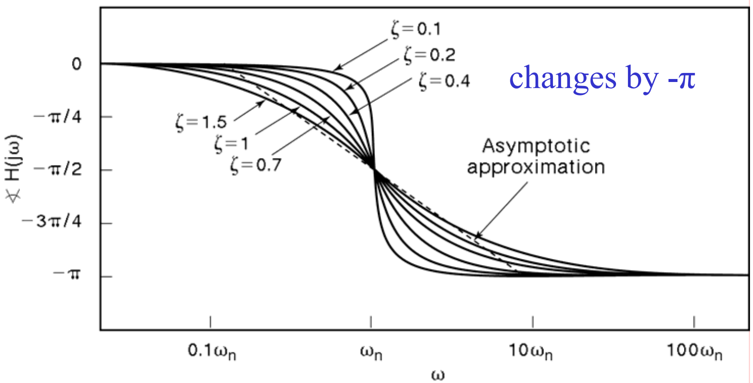
$\zeta > 1$ $\Rightarrow$ 2 poles on negative real axis
— *Over*damped

Demo Pole-zero diagrams, frequency response, and step response of first-order and second-order CT causal systems



Bode Plot of a Second-Order System

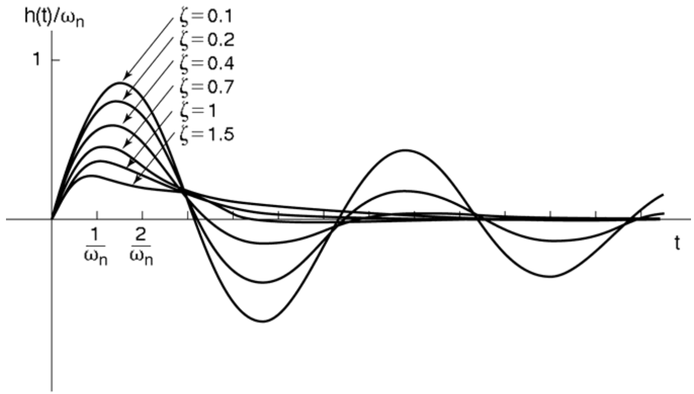




Top is flat when $\zeta = 1/\sqrt{2} = 0.707$ $\Rightarrow$ a LPF for

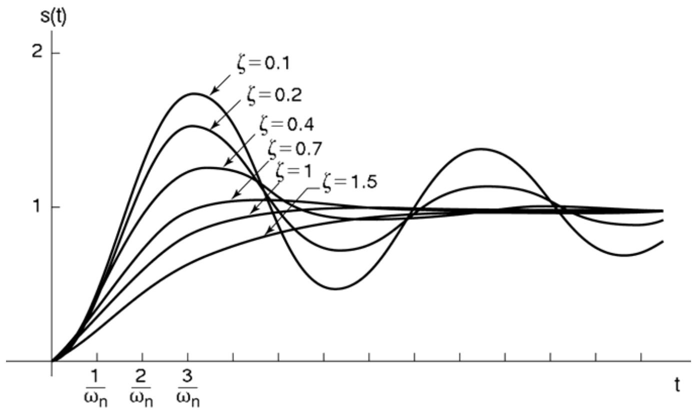
$\omega < \omega_n$

Unit-Impulse and Unit-Step Response of a Second-Order System



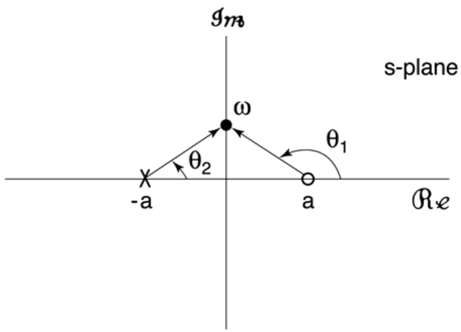
No oscillations when $\zeta \geq 1$

$\Rightarrow$ Critically (=) and over (>) damped.

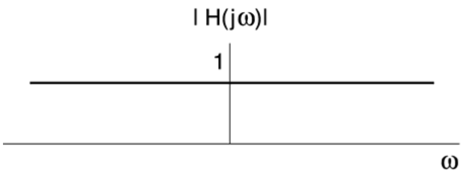


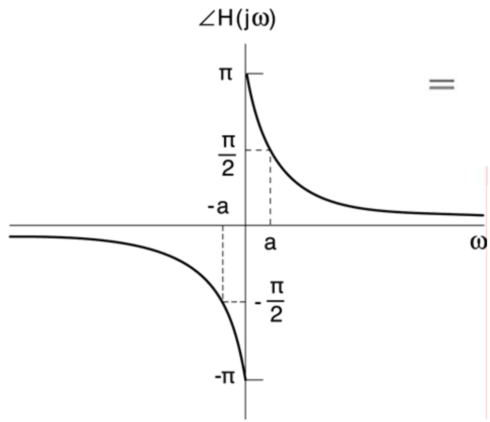
First-Order All-Pass System

$$H(s) = \frac{s-a}{s+a}, \quad \Re e\{s\} > -a \quad (a > 0)$$



1. Two vectors have the same lengths
2. $$\begin{aligned} \angle H(j\omega) &= \theta_1 - \theta_2 \\ &= (\pi - \theta_2) - \theta_2 \\ &= \pi - 2\theta_2 \\ &= \begin{cases} \pi & \omega = 0 \\ \pi/2 & \omega = a \\ \backsim 0 & \omega >> a \end{cases} \end{aligned}$$





تبدیل لاپلاس (۲)

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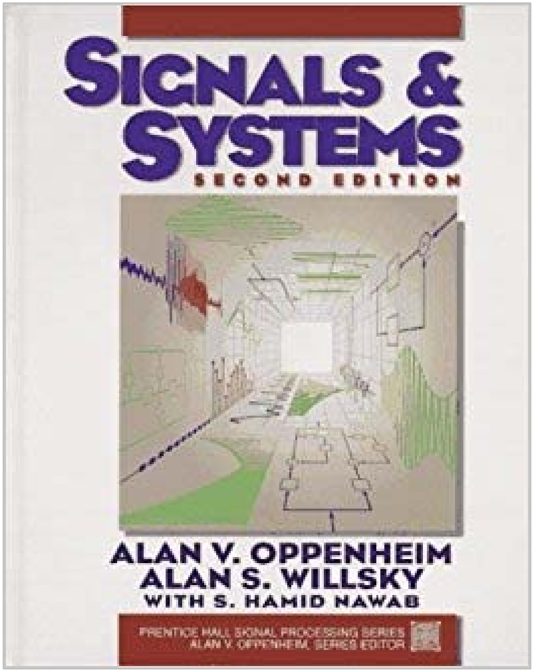
منابع

منبع اصلی



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Chapter 9