سیگنال‌ها و سیستم‌ها

درس ۱۹

مشخصه‌های زمانی و فرکانسی (۲)

Time and Frequency Characterization (2)

کاظم فولادی قلعه
دانشکده مهندسی، پردیس فارابی
دانشگاه تهران

http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

پاسخ‌های فرکانسی و نمودار بودی
Frequency Responses and Bode Diagram

سیگنال‌ها و سیستم‌ها

مشخصه‌های زمانی و فرکانسی (۲)

۱

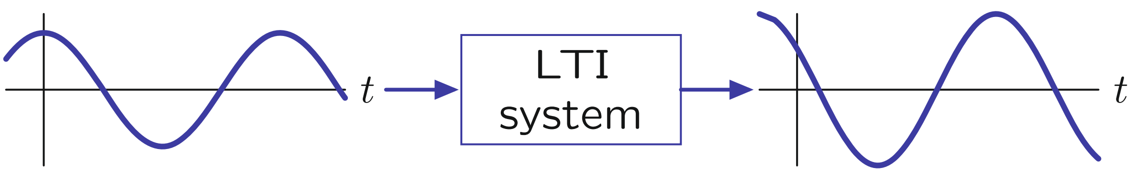
پاسخ‌های فرکانسی و نمودار بودی

پاسخ فرکانسی

Frequency Response

اگر ورودی به یک سیستم خطی تغییرناپذیر با زمان یک سینوسی دائمی باشد،
آن‌گاه خروجی آن نیز یک سینوسی دائمی است:
* با همان فرکانس،
* احتمالاً دامنه‌ی متفاوت، و
* احتمالاً زاویه فاز متفاوت.

$x(t) = \cos(\omega t)$ $\qquad$ $y(t) = M\cos(\omega t + \phi)$



پاسخ فرکانسی، ترسیم اندازه M و زاویه‌ی ϕ به‌صورت تابعی از فرکانس ω است.

Prepared by Kazim Fouladi | Fall 2017 | 2nd Edition
کاظم فولادی

پاسخ فرکانسی

روش محاسبه

FREQUENCY RESPONSE

Calculate the frequency response.

Methods

- solve differential equation
  → find particular solution for $x(t) = \cos \omega_0 t$
- find impulse response of system
  → convolve with $x(t) = \cos \omega_0 t$

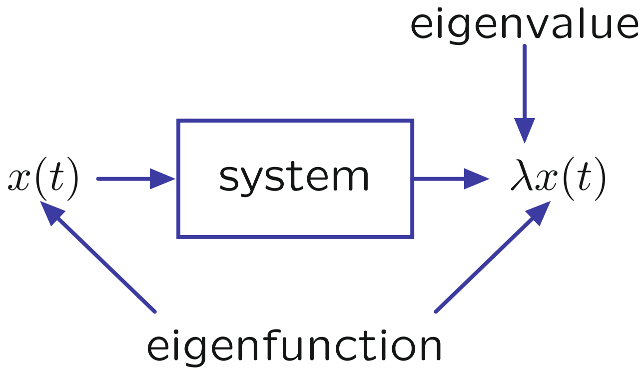
New method

- use eigenfunctions and eigenvalues

توابع ویژه

EIGENFUNCTIONS

اگر سیگنال خروجی یک ضریب اسکالر از سیگنال ورودی باشد،
به آن سیگنال **تابع ویژه** و به ضریب آن **مقدار ویژه** می‌گوییم.



توابع ویژه

مثال (۱ از ۳)

EIGENFUNCTIONS

Consider the system described by

$$\dot{y}(t) + 2y(t) = x(t).$$

Determine if each of the following functions is an eigen-function of this system. If it is, find its eigenvalue.

1. e^{-t} for all time
2. e^{t} for all time
3. e^{jt} for all time
4. $\cos(t)$ for all time
5. $u(t)$ for all time

توابع ویژه

مثال (۲ از ۳)

EIGENFUNCTIONS

$$\dot{y}(t) + 2y(t) = x(t)$$

1. $e^{-t}: \quad -\lambda e^{-t} + 2\lambda e^{-t} = e^{-t} \rightarrow \lambda = 1$

2. $e^{t}: \quad \lambda e^{t} + 2\lambda e^{t} = e^{t} \rightarrow \lambda = \frac{1}{3}$

3. $e^{jt}: \quad j\lambda e^{jt} + 2\lambda e^{jt} = e^{jt} \rightarrow \lambda = \frac{1}{j+2}$

4. $\cos t: \quad -\lambda \sin t + 2\lambda \cos t = \cos t \rightarrow$ not possible!

5. $u(t): \quad \lambda \delta(t) + 2\lambda u(t) = u(t) \rightarrow$ not possible!

توابع ویژه

مثال (۳ از ۳)

EIGENFUNCTIONS

Consider the system described by

$$\dot{y}(t) + 2y(t) = x(t).$$

Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

1. e^{-t} for all time $\surd$ $\lambda = 1$
2. e^{t} for all time $\surd$ $\lambda = \frac{1}{3}$
3. e^{jt} for all time $\surd$ $\lambda = \frac{1}{j+2}$
4. $\cos(t)$ for all time $\times$
5. $u(t)$ for all time $\times$

توابع ویژه

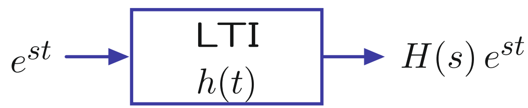
نمایی‌های مختلط

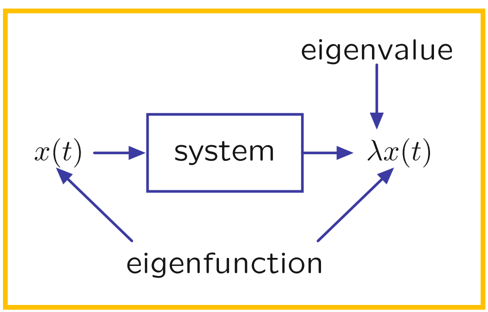
Complex Exponentials

نمایی‌های مختلط، توابع ویژه‌ی سیستم‌های LTI هستند.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s)\, e^{st}$$





توابع ویژه

نمایی‌های مختلط

COMPLEX EXPONENTIALS

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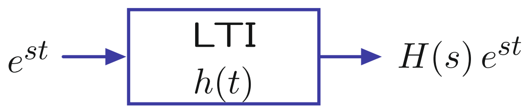


Eternal sinusoids are sums of complex exponentials.

$$\cos \omega_0 t = \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$

Furthermore, the eigenvalue associated with e^{st} is $H(s)$!

توابع ویژه

توابع سیستم گویا

COMPLEX EXPONENTIALS

ارزیابی مقادیر ویژه برای سیستم‌های بازنمایی شده با معادلات دیفرانسیل خطی با ضرایب ثابت ساده است:
در این حالت، تابع سیستم گویا (نسبت دو چندجمله‌ای برحسب s) است.

Example:

$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2\ddot{x}(t) + 7\dot{x}(t) + 8x(t)$$

Then

$$H(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 4} \equiv \frac{N(s)}{D(s)}$$

دیاگرام‌های برداری

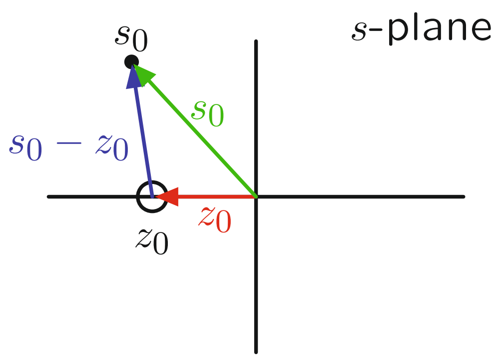
توابع سیستم گویا

VECTOR DIAGRAMS

مقدار $H(s)$ در نقطه‌ی $s = s_0$ می‌تواند به‌صورت گرافیکی با استفاده از تحلیل برداری تعیین شود:

صورت و مخرج تابع سیستم را تجزیه (فاکتوریابی) می‌کنیم تا قطب‌ها و صفرها واضح شوند.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$



هر فاکتور در صورت متناظر با یک بردار از یک صفر (z_0) به s_0 (نقطه‌ی مورد نظر) در صفحه‌ی S است.
هر فاکتور در مخرج متناظر با یک بردار از یک قطب (p_0) به s_0 (نقطه‌ی مورد نظر) در صفحه‌ی S است.

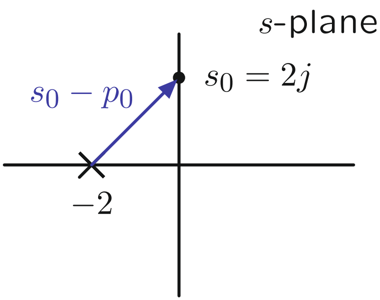
مثال

VECTOR DIAGRAMS

Example: Find the response of the system described by

$$H(s) = \frac{1}{s+2}$$

to the input $x(t) = e^{2jt}$ (for all time).



The denominator of $H(s)|_{s=2j}$ is $2j+2$, a vector with length $2\sqrt{2}$ and angle $\pi/4$. Therefore, the response of the system is

$$y(t) = H(2j)e^{2jt} = \frac{1}{2\sqrt{2}}e^{-\frac{j\pi}{4}}e^{2jt}.$$

دیاگرام‌های برداری

VECTOR DIAGRAMS

مقدار $H(s)$ در نقطه‌ی $s = s_0$ می‌تواند با ترکیب اثرگذاری بردارهای متناظر با قطب‌ها و صفرها تعیین شود.

$$H(s_0) = K\frac{(s_0-z_0)(s_0-z_1)(s_0-z_2)\cdots}{(s_0-p_0)(s_0-p_1)(s_0-p_2)\cdots}$$

اندازه توسط ضرب اندازه‌ها تعیین می‌شود:

$$|H(s_0)| = |K|\frac{|(s_0-z_0)||(s_0-z_1)||(s_0-z_2)|\cdots}{|(s_0-p_0)||(s_0-p_1)||(s_0-p_2)|\cdots}$$

زاویه توسط جمع زاویه‌ها تعیین می‌شود:

$$\angle H(s_0) = \angle K + \angle(s_0-z_0) + \angle(s_0-z_1) + \cdots - \angle(s_0-p_0) - \angle(s_0-p_1) - \cdots$$

تقارن مزدوج

CONJUGATE SYMMETRY

The complex conjugate of $H(j\omega)$ is $H(-j\omega)$.

The system function is the Laplace transform of the impulse response:

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where $h(t)$ is a real-valued function of t for physical systems.

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

$$H(-j\omega) = \int_{-\infty}^{\infty} h(t)e^{j\omega t}dt \equiv \big(H(j\omega)\big)^*$$

پاسخ فرکانسی

FREQUENCY RESPONSE

Response to eternal sinusoids.

Let $x(t) = \cos\omega_0 t$ (for all time), which can be written as

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

The response to a sum is the sum of the responses,

$$\begin{aligned}
y(t) &= \frac{1}{2}\left(H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right) \\
&= \mathrm{Re}\left\{H(j\omega_0)e^{j\omega_0 t}\right\} \\
&= \mathrm{Re}\left\{|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t}\right\} \\
&= |H(j\omega_0)|\mathrm{Re}\left\{e^{j\omega_0 t + j\angle H(j\omega_0)}\right\}
\end{aligned}$$

$$y(t) = |H(j\omega_0)| \cos\left(\omega_0 t + \angle\left(H(j\omega_0)\right)\right).$$

پاسخ فرکانسی

Frequency Response

اندازه و فاز پاسخ یک سیستم به یک سیگنال کسینوسی دائمی،
برابر است با اندازه و فاز تابع سیستم که در $s = j\omega$ ارزیابی شود.

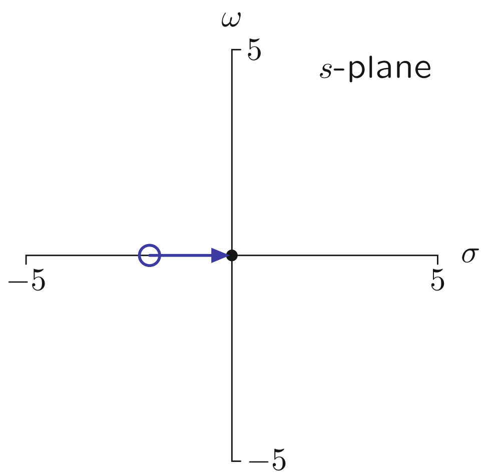
$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)|\cos\left(\omega t + \angle H(j\omega)\right)$$

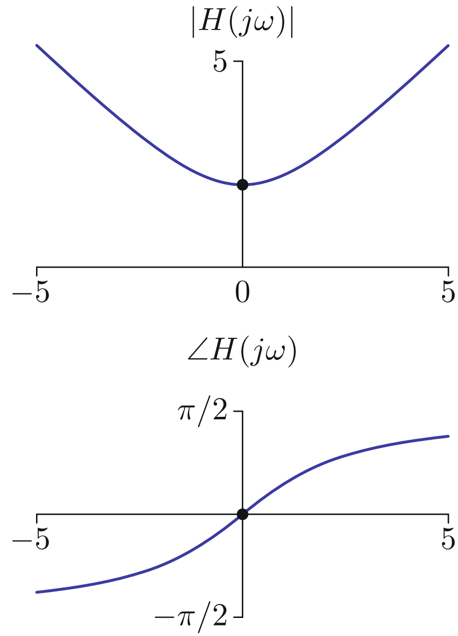
پاسخ فرکانسی

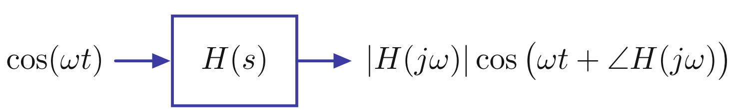
دیاگرام‌های برداری

Vector Diagrams

$$H(s) = s - z_1$$







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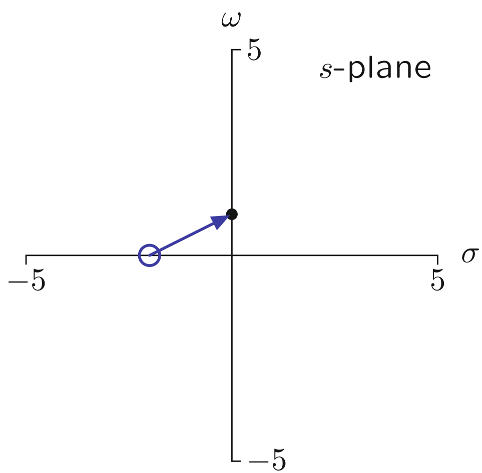

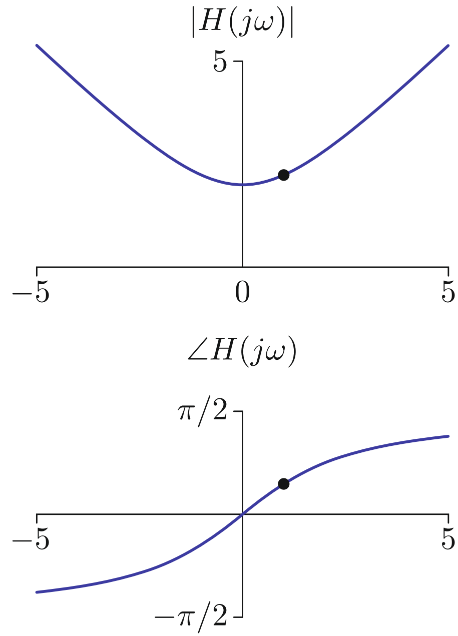

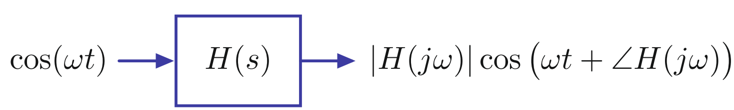

پاسخ فرکانسی

دیاگرام‌های برداری

VECTOR DIAGRAMS

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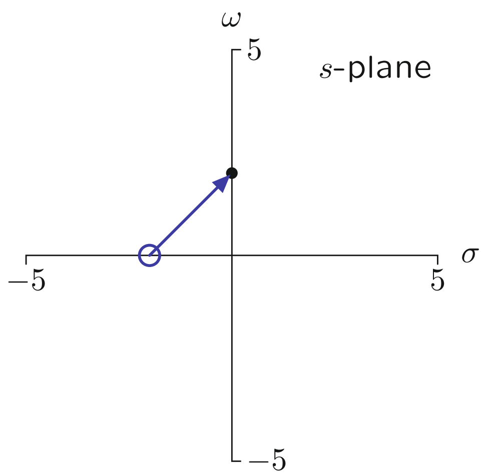

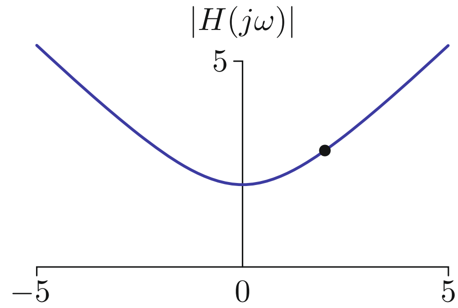

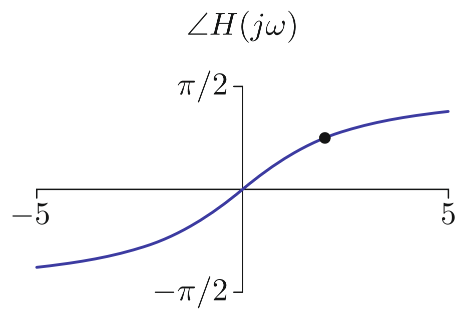

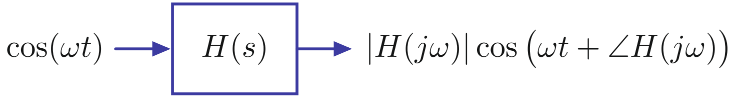

$$\cos(\omega t) \rightarrow \boxed{H(s)} \rightarrow |H(j\omega)| \cos\big(\omega t + \angle H(j\omega)\big)$$

پاسخ فرکانسی

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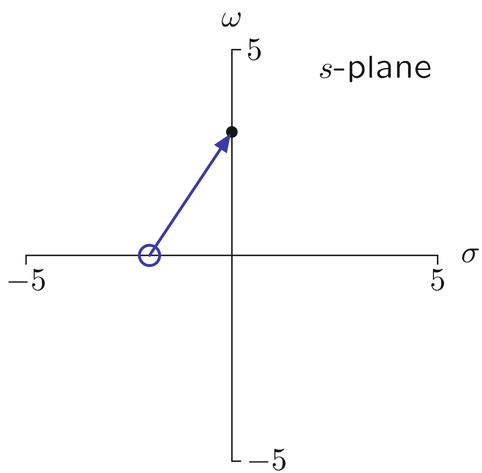

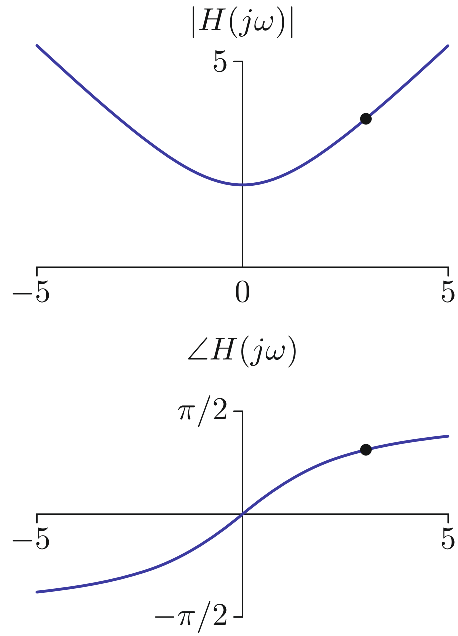

$$\cos(\omega t) \rightarrow \boxed{H(s)} \rightarrow |H(j\omega)| \cos\big(\omega t + \angle H(j\omega)\big)$$

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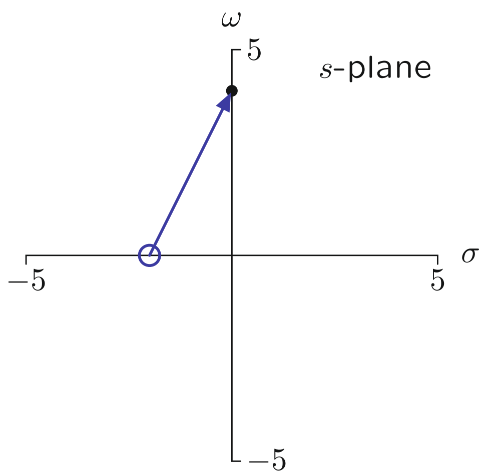

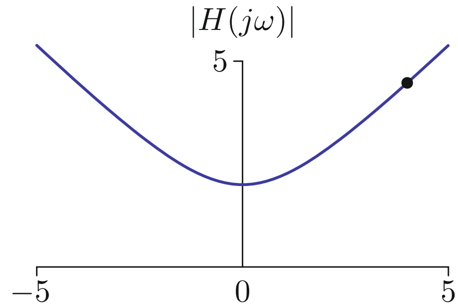

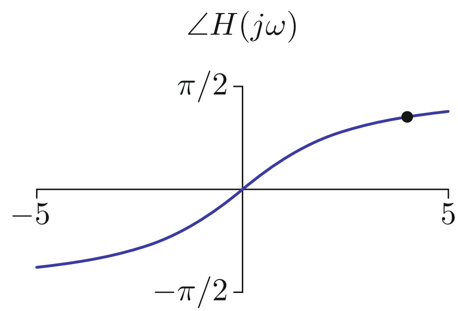

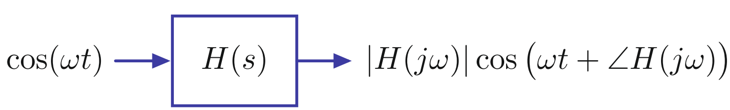

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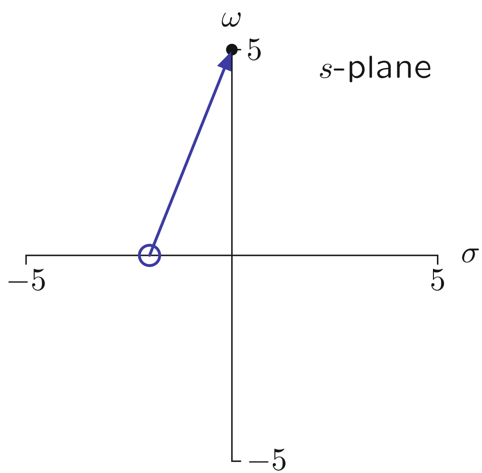

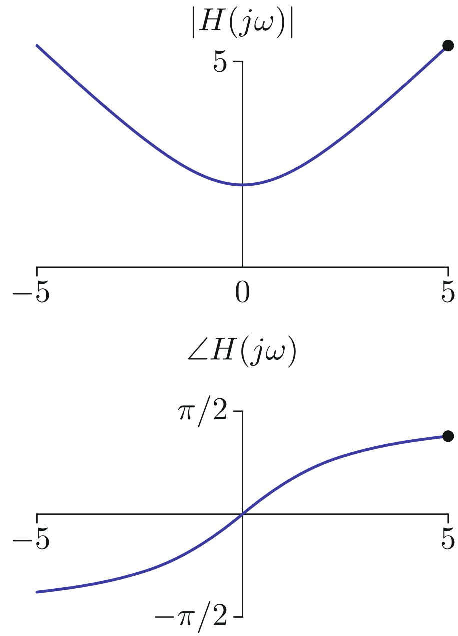

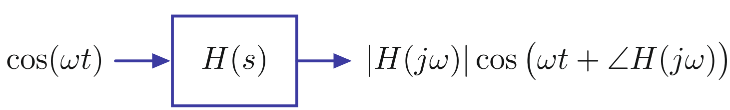

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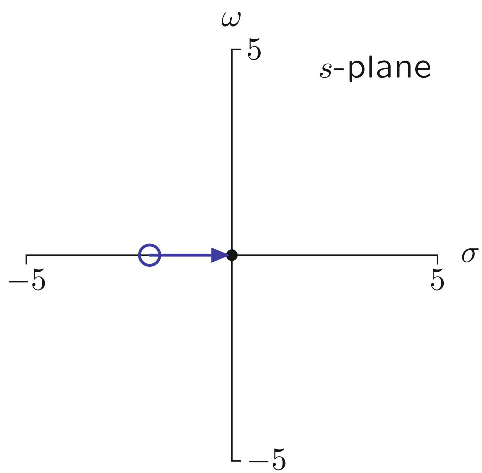

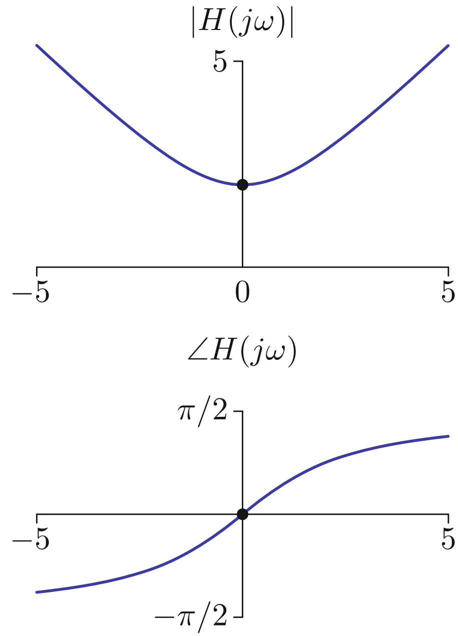

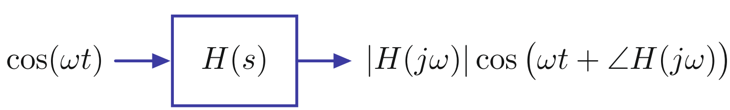

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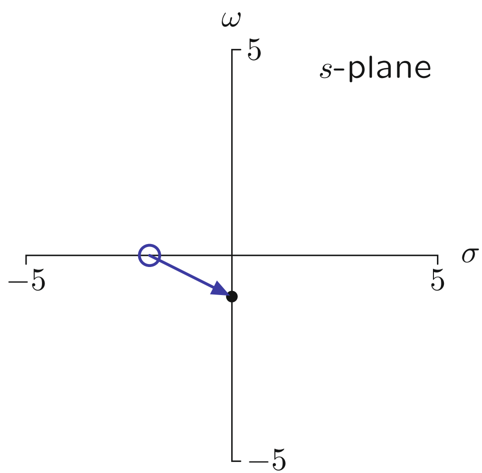

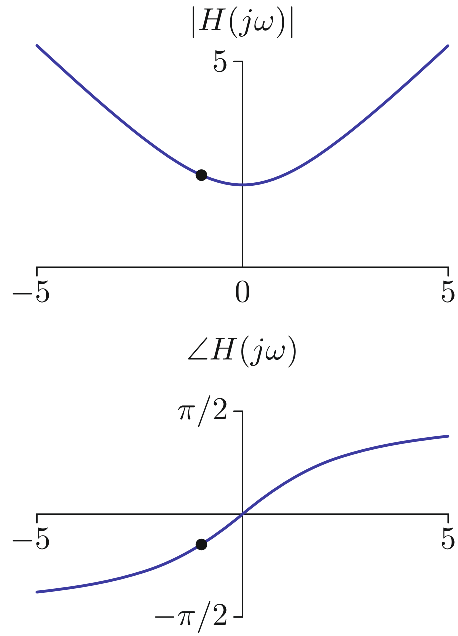

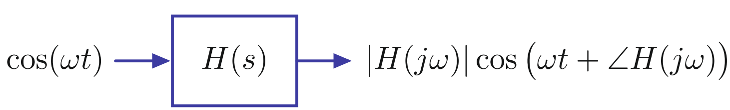

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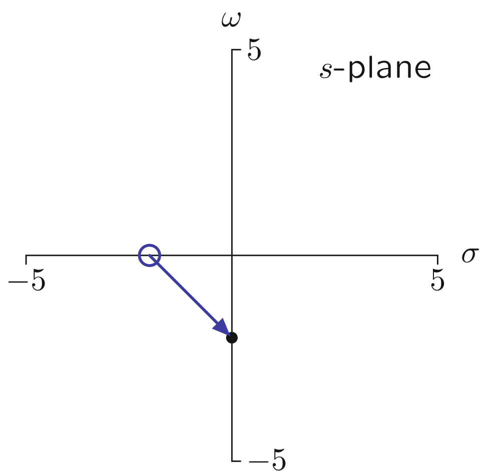

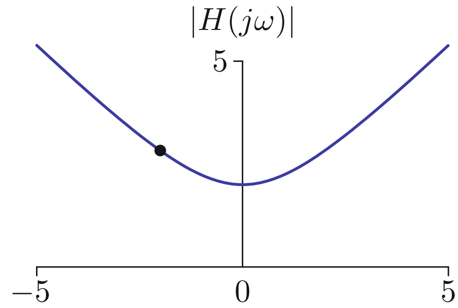

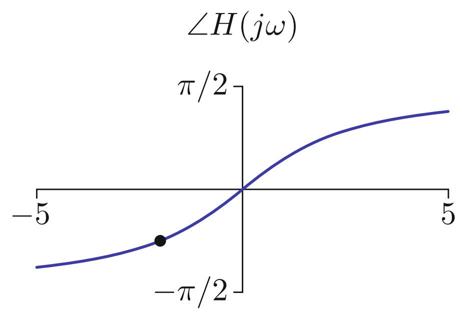

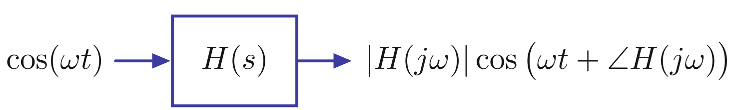

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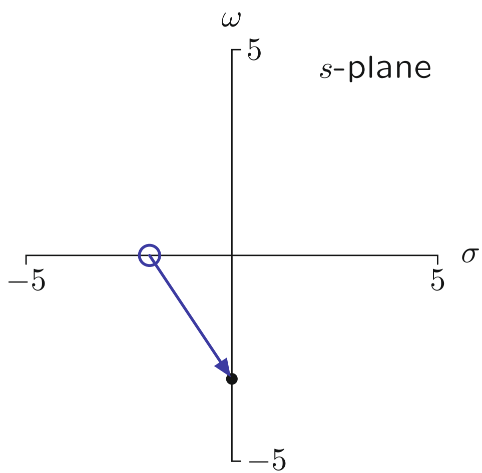
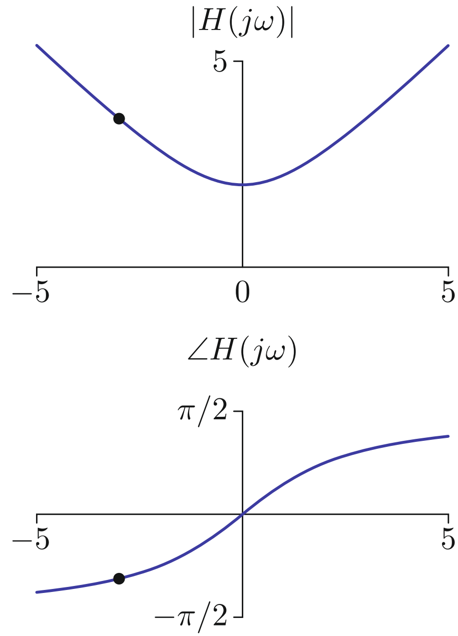
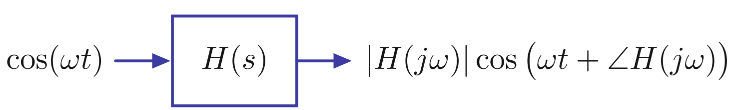




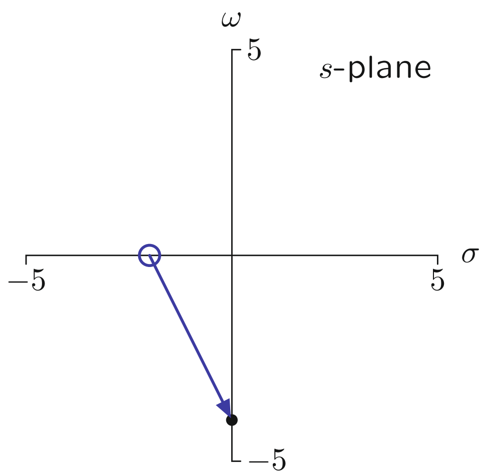


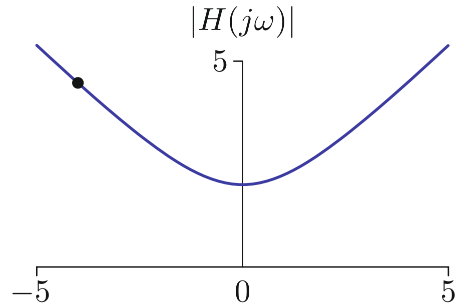
پاسخ فرکانسی

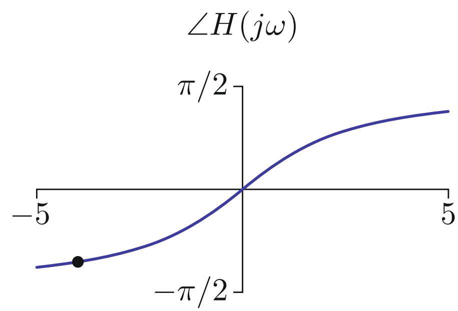
دیاگرام‌های برداری

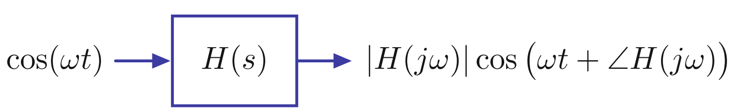
VECTOR DIAGRAMS

$$H(s) = s - z_1$$







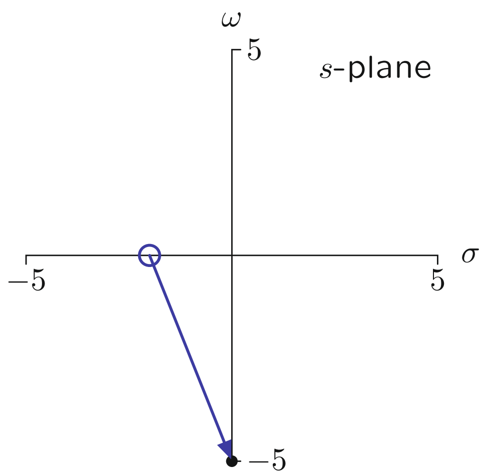


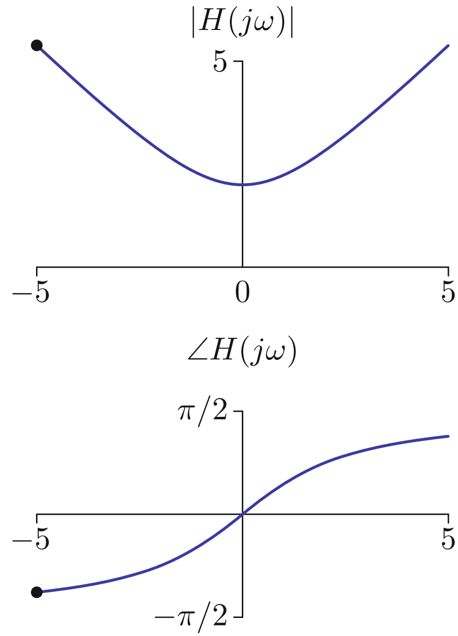
پاسخ فرکانسی

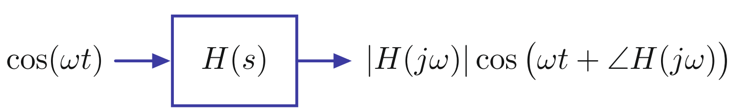
دیاگرام‌های برداری

Vector Diagrams

$$H(s) = s - z_1$$





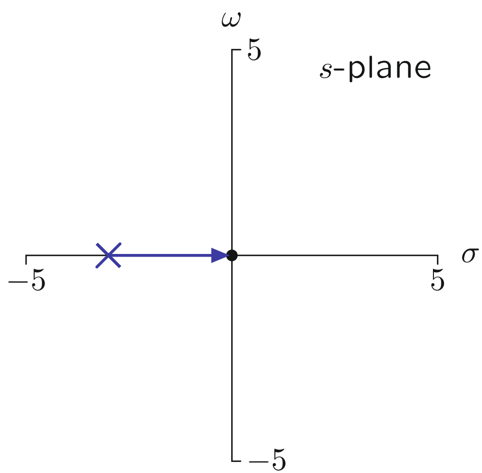


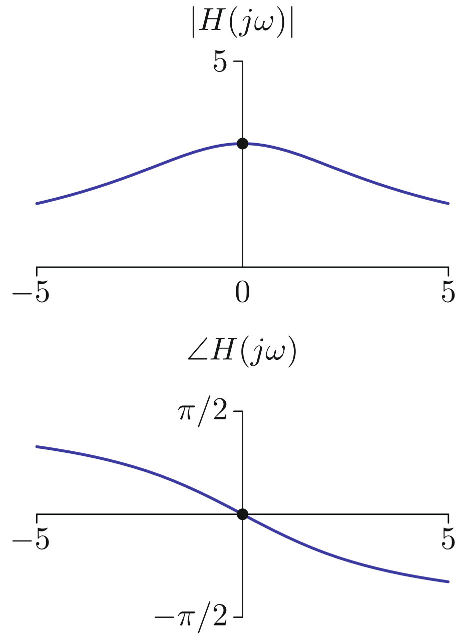
پاسخ فرکانسی

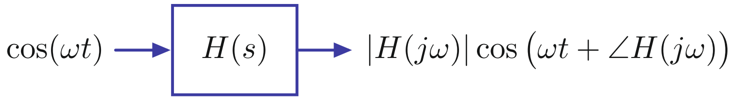
دیاگرام‌های برداری

VECTOR DIAGRAMS

$$H(s) = \frac{9}{s - p_1}$$





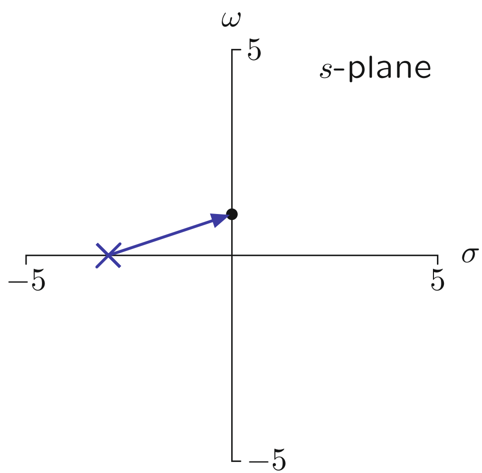


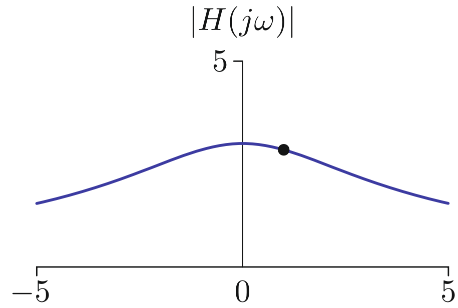
پاسخ فرکانسی

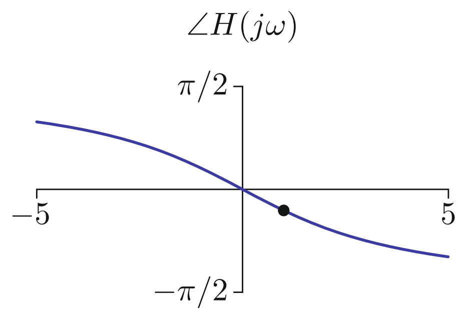
دیاگرام‌های برداری

VECTOR DIAGRAMS

$$H(s) = \frac{9}{s - p_1}$$







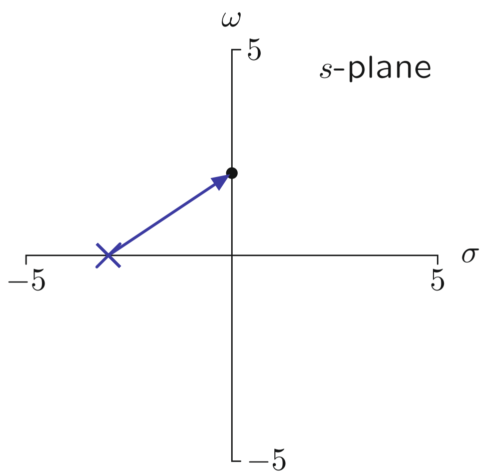
$$\cos(\omega t) \rightarrow \boxed{H(s)} \rightarrow |H(j\omega)| \cos\big(\omega t + \angle H(j\omega)\big)$$

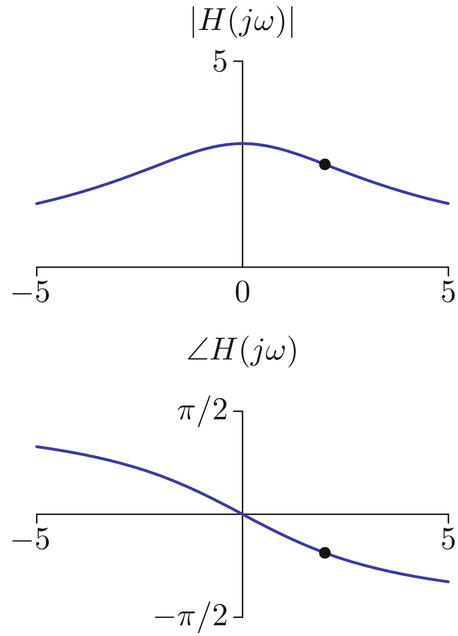
پاسخ فرکانسی

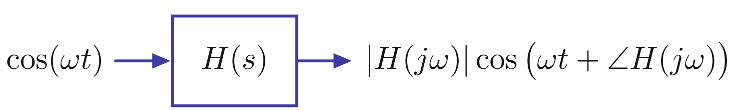
دیاگرام‌های برداری

VECTOR DIAGRAMS

$$H(s) = \frac{9}{s - p_1}$$





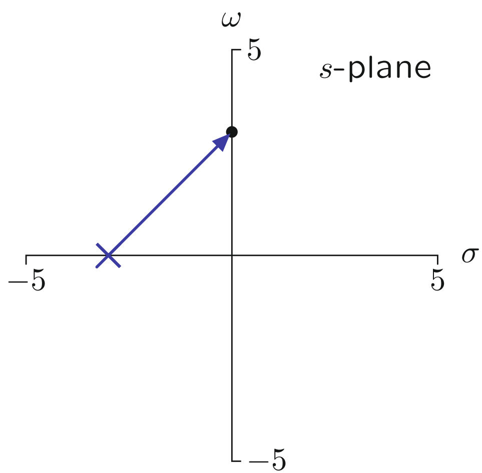


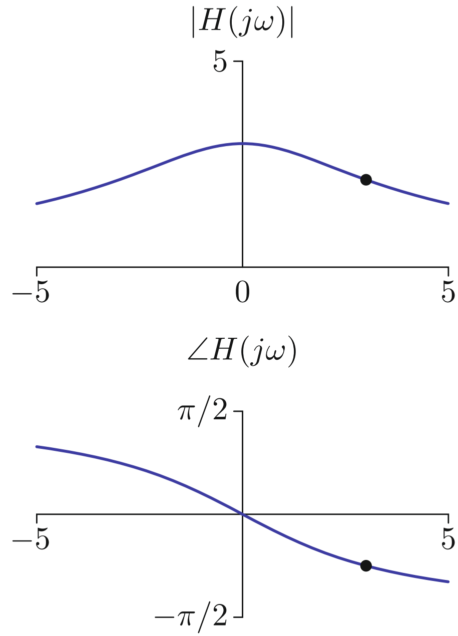
پاسخ فرکانسی

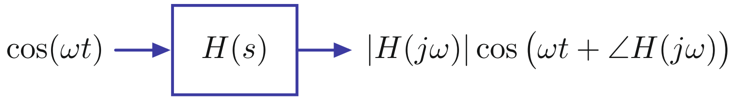
دیاگرام‌های برداری

Vector Diagrams

$$H(s) = \frac{9}{s - p_1}$$







پاسخ فرکانسی

دیاگرام‌های برداری

Vector Diagrams

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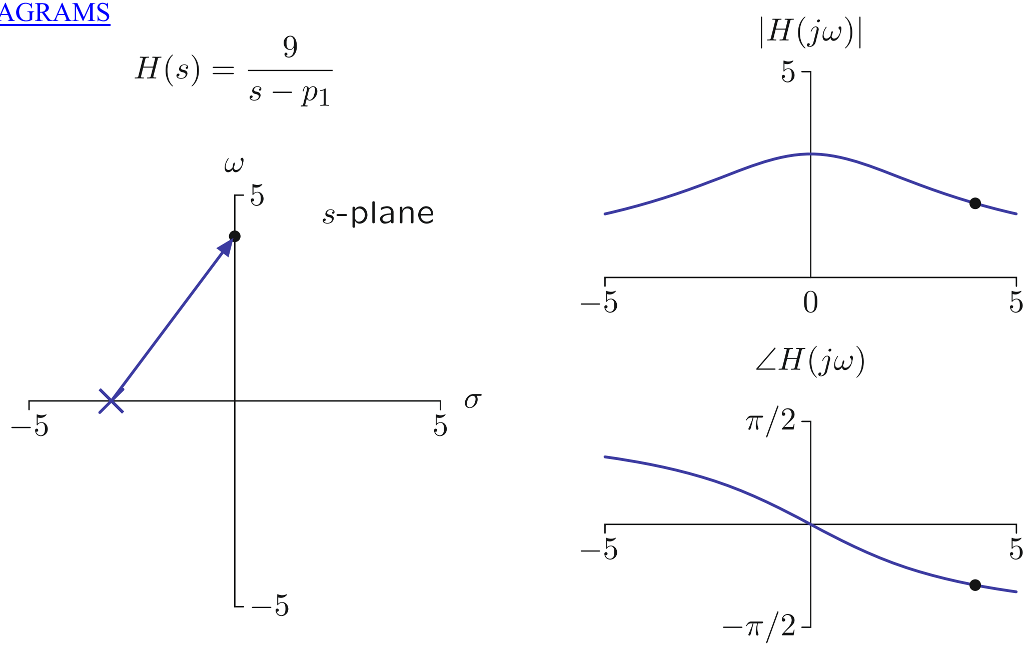

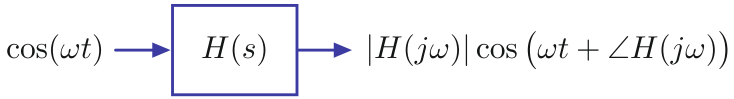

پاسخ فرکانسی

دیاگرام‌های برداری

Vector Diagrams

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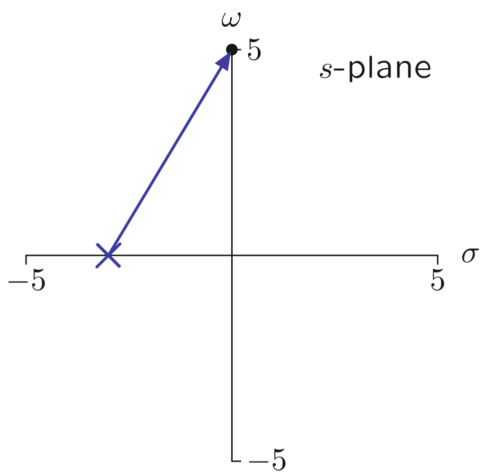
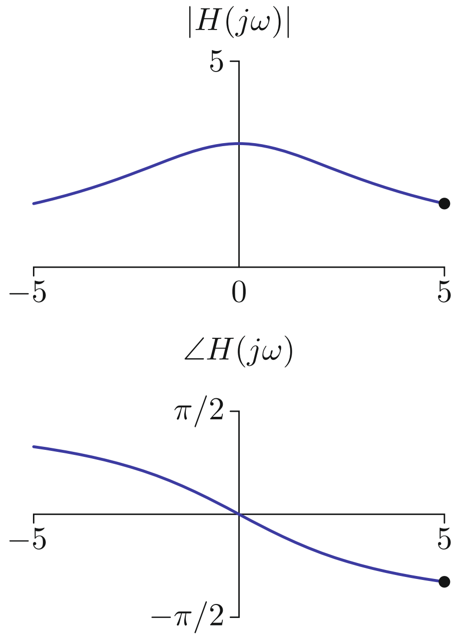
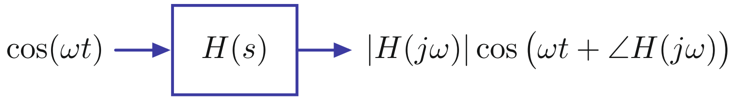




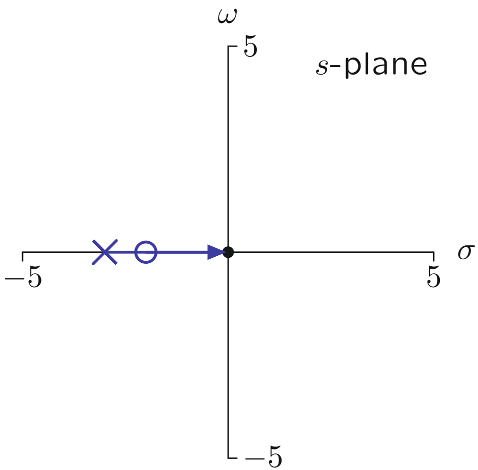


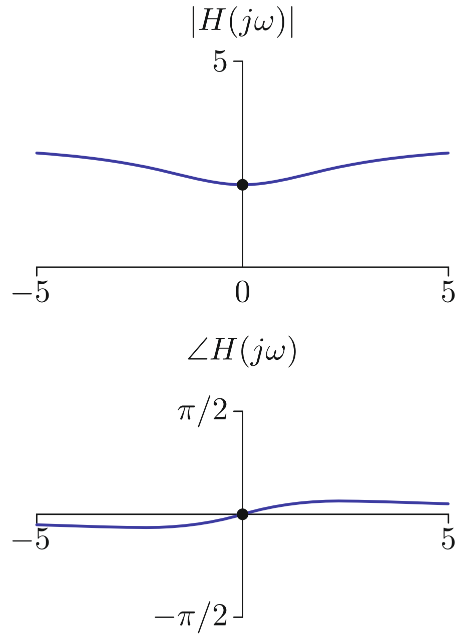
پاسخ فرکانسی

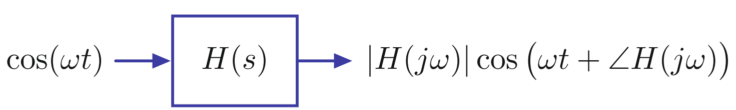
دیاگرام‌های برداری

Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p1}$$







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دیاگرام‌های برداری

Vector Diagrams

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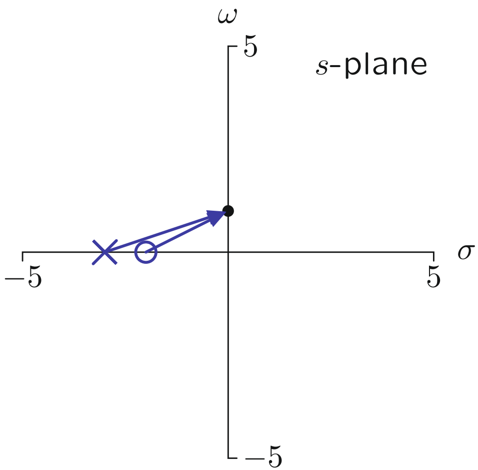

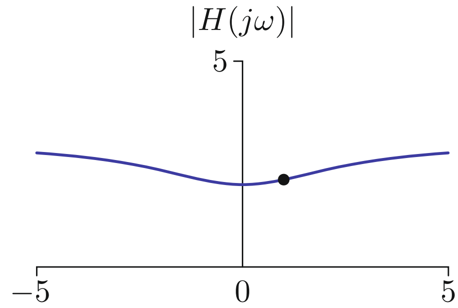

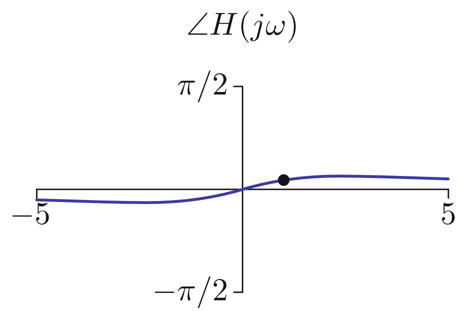

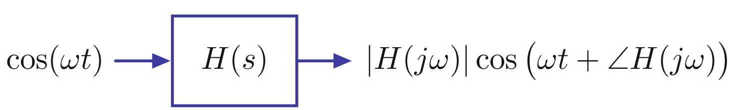

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دیاگرام‌های برداری

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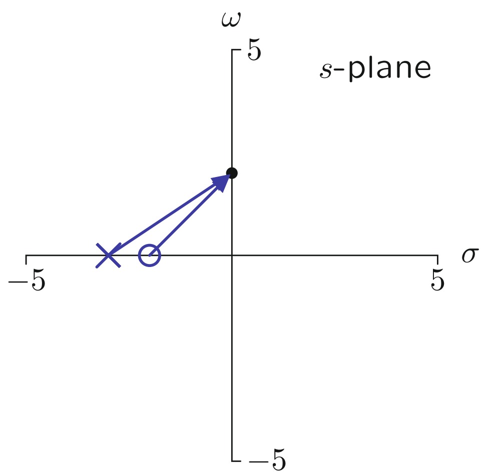

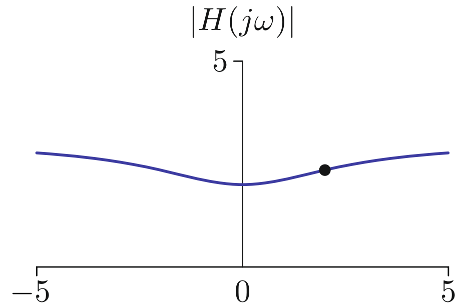

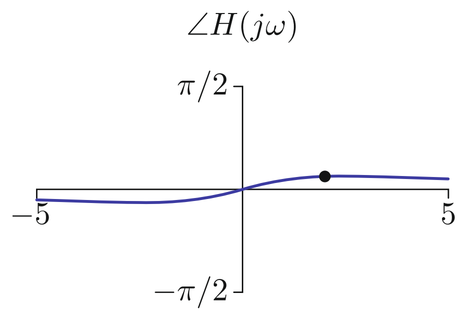

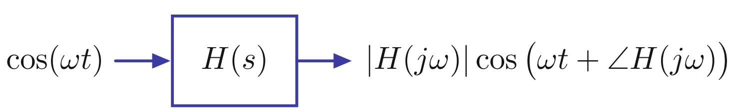

پاسخ فرکانسی

دیاگرام‌های برداری

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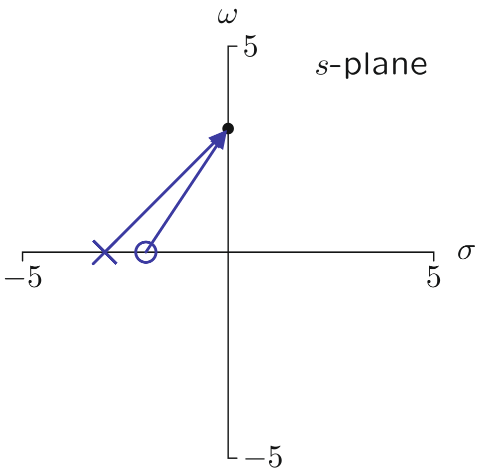
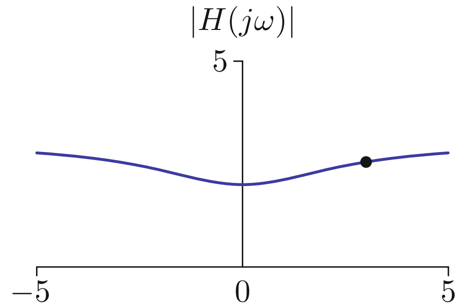
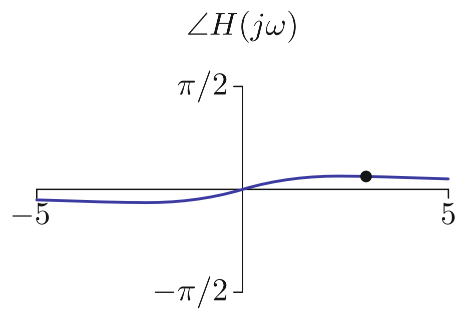
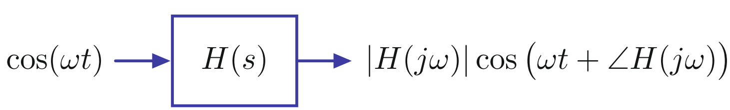






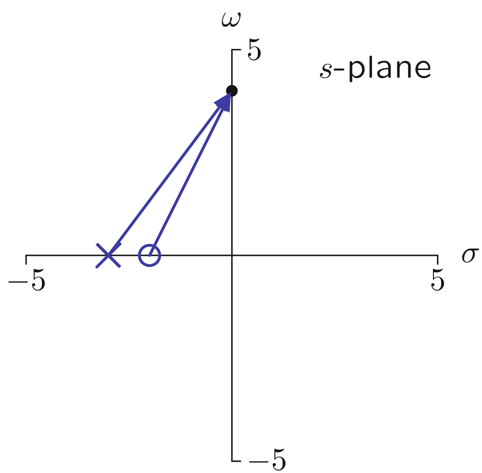


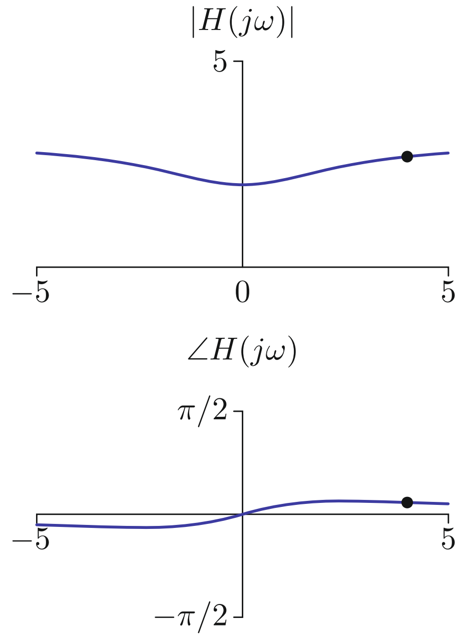
پاسخ فرکانسی

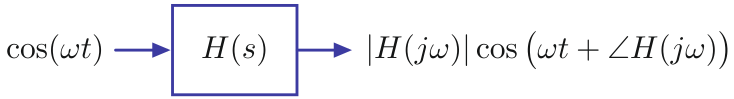
دیاگرام‌های برداری

VECTOR DIAGRAMS

$$H(s) = 3\frac{s - z_1}{s - p1}$$







پاسخ فرکانسی

دیاگرام‌های برداری

VECTOR DIAGRAMS

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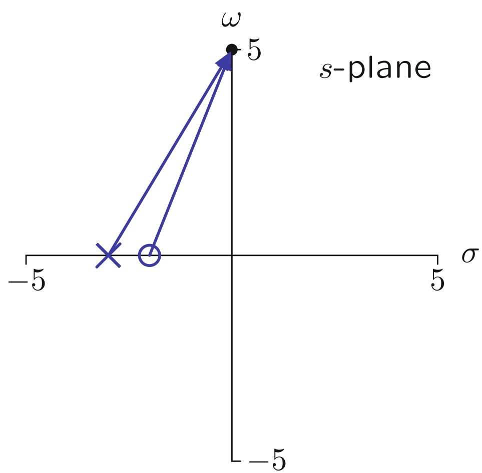
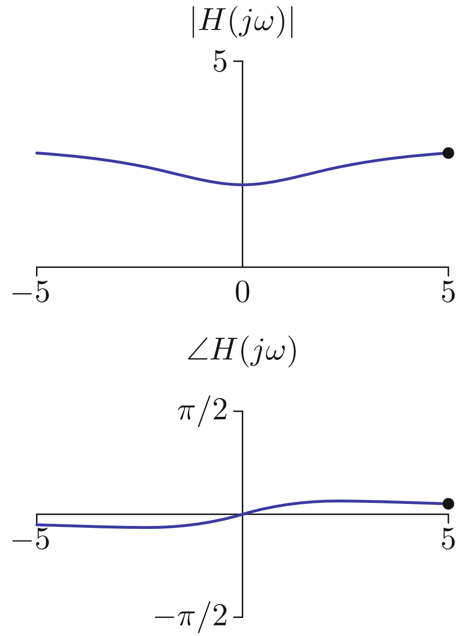
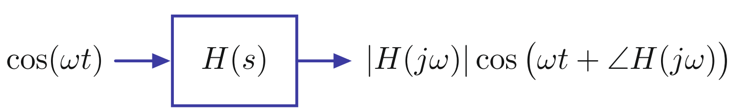


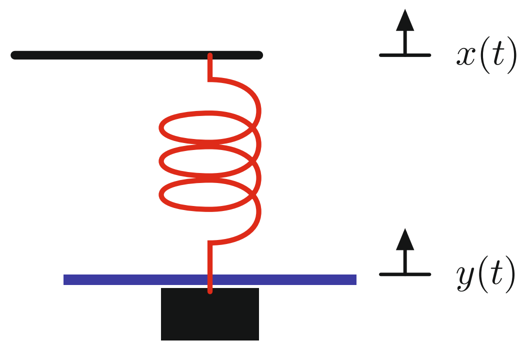




پاسخ فرکانسی

مثال: جرم، فنر و دمپر

Example: Mass, Spring, and Dashpot



$$F = Ma = M\ddot{y}(t) = K(x(t) - y(t)) - B\dot{y}(t)$$

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = Kx(t)$$

$$(s^2M + sB + K)\ Y(s) = KX(s)$$

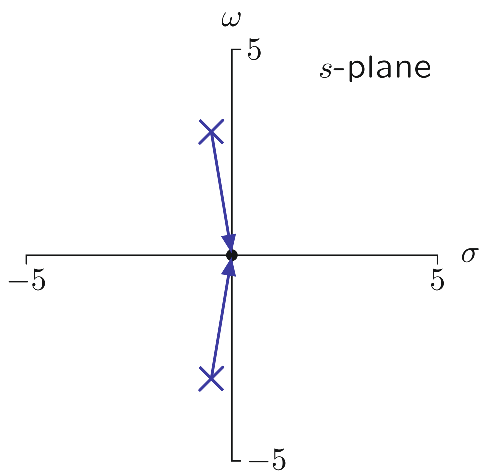
$$H(s) = \frac{K}{s^2M + sB + K}$$

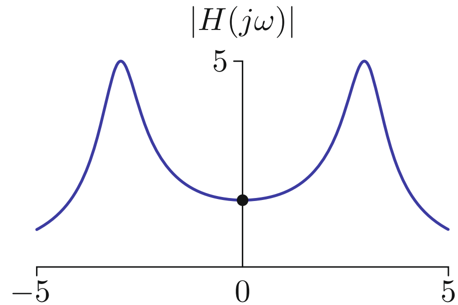
پاسخ فرکانسی

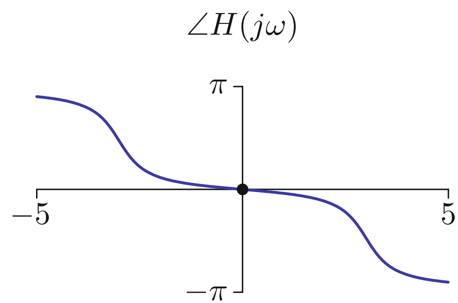
دیاگرام‌های برداری

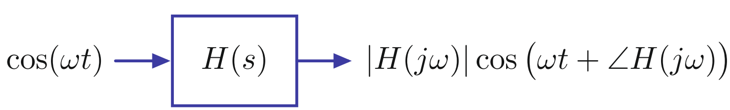
Vector Diagrams

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$









پاسخ فرکانسی

دیاگرام‌های برداری

VECTOR DIAGRAMS

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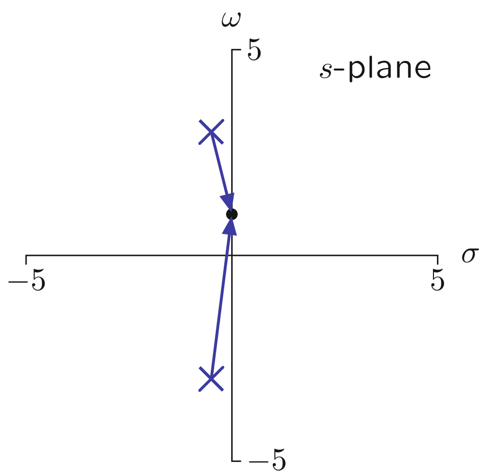

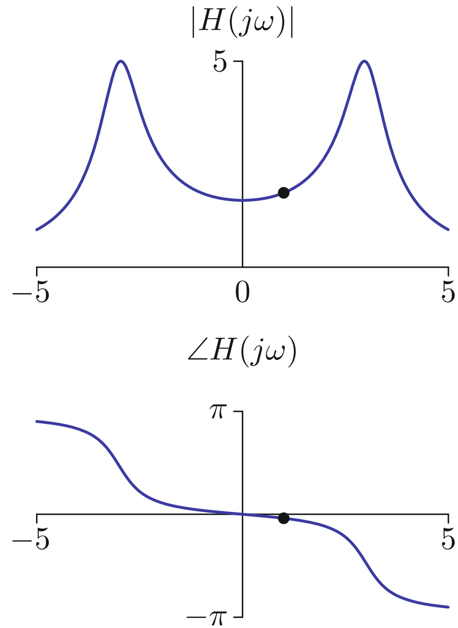

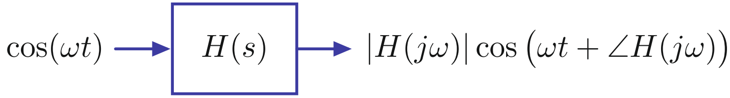

پاسخ فرکانسی

دیاگرام‌های برداری

Vector Diagrams

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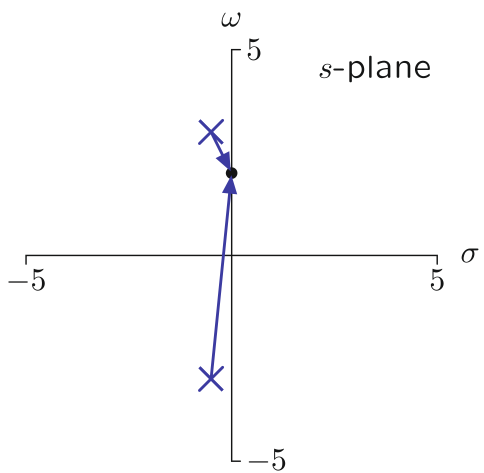

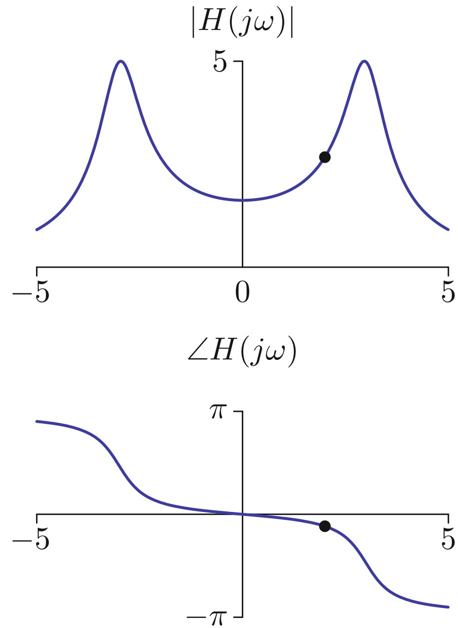

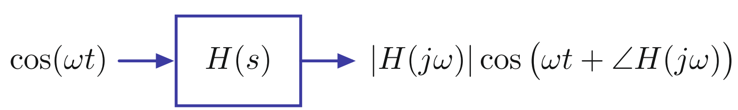

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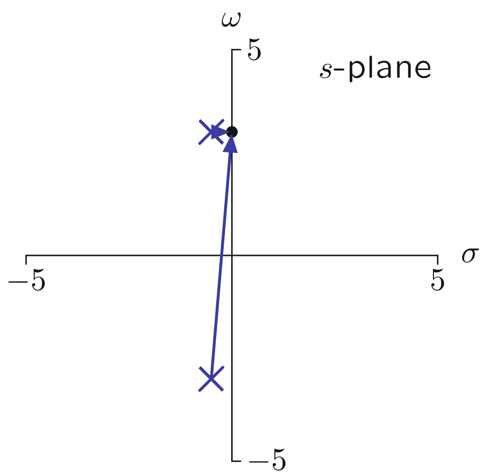

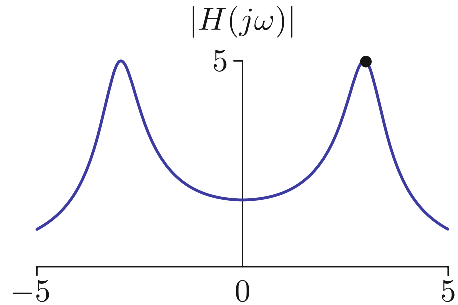

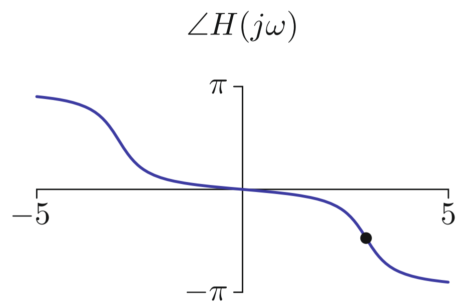

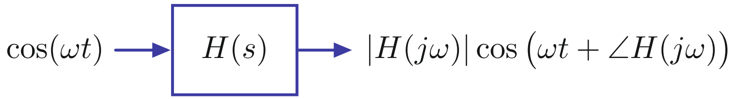

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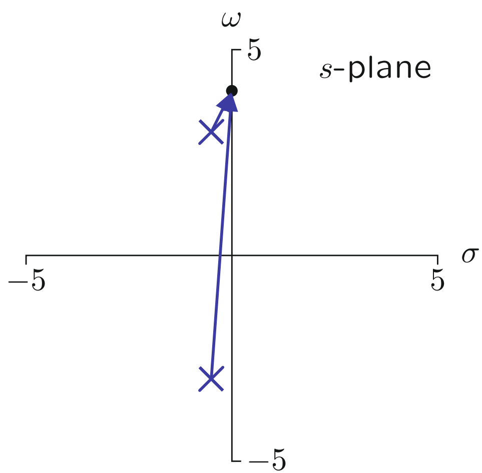

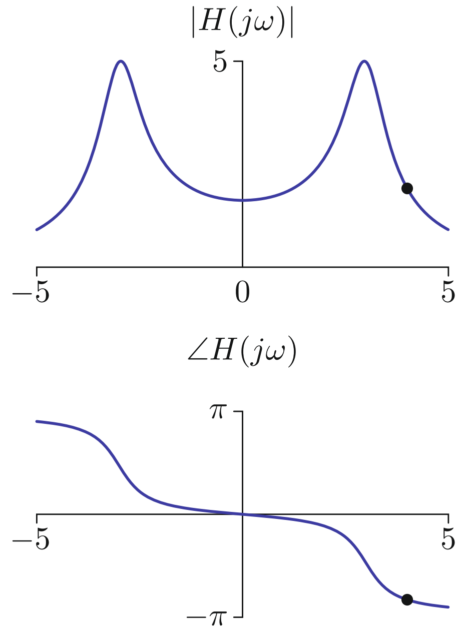

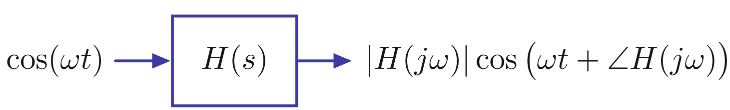

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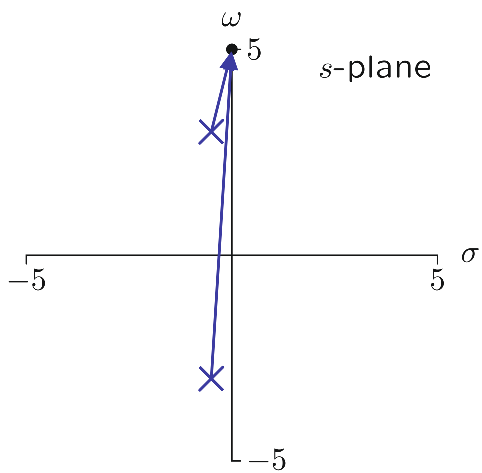
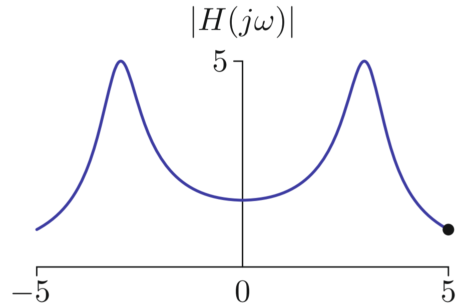
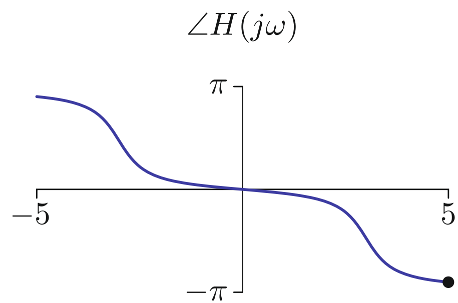
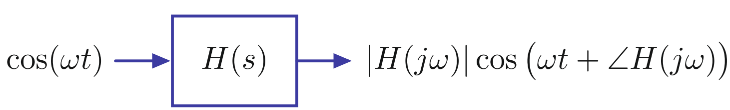
Vector Diagrams

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$







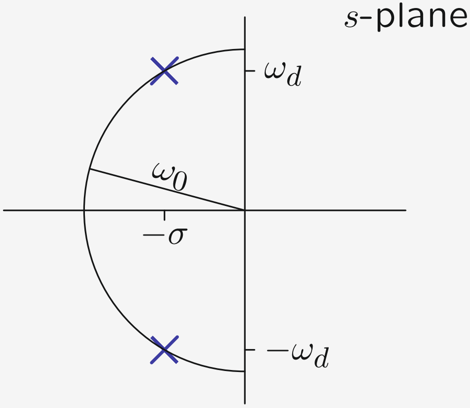


پاسخ فرکانسی

مثال (۱ از ۵)

VECTOR DIAGRAMS

Consider the system represented by the following poles.



Find the frequency ω at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos \omega t$.

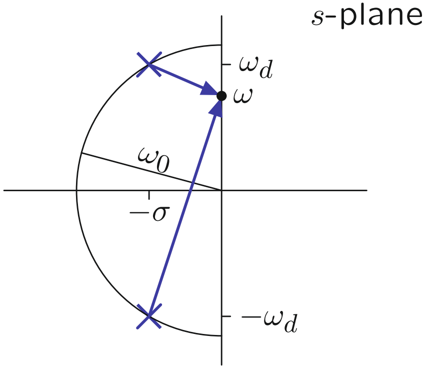
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $0 < \omega < \omega_d$
4. none of the above

پاسخ فرکانسی

مثال (۲ از ۵)

VECTOR DIAGRAMS

Analyze with vectors.



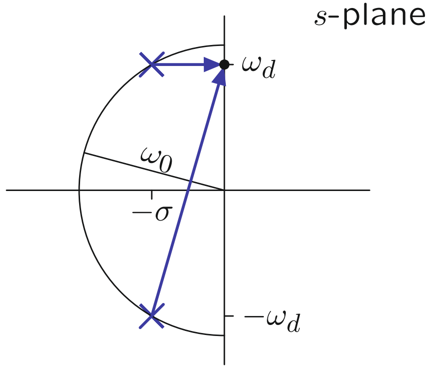
The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

پاسخ فرکانسی

مثال (۳ از ۵)

VECTOR DIAGRAMS

Analyze with vectors.



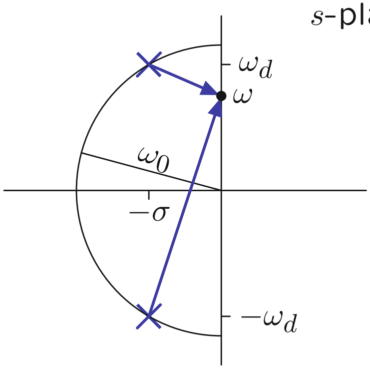
The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

Decreasing ω from ω_d to $\omega_d - \epsilon$ decreases the product since length of bottom vector decreases as ϵ while length of top vector increases only ϵ^2.

مثال (۴ از ۵)

VECTOR DIAGRAMS

More mathematically ...



The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

Maximum occurs where derivative of squared lengths is zero.

$$\frac{d}{d\omega}\left((\omega+\omega_d)^2+\sigma^2\right)\left((\omega-\omega_d)^2+\sigma^2\right)=0$$

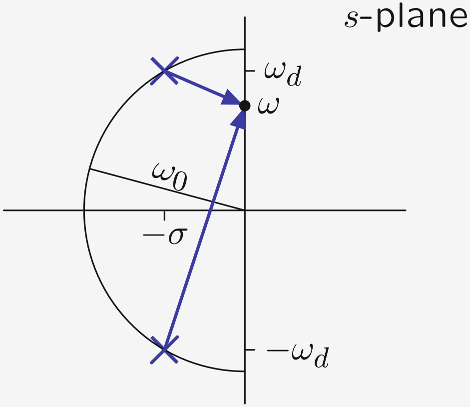
$$\rightarrow \quad \omega^2=\omega_d^2-\sigma^2=\omega_0^2-2\sigma^2\,.$$

پاسخ فرکانسی

مثال (۵ از ۵)

VECTOR DIAGRAMS

Consider the system represented by the following poles.



Find the frequency ω at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos \omega t$. 3

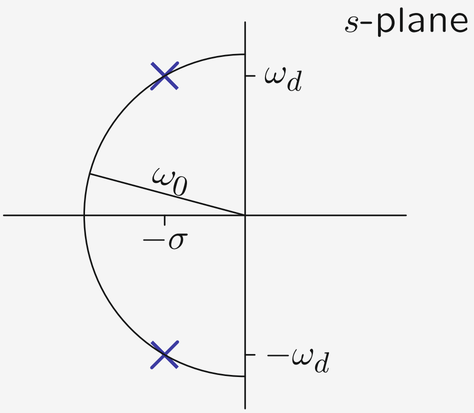
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $0 < \omega < \omega_d$
4. none of the above

پاسخ فرکانسی

مثال (۱ از ۶)

VECTOR DIAGRAMS

Consider the system represented by the following poles.



Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos\omega t$.

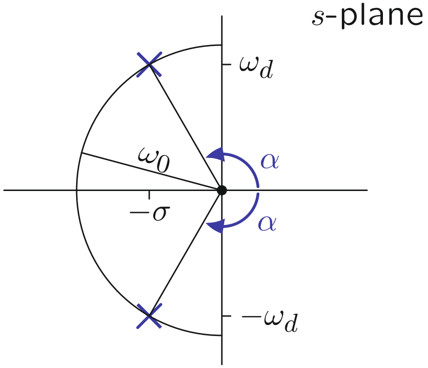
0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none

پاسخ فرکانسی

مثال (۲ از ۶)

VECTOR DIAGRAMS

The phase is 0 when $\omega = 0$.

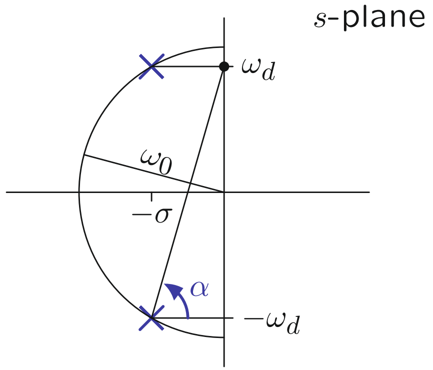


پاسخ فرکانسی

مثال (۳ از ۶)

VECTOR DIAGRAMS

The phase is less than $\pi/2$ when $\omega = \omega_d$.

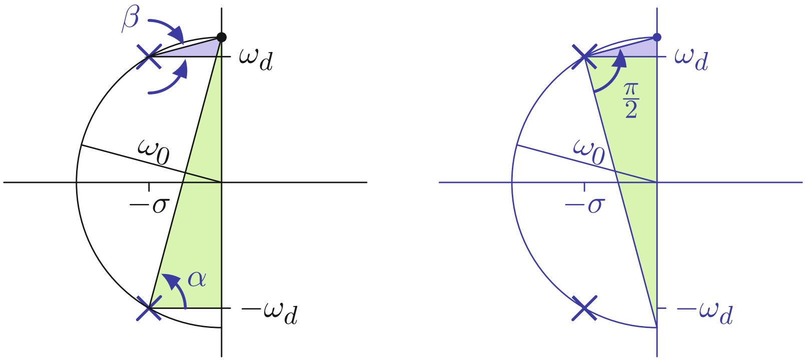


پاسخ فرکانسی

مثال (۴ از ۶)

VECTOR DIAGRAMS

The phase at $\omega = \omega_0$ is $-\pi/2$.



پاسخ فرکانسی

مثال (۵ از ۶)

VECTOR DIAGRAMS

Check result by evaluating the system function.

Substitute $s = j\omega_0 = j\sqrt{\frac{K}{M}}$ into

$$H(s) = \frac{K}{s^2M + sB + K} = \frac{K}{-\frac{K}{M}M + j\omega_0 B + K} = \frac{K}{j\omega_0 B}$$

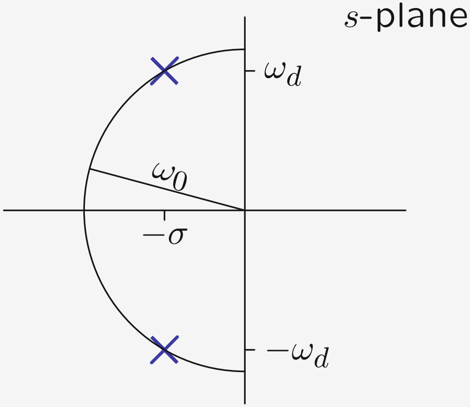
The phase is $-\frac{\pi}{2}$.

پاسخ فرکانسی

مثال (۶ از ۶)

VECTOR DIAGRAMS

Consider the system represented by the following poles.



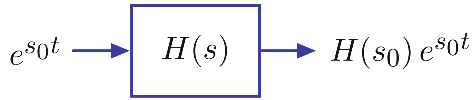
Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos \omega t$. 3

0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none

پاسخ فرکانسی

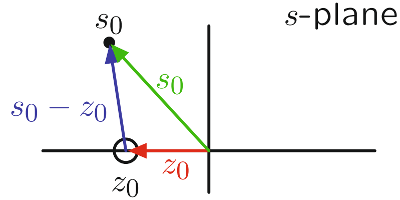
خلاصه

Complex exponentials are eigenfunctions of LTI systems.

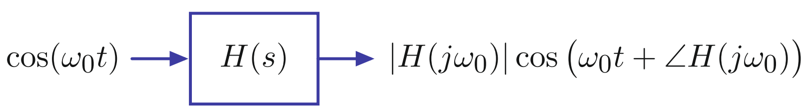


$H(s_0)$ can be determined graphically using vectorial analysis.

$$H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2) \cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2) \cdots}$$



Response of an LTI system to an eternal cosine is an eternal cosine: same frequency, but scaled and shifted.



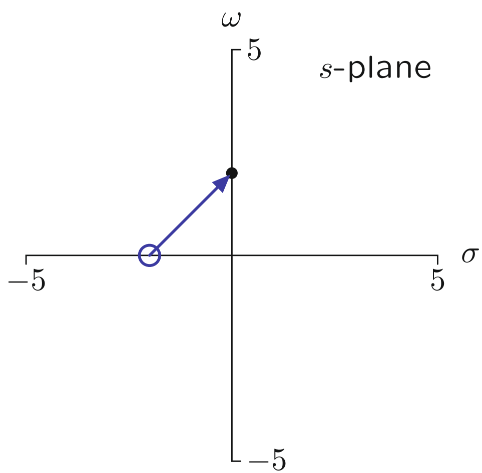
Frequency Response: $H(s)|_{s \leftarrow j\omega}$

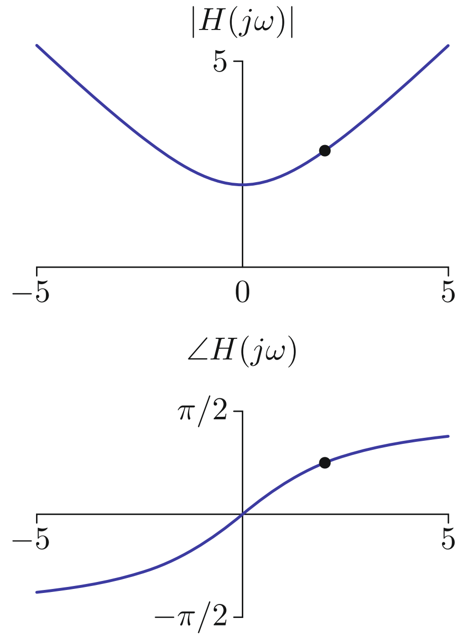
پاسخ فرکانسی

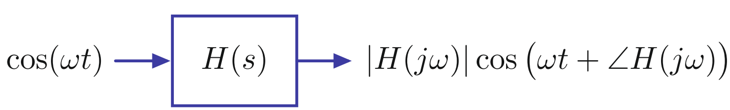
دیاگرام‌های برداری

Vector Diagrams

$$H(s) = s - z_1$$





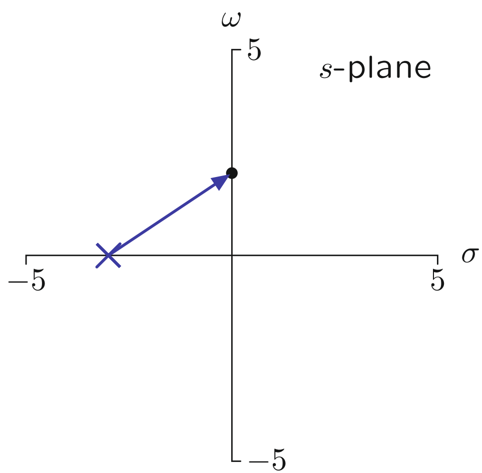


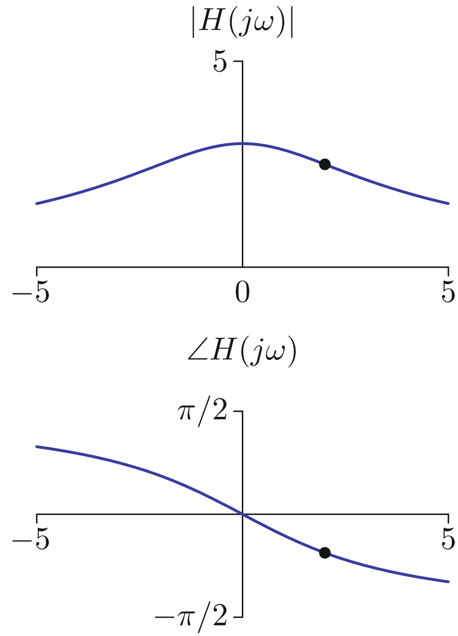
پاسخ فرکانسی

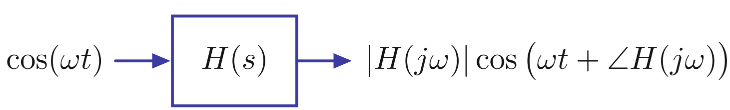
دیاگرام‌های برداری

VECTOR DIAGRAMS

$$H(s) = \frac{9}{s - p_1}$$





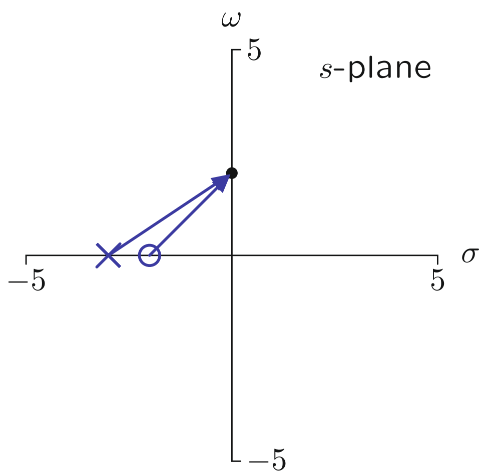


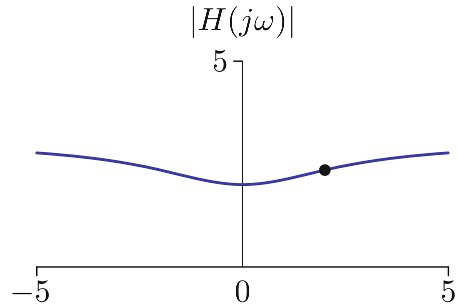
پاسخ فرکانسی

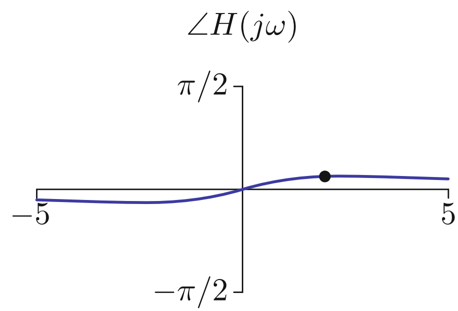
دیاگرام‌های برداری

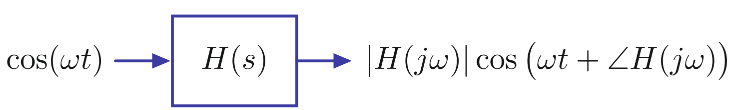
Vector Diagrams

$$H(s) = 3\,\frac{s - z_1}{s - p1}$$









صفرها و قطب‌ها

POLES AND ZEROS

فکر کردن در مورد سیستم‌ها به‌صورت گردایه‌ای از قطب‌ها و صفرها یک مفهوم مهم طراحی است.

ساده: تعداد کمی عدد، کل سیستم را مشخص می‌کند.

قدرتمند: اطلاعات کامل در مورد پاسخ فرکانسی سیستم را دربردارد.

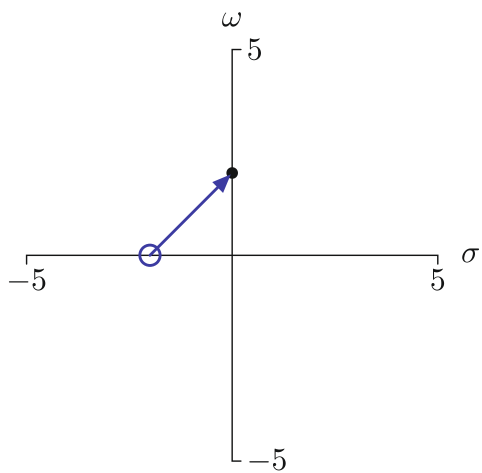
رفتار مجانبی

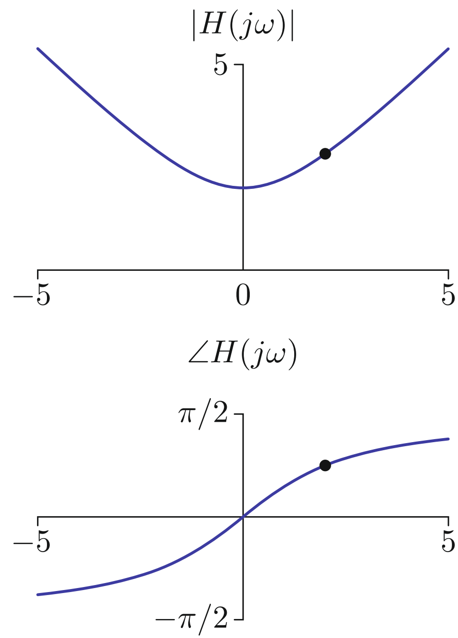
صفر تنها (۱ از ۴)

Asymptotic Behavior: Isolated Zero

The magnitude response is simple at low and high frequencies.

$$H(j\omega) = j\omega - z_1$$





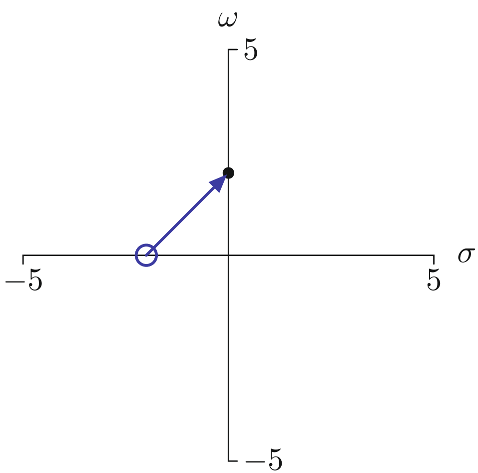
رفتار مجانبی

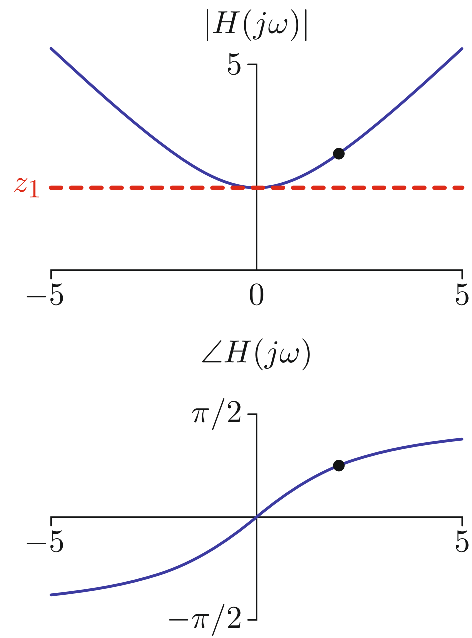
صفر تنها (۲ از ۴)

ASYMPTOTIC BEHAVIOR: ISOLATED ZERO

The magnitude response is simple at low and high frequencies.

$$H(j\omega) = j\omega - z_1$$





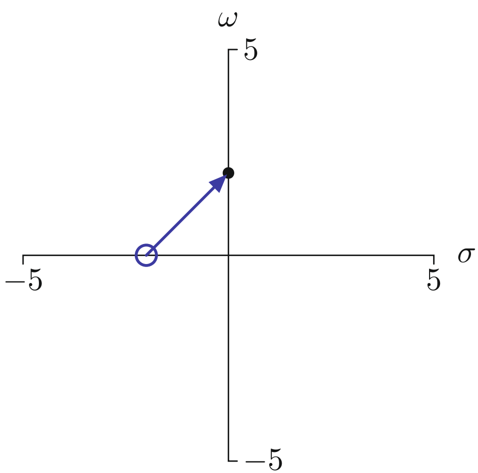
رفتار مجانبی

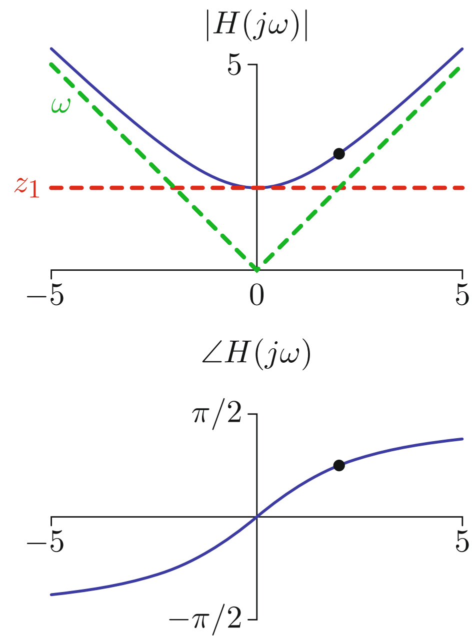
صفر تنها (۳ از ۴)

Asymptotic Behavior: Isolated Zero

The magnitude response is simple at low and high frequencies.

$$H(j\omega) = j\omega - z_1$$





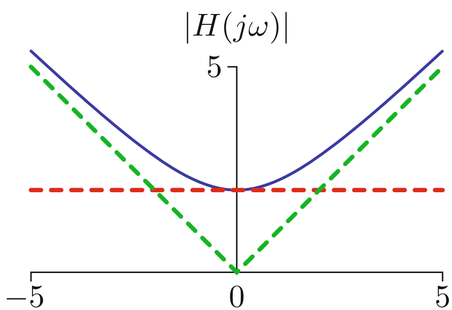
رفتار مجانبی

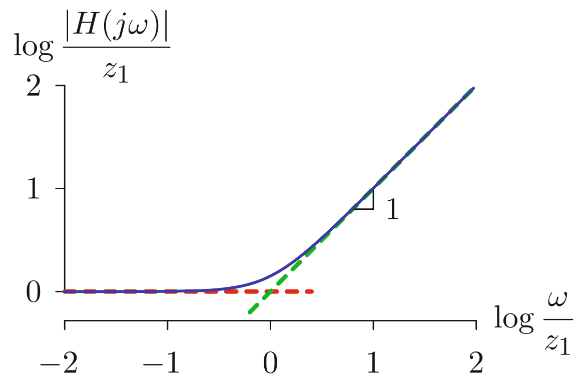
صفر تنها (۴ از ۴)

Asymptotic Behavior: Isolated Zero

Two asymptotes provide a good approxmation on log-log axes.

$$H(s) = s - z_1$$





$$\lim_{\omega \to 0} |H(j\omega)| = z_1$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \omega$$

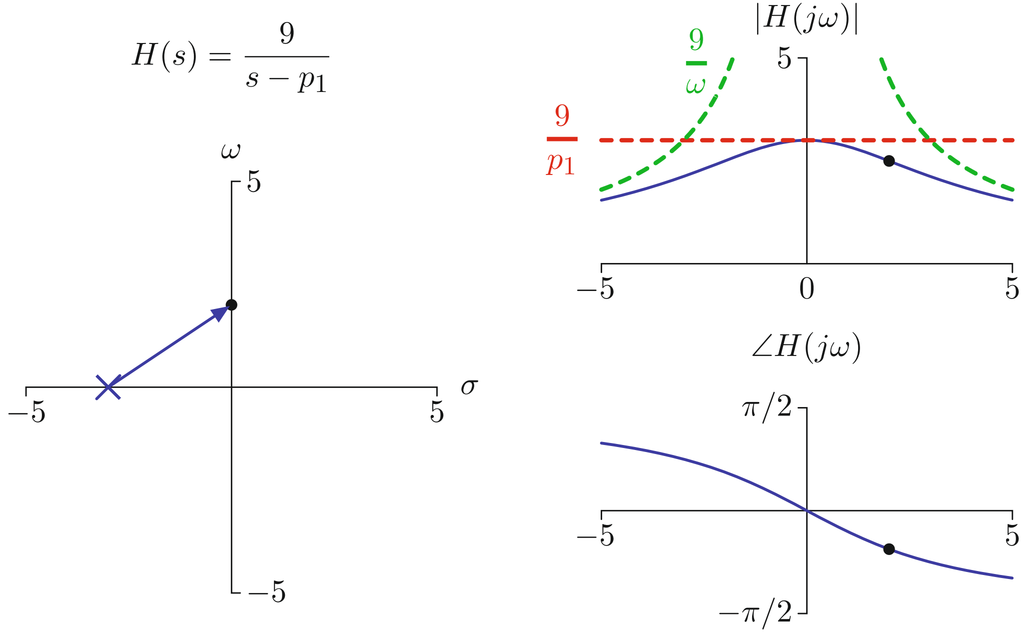
رفتار مجانبی

قطب تنها (۱ از ۲)

Asymptotic Behavior: Isolated Pole

The magnitude response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$



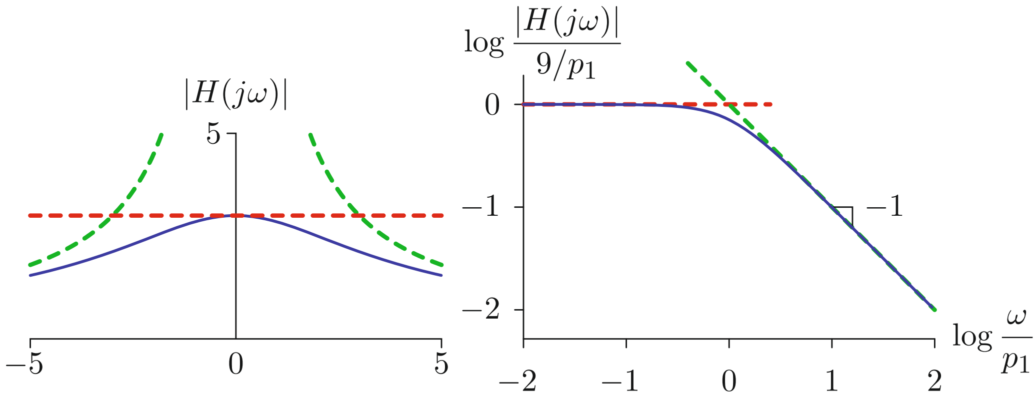
رفتار مجانبی

قطب تنها (۱ از ۲)

ASYMPTOTIC BEHAVIOR: ISOLATED POLE

Two asymptotes provide a good approxmation on log-log axes.

$$H(s) = \frac{9}{s - p_1}$$



$$\lim_{\omega \to 0} |H(j\omega)| = \frac{9}{p_1}$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \frac{9}{\omega}$$

رفتار مجانبی

مثال (۱ از ۳)

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$

The former can be transformed into the latter by

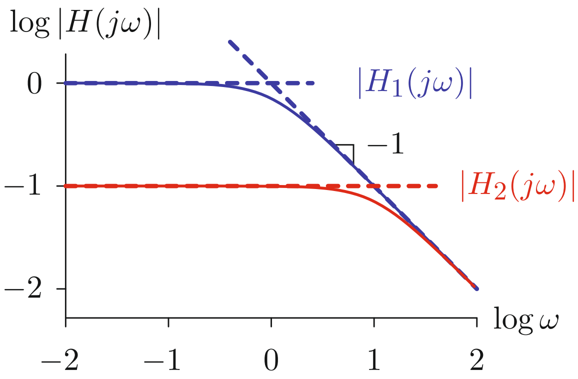
1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

رفتار مجانبی

مثال (۲ از ۳)

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$



رفتار مجانبی

مثال (۳ از ۳)

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$

The former can be transformed into the latter by 3

1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

no scaling in either vertical or horizontal directions !

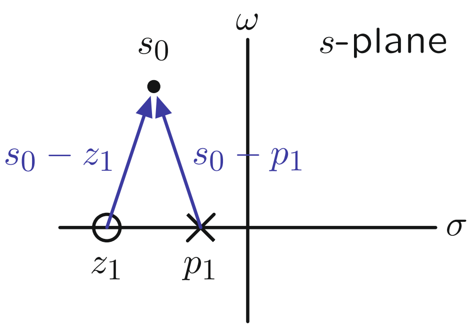
رفتار مجانبی

سیستم‌های پیچیده‌تر

Asymptotic Behavior of More Complicated Systems

Constructing $H(s_0)$.

$$H(s_0) = K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \quad \begin{array}{l} \leftarrow \text{ product of vectors for zeros} \\ \\ \leftarrow \text{ product of vectors for poles} \end{array}$$

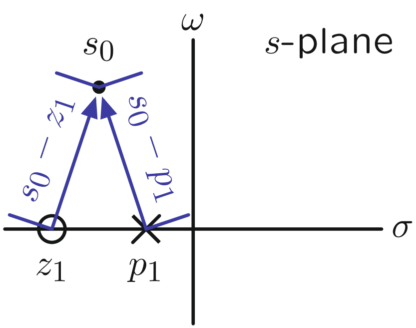


سیستم‌های پیچیده‌تر

Asymptotic Behavior of More Complicated Systems

The magnitude of a product is the product of the magnitudes.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q}|s_0 - z_q|}{\prod_{p=1}^{P}|s_0 - p_p|}$$



سیستم‌های پیچیده‌تر

ASYMPTOTIC BEHAVIOR OF MORE COMPLICATED SYSTEMS

The log of the magnitude is a sum of logs.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q}|s_0 - z_q|}{\prod_{p=1}^{P}|s_0 - p_p|}$$

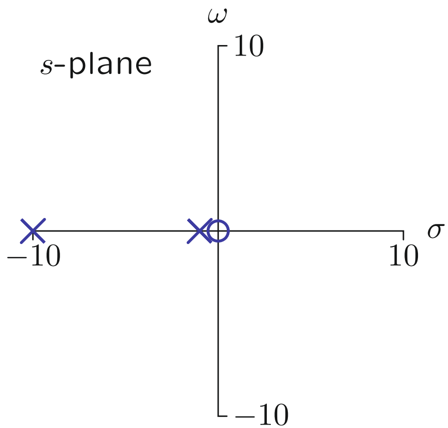
$$\log|H(j\omega)| = \log|K| + \sum_{q=1}^{Q}\log|j\omega - z_q| - \sum_{p=1}^{P}\log|j\omega - p_p|$$

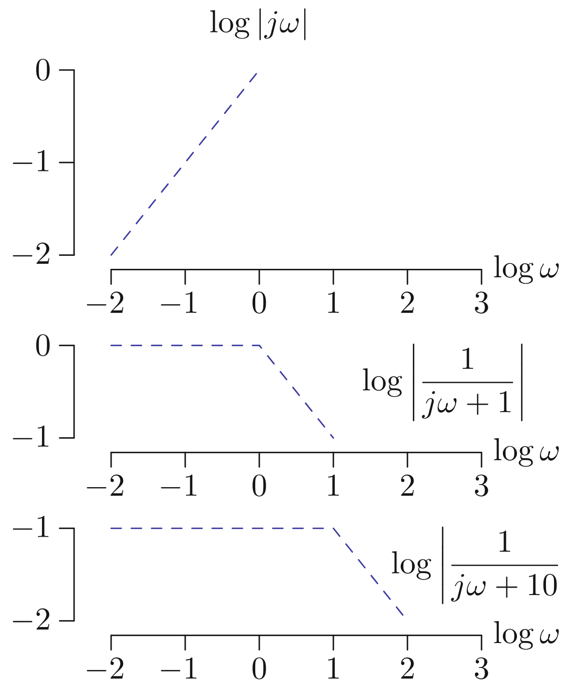
نمودار بودی

جمع به‌جای ضرب

Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$



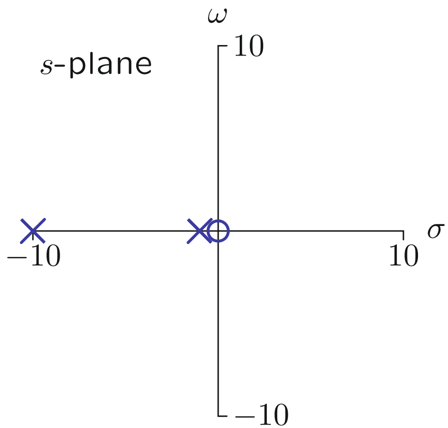


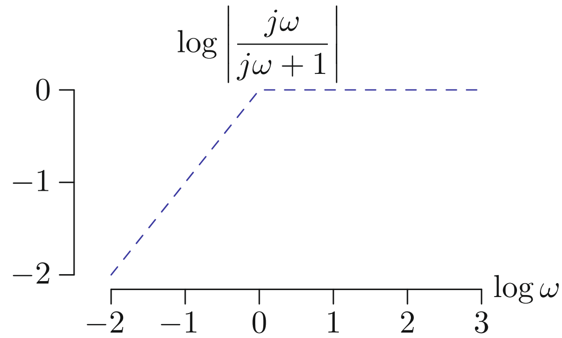
نمودار بودی

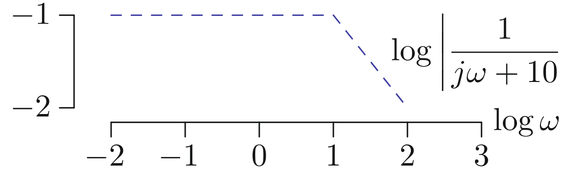
جمع به‌جای ضرب

Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$





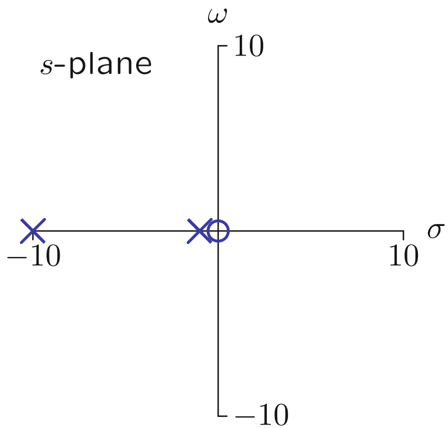


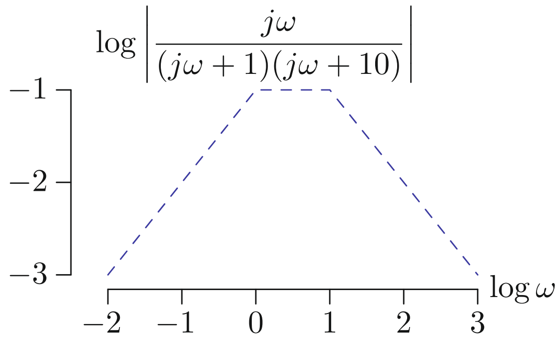
نمودار بودی

جمع به‌جای ضرب

Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$



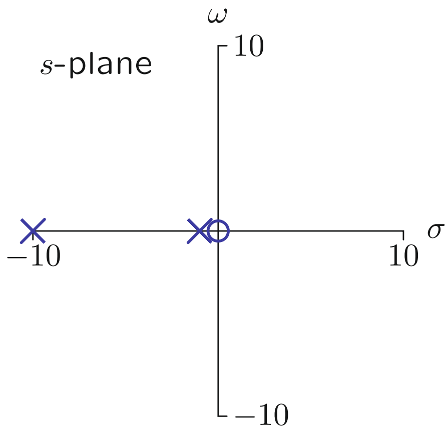


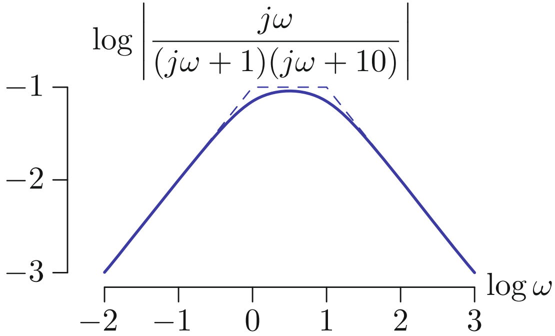
نمودار بودی

جمع به‌جای ضرب

Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$





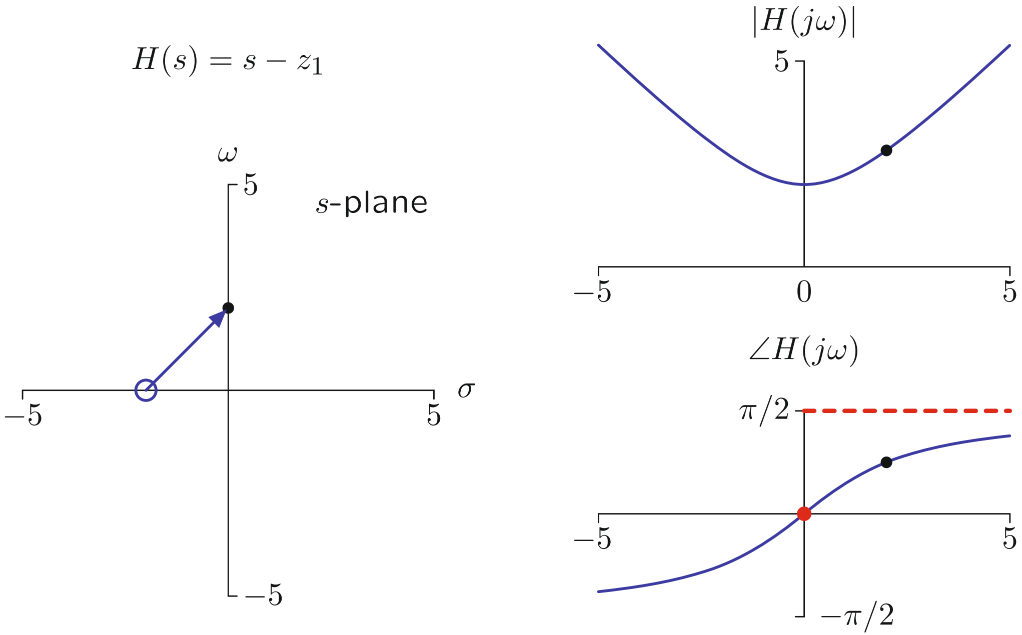
رفتار مجانبی

صفر تنها

Asymptotic Behavior: Isolated Zero

The angle response is simple at low and high frequencies.

$H(s) = s - z_1$



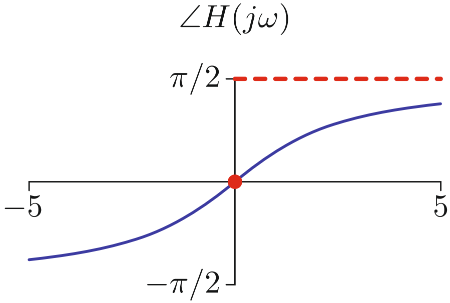
رفتار مجانبی

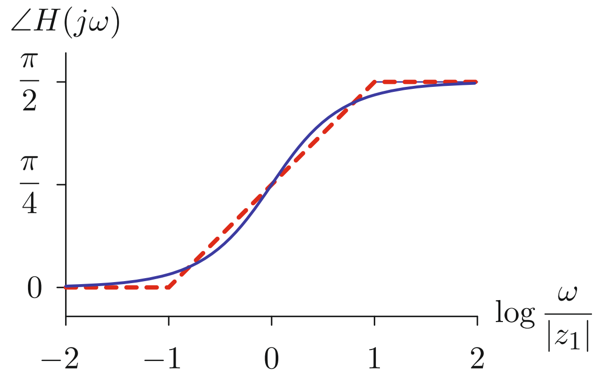
صفر تنها

Asymptotic Behavior: Isolated Zero

Three straight lines provide a good approxmation versus log ω.

$$H(s) = s - z_1$$





$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = \pi/2$$

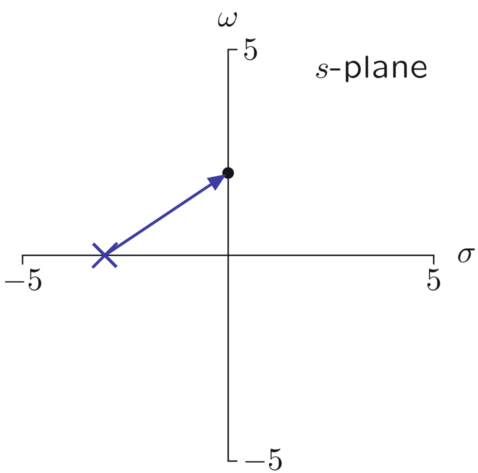
رفتار مجانبی

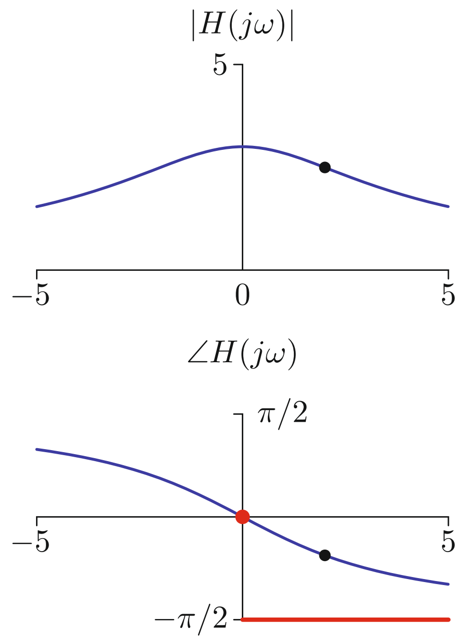
قطب تنها

Asymptotic Behavior: Isolated Pole

The angle response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$





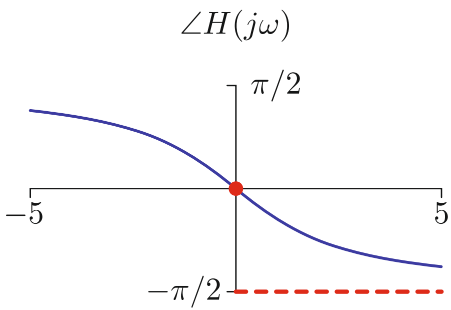
رفتار مجانبی

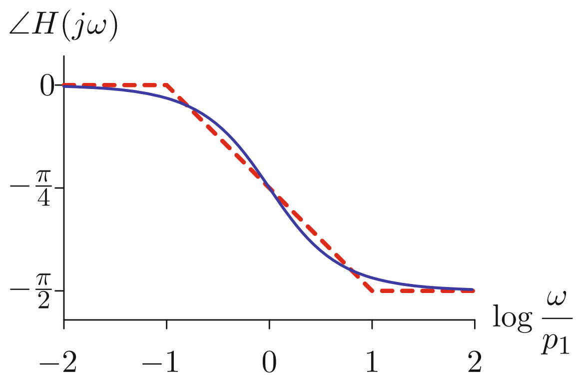
قطب تنها

Asymptotic Behavior: Isolated Pole

Three straight lines provide a good approxmation versus log ω.

$$H(s) = \frac{9}{s - p_1}$$





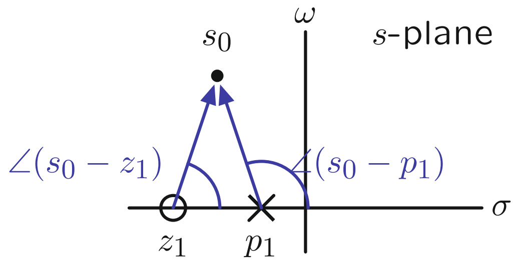
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = -\pi/2$$

نمودار بودی

The angle of a product is the sum of the angles.

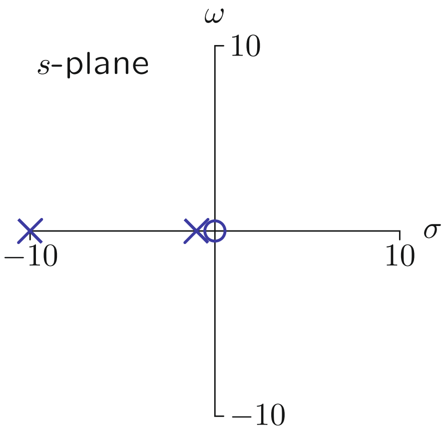
$$\angle H(s_0) = \angle \left(K \frac{\prod_{q=1}^{Q} (s_0 - z_q)}{\prod_{p=1}^{P} (s_0 - p_p)} \right) = \angle K + \sum_{q=1}^{Q} \angle (s_0 - z_q) - \sum_{p=1}^{P} \angle (s_0 - p_p)$$

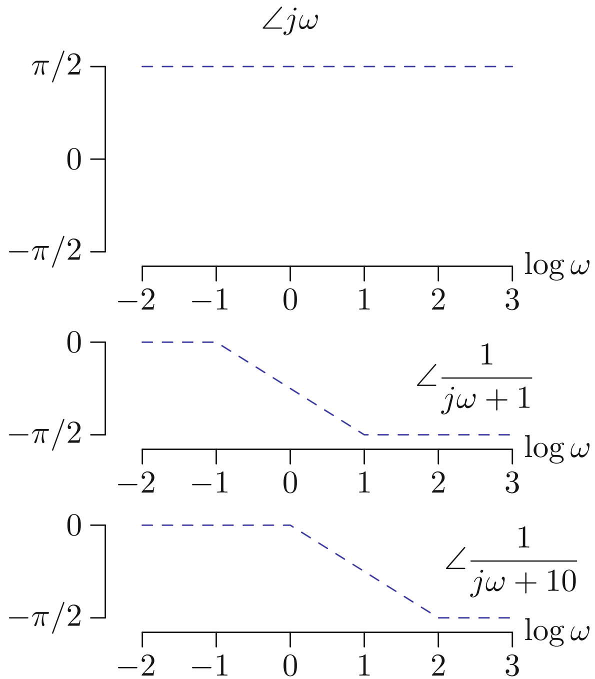


The angle of K can be 0 or π for systems described by linear differential equations with constant, real-valued coefficients.

نمودار بودی

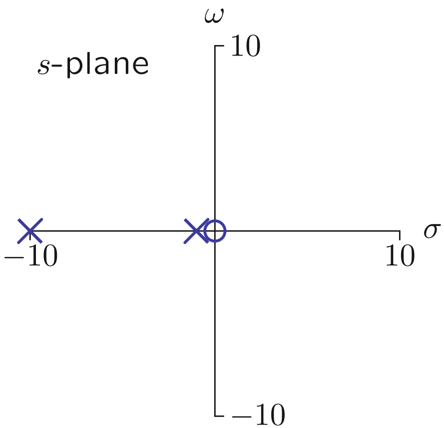
$$H(s) = \frac{s}{(s+1)(s+10)}$$

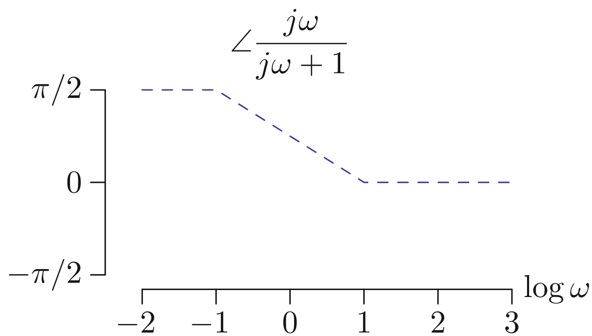


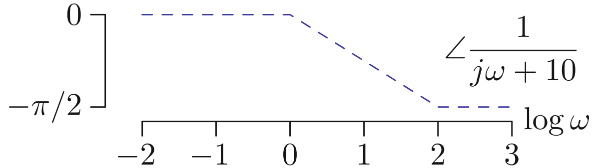


نمودار بودی

$$H(s) = \frac{s}{(s+1)(s+10)}$$

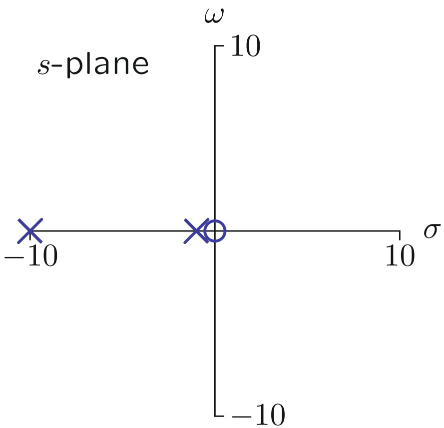


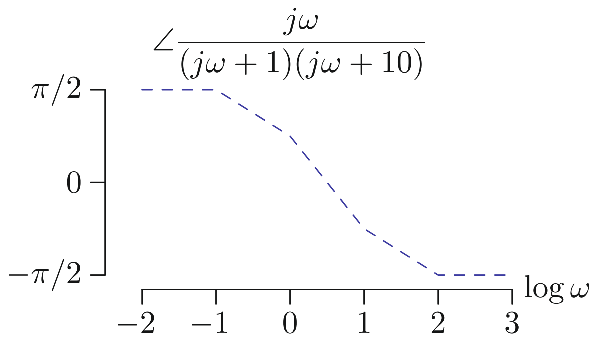




نمودار بودی

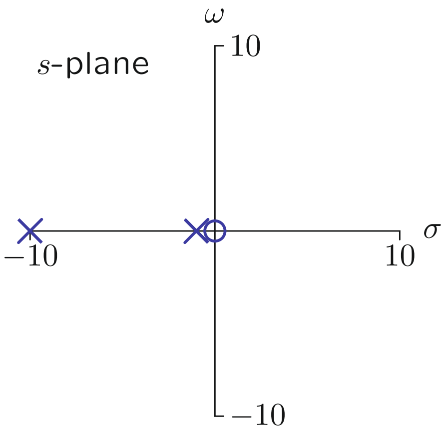
$$H(s) = \frac{s}{(s+1)(s+10)}$$

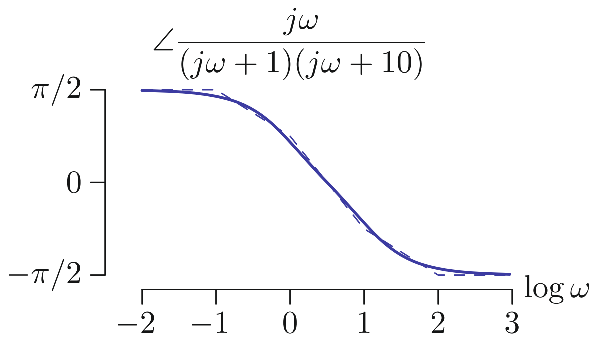




نمودار بودی

$$H(s) = \frac{s}{(s+1)(s+10)}$$





از پاسخ فرکانسی تا نمودار بودی

From Frequency Response to Bode Plot

The magnitude of $H(j\omega)$ is a product of magnitudes.

$$|H(j\omega)| = |K| \frac{\prod_{q=1}^{Q} |j\omega - z_q|}{\prod_{p=1}^{P} |j\omega - p_p|}$$

The log of the magnitude is a sum of logs.

$$\log|H(j\omega)| = \log|K| + \sum_{q=1}^{Q} \log|j\omega - z_q| - \sum_{p=1}^{P} \log|j\omega - p_p|$$

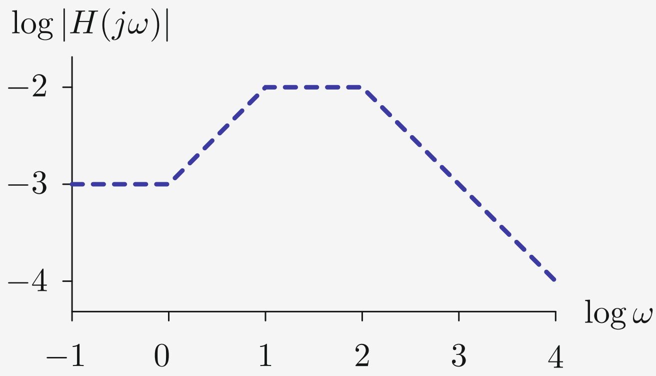
The angle of $H(j\omega)$ is a sum of angles.

$$\angle H(j\omega) = \angle K + \sum_{q=1}^{Q} \angle(j\omega - z_q) - \sum_{p=1}^{P} \angle(j\omega - p_p)$$

از پاسخ فرکانسی تا نمودار بودی

مثال

From Frequency Response to Bode Plot



Which corresponds to the Bode approximation above? 2

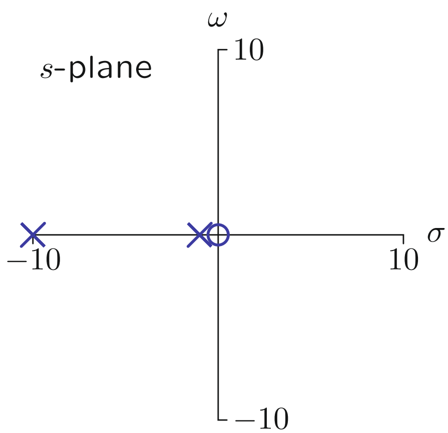
1. $\dfrac{1}{(s+1)(s+10)(s+100)}$
2. $\dfrac{s+1}{(s+10)(s+100)}$
3. $\dfrac{(s+10)(s+100)}{s+1}$
4. $\dfrac{s+100}{(s+1)(s+10)}$
5. none of the above

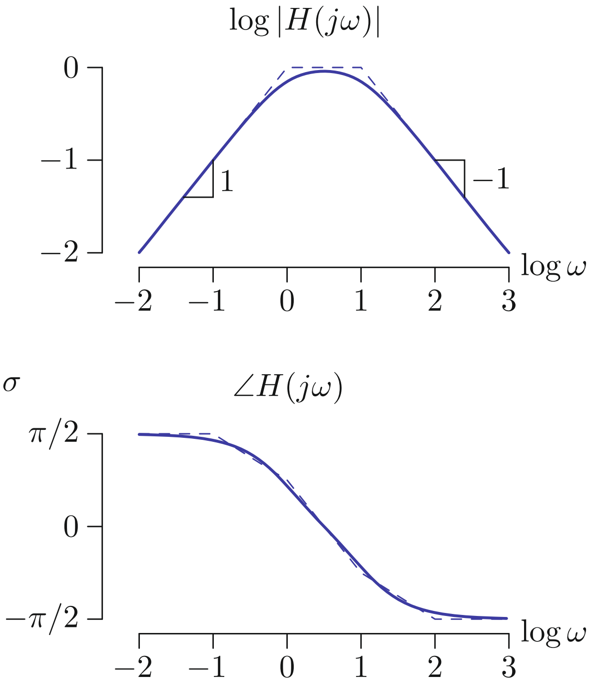
از پاسخ فرکانسی تا نمودار بودی

مثال (db: دسی‌بل) (۱ از ۴)

FROM FREQUENCY RESPONSE TO BODE PLOT

$$H(s) = \frac{10s}{(s+1)(s+10)}$$



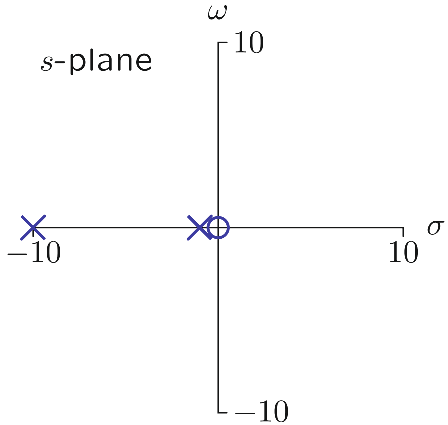


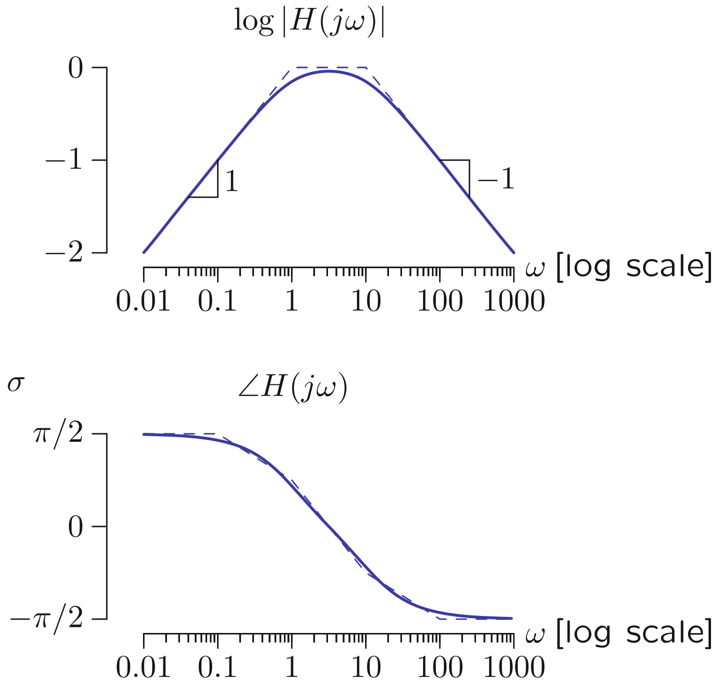
از پاسخ فرکانسی تا نمودار بودی

مثال (db: دسی‌بل) (۲ از ۴)

FROM FREQUENCY RESPONSE TO BODE PLOT

$$H(s) = \frac{10s}{(s+1)(s+10)}$$



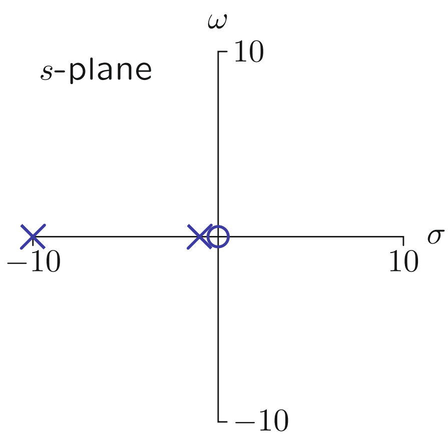


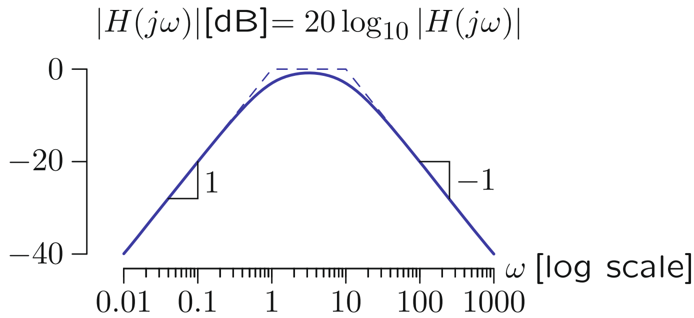
از پاسخ فرکانسی تا نمودار بودی

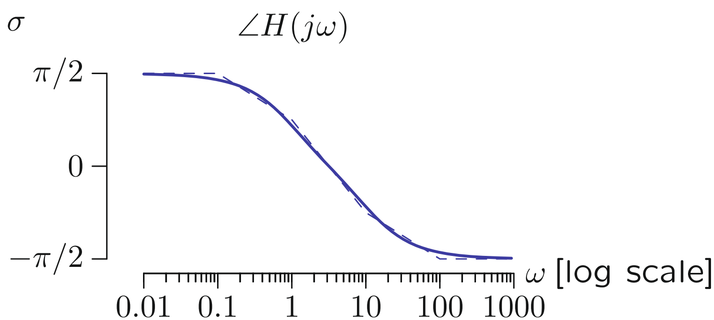
مثال (db: دسی‌بل) (۳ از ۴)

From Frequency Response to Bode Plot

$$H(s) = \frac{10s}{(s+1)(s+10)}$$





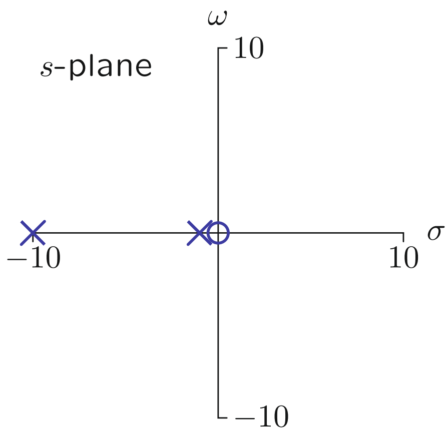


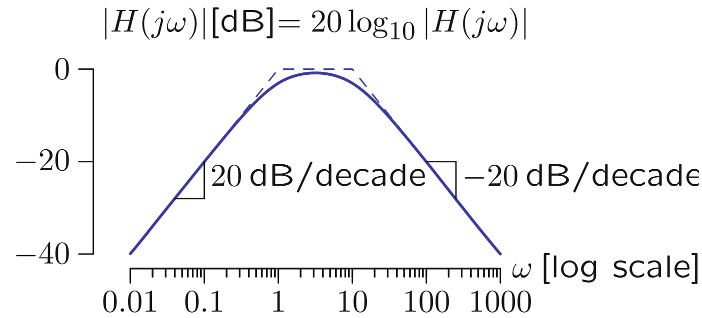
از پاسخ فرکانسی تا نمودار بودی

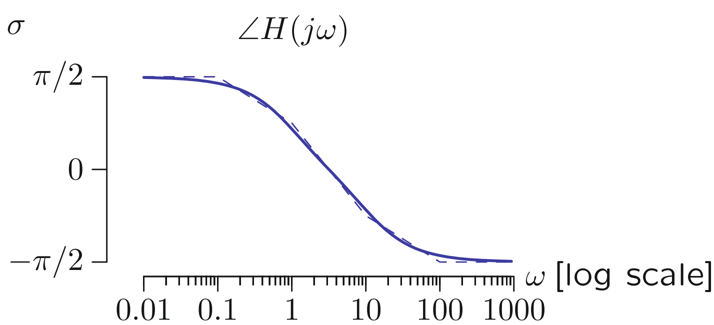
مثال (db: دسی‌بل) (۴ از ۴)

FROM FREQUENCY RESPONSE TO BODE PLOT

$$H(s) = \frac{10s}{(s+1)(s+10)}$$







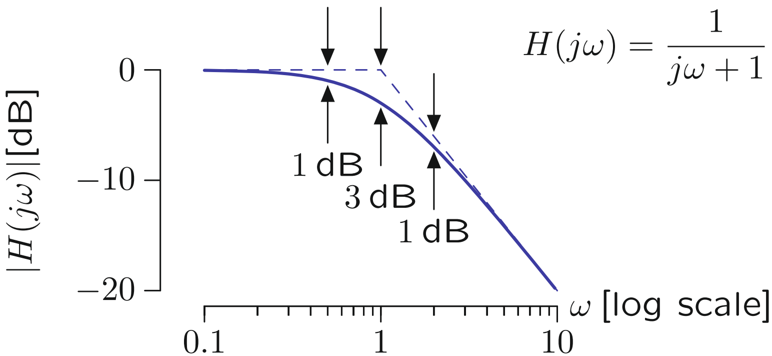
نمودار بودی

دقت

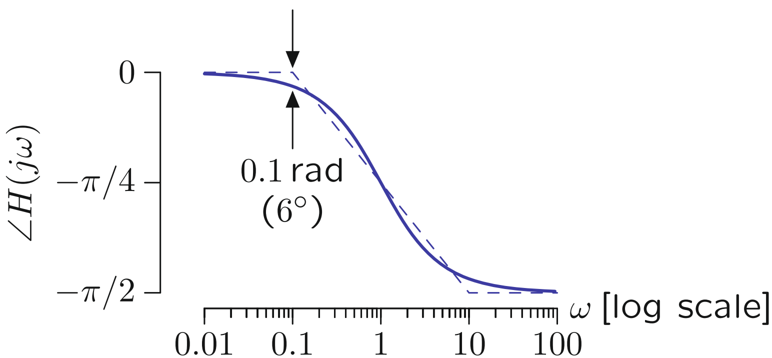
Bode Plot: Accuracy

The straight-line approximations are surprisingly accurate.

$$H(j\omega) = \frac{1}{j\omega + 1}$$



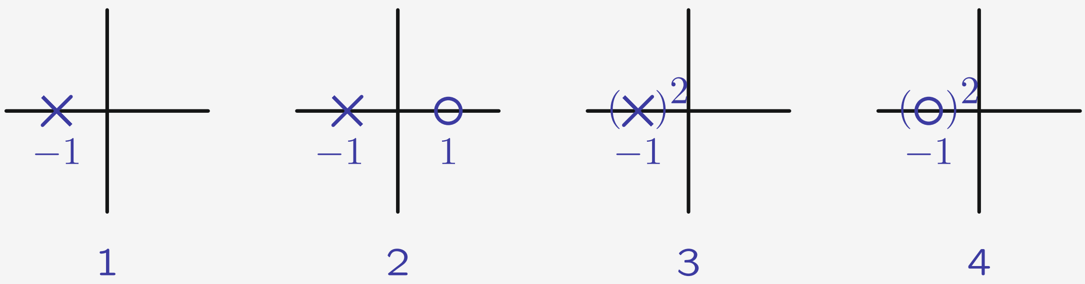
X	$20\log_{10} X$
1	0 dB
$\sqrt{2}$	≈ 3 dB
2	≈ 6 dB
10	20 dB
100	40 dB



نمودار بودی

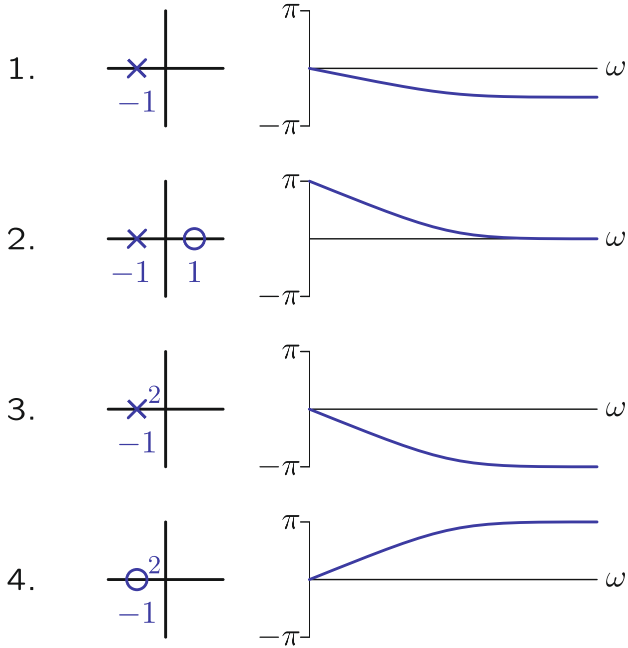
مثال (۱ از ۴)

Could the phase plots of any of these systems be equal to each other? [caution: this is a trick question]



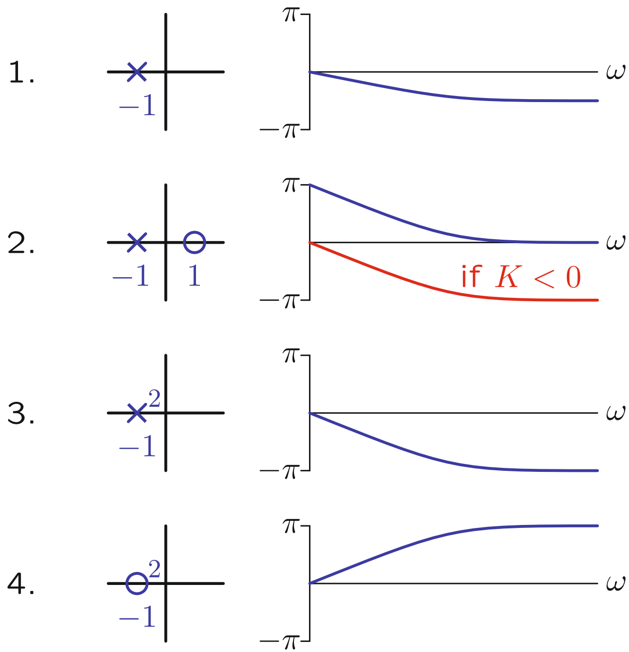
نمودار بودی

مثال (۲ از ۴)



نمودار بودی

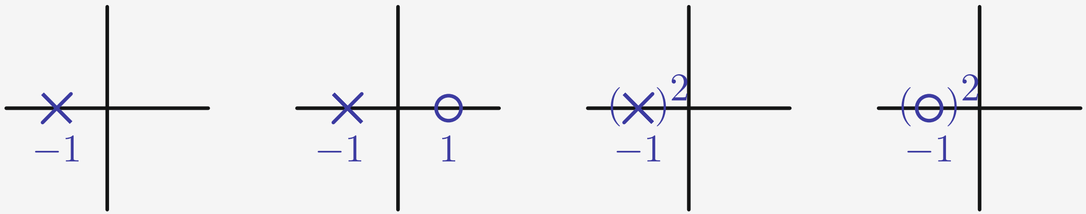
مثال (۳ از ۴)



نمودار بودی

مثال (۴ از ۴)

Could the phase plots of any of these systems be equal to each other? [caution: this is a trick question] yes



1 2 3 4

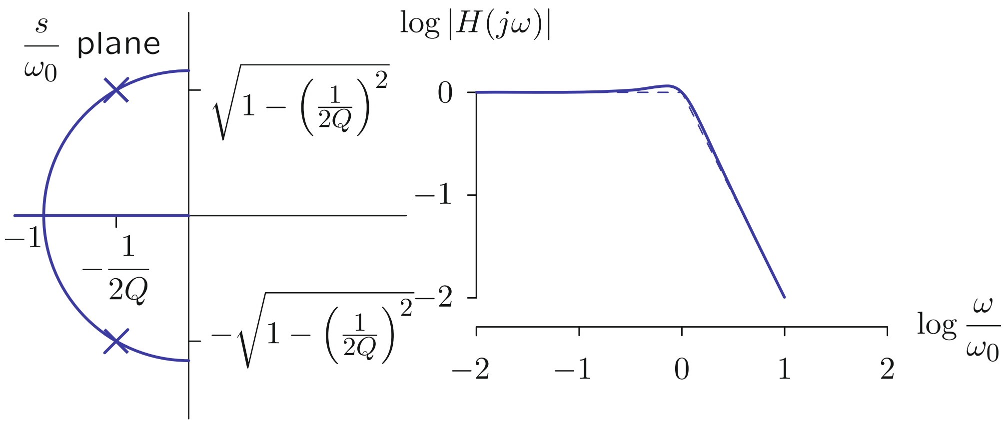
phase of 2 could be same as phase of 3: depends on sign of K

پاسخ فرکانسی یک سیستم با Q بالا

Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



پاسخ فرکانسی یک سیستم با Q بالا

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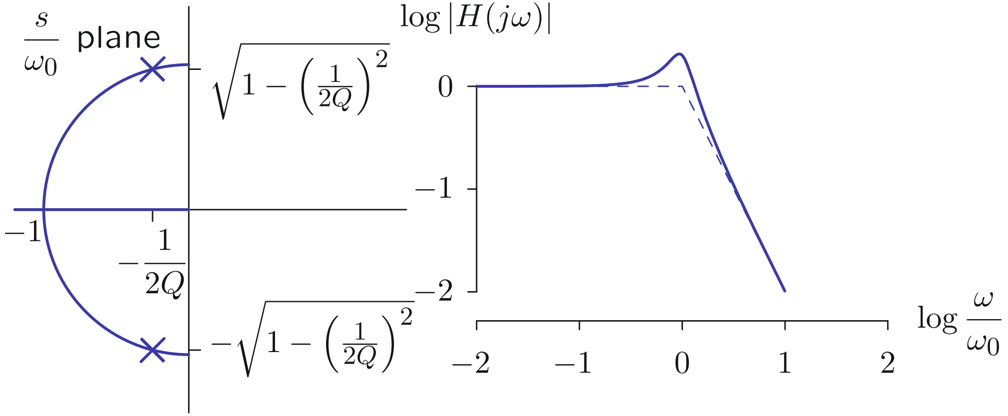

پاسخ فرکانسی یک سیستم با Q بالا

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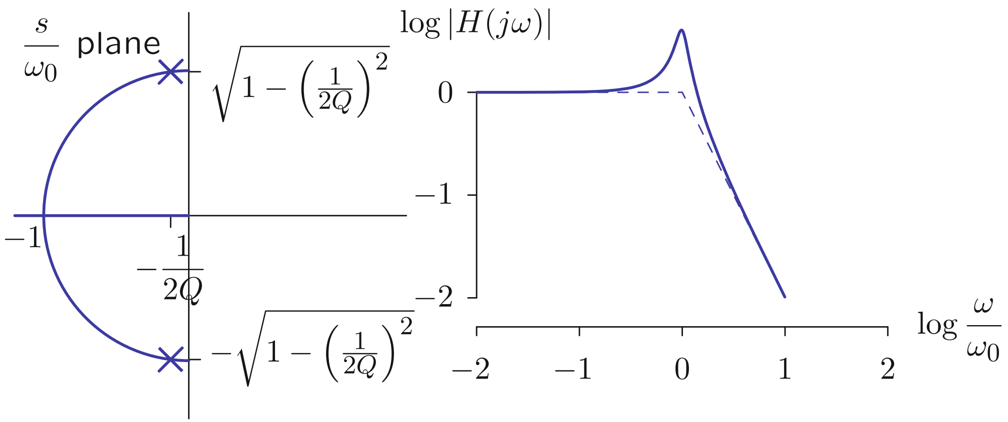

پاسخ فرکانسی یک سیستم با Q بالا

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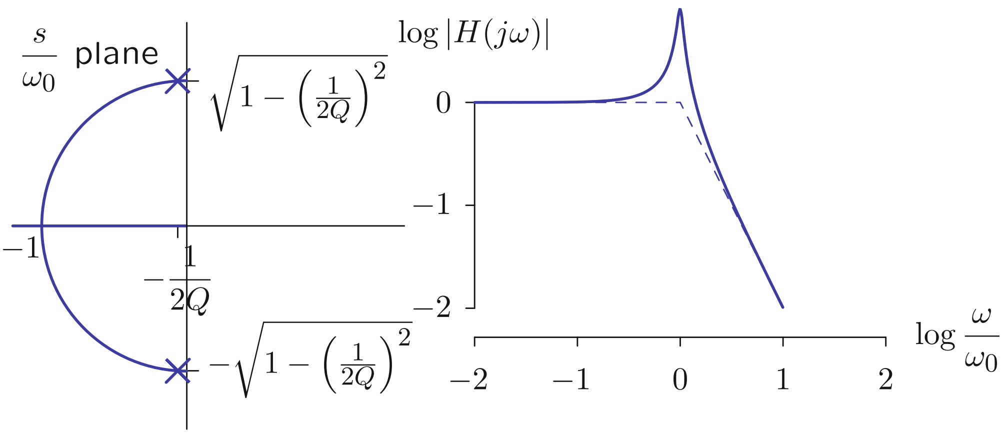

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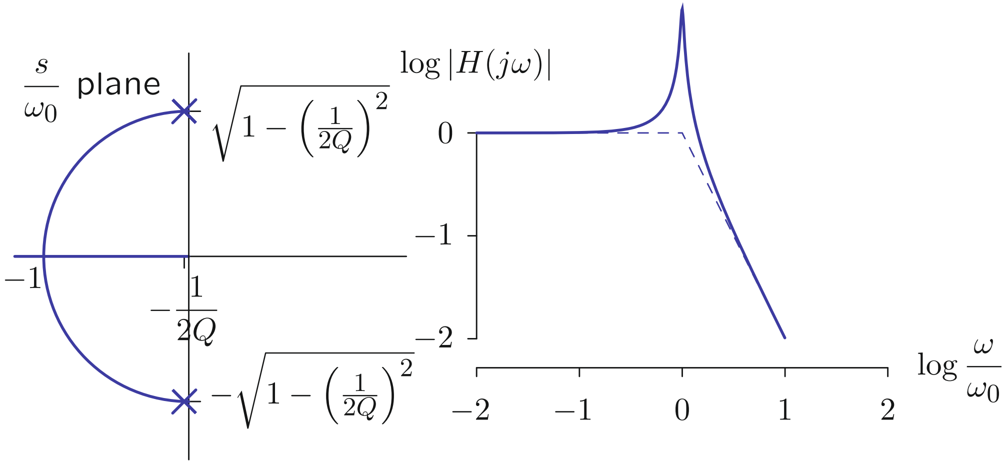


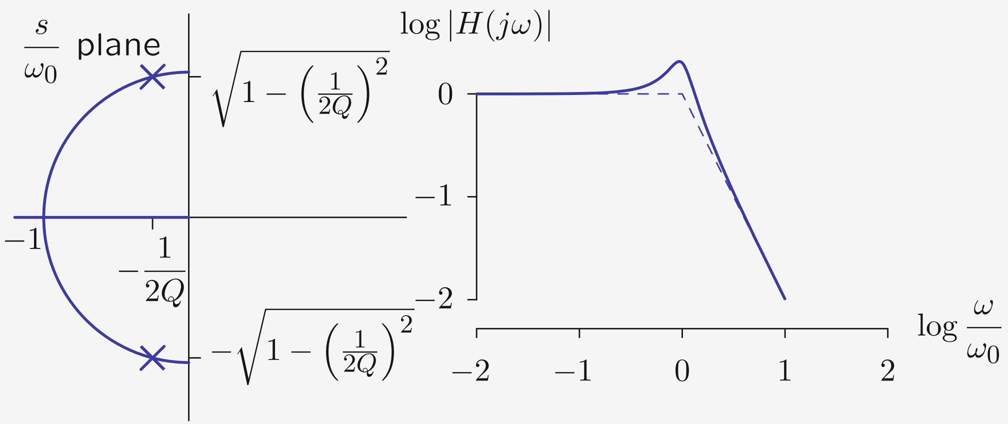
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۱ از ۷)

FREQUENCY RESPONSE OF A HIGH-Q SYSTEM

Find dependence of peak magnitude on Q (assume $Q > 3$).

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



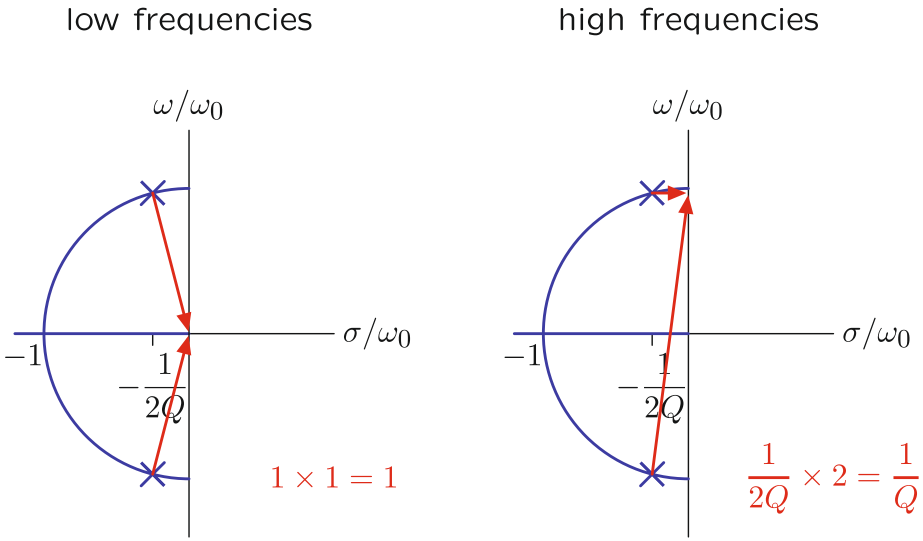
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۲ از ۷)

FREQUENCY RESPONSE OF A HIGH-Q SYSTEM

Find dependence of peak magnitude on Q (assume $Q > 3$).

Analyze with vectors.



Peak magnitude increases with Q!

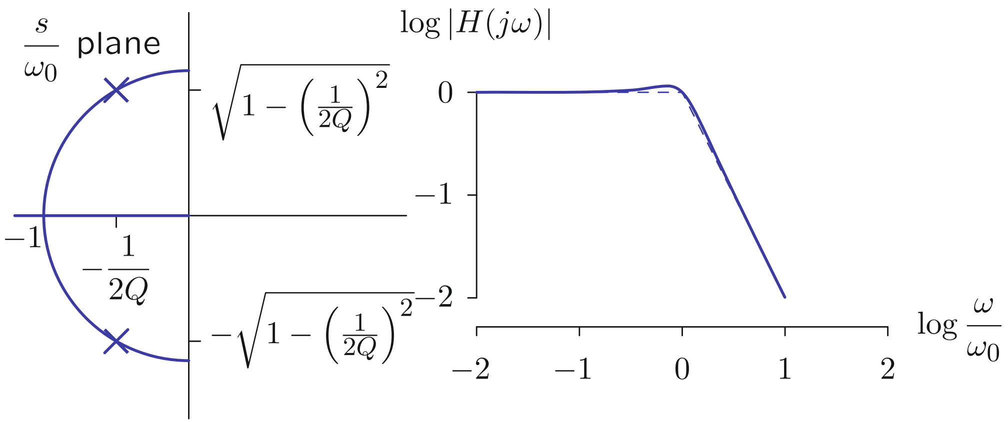
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۳ از ۷)

Frequency Response of a High-Q System

As Q increases, the width of the peak narrows.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



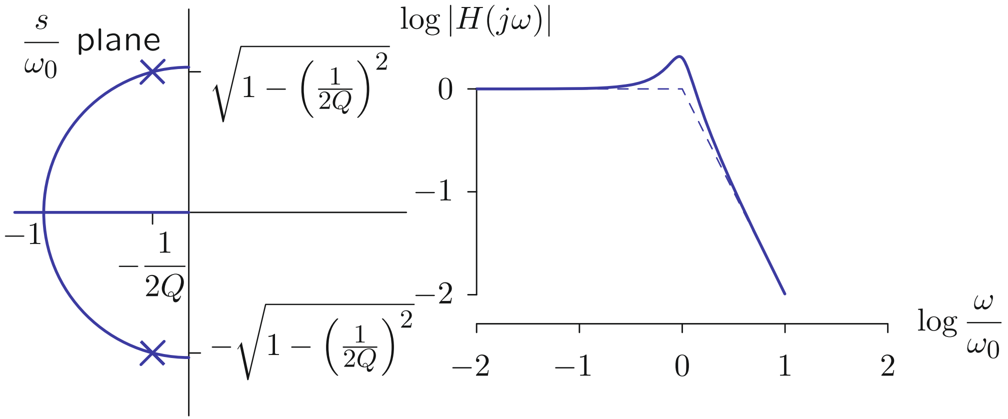
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۴ از ۷)

Frequency Response of a High-Q System

As Q increases, the width of the peak narrows.

$$H(s)=\frac{1}{1+\frac{1}{Q}\frac{s}{\omega_0}+\left(\frac{s}{\omega_0}\right)^2}$$



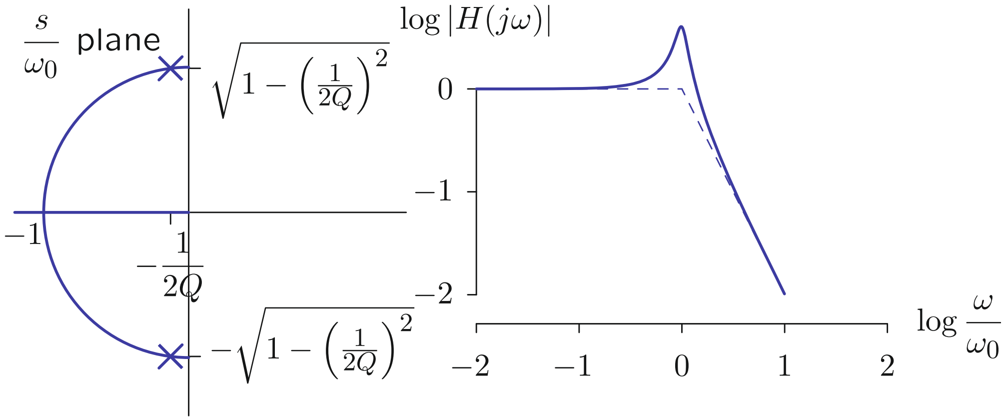
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۵ از ۷)

FREQUENCY RESPONSE OF A HIGH-Q SYSTEM

As Q increases, the width of the peak narrows.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



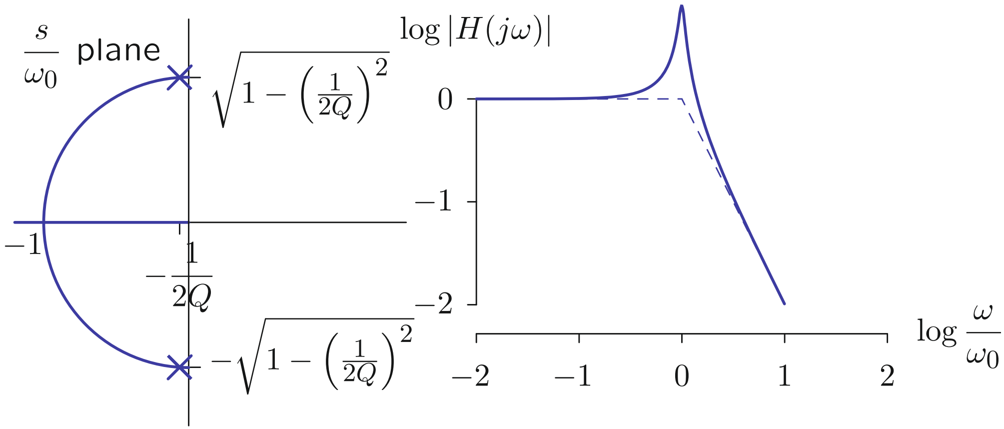
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۶ از ۷)

Frequency Response of a High-Q System

As Q increases, the width of the peak narrows.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



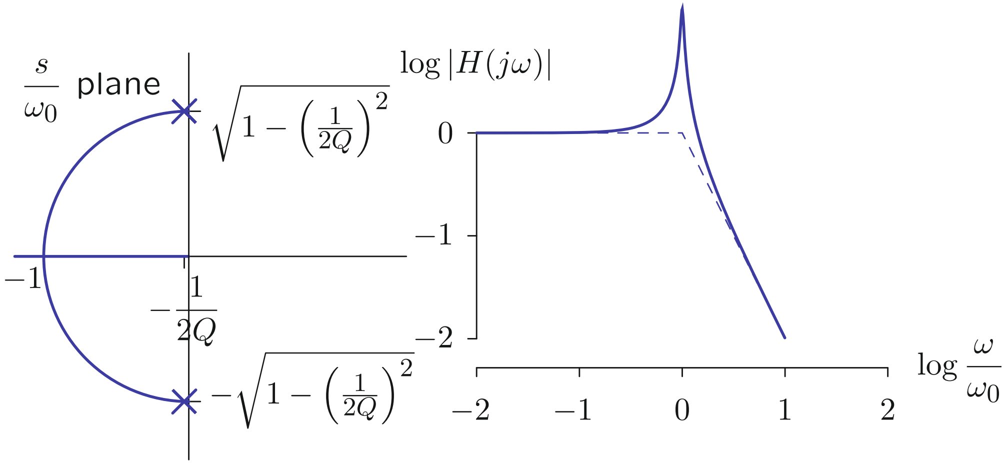
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۷ از ۷)

Frequency Response of a High-Q System

As Q increases, the width of the peak narrows.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



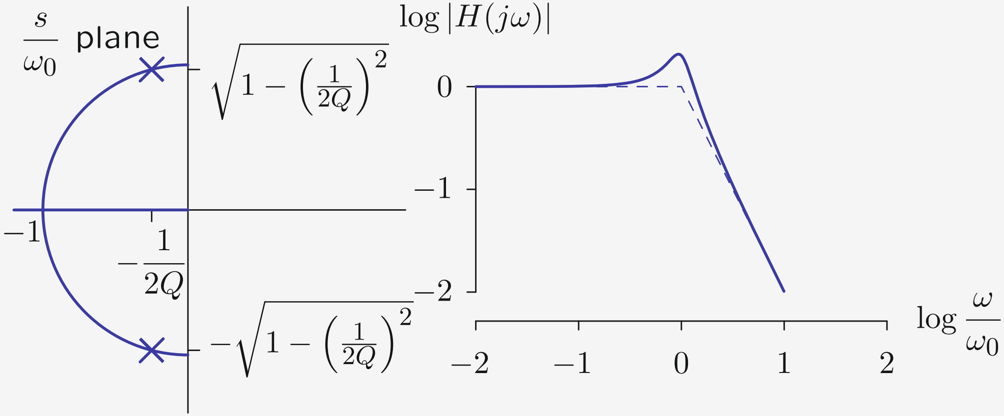
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۱ از ۷)

FREQUENCY RESPONSE OF A HIGH-Q SYSTEM

Estimate the "3dB bandwidth" of the peak (assume $Q > 3$).

Let ω_l (or ω_h) represent the lowest (or highest) frequency for which the magnitude is greater than the peak value divided by $\sqrt{2}$. The 3dB bandwidth is then $\omega_h - \omega_l$.



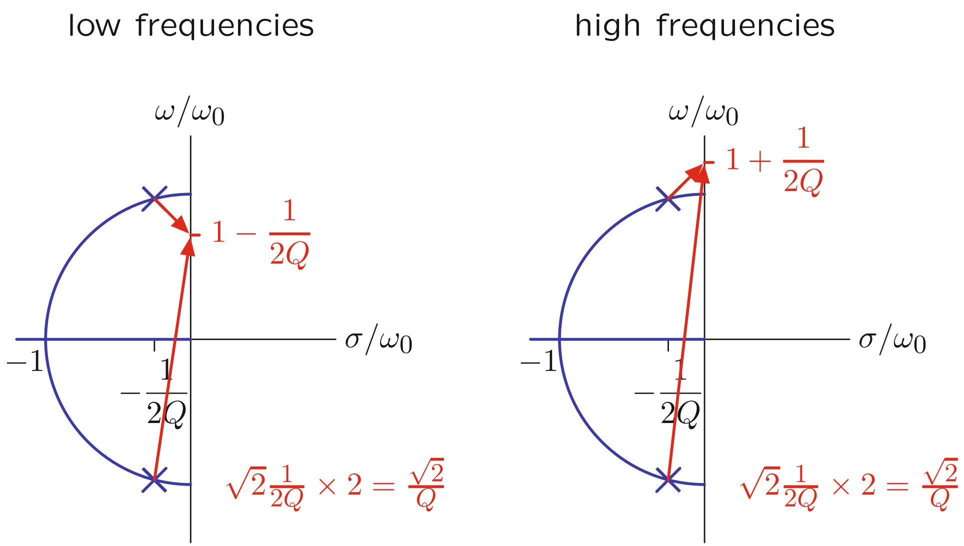
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۲ از ۷)

Frequency Response of a High-Q System

Estimate the "3dB bandwidth" of the peak (assume $Q > 3$).

Analyze with vectors.



Bandwidth approximately $\frac{1}{Q}$

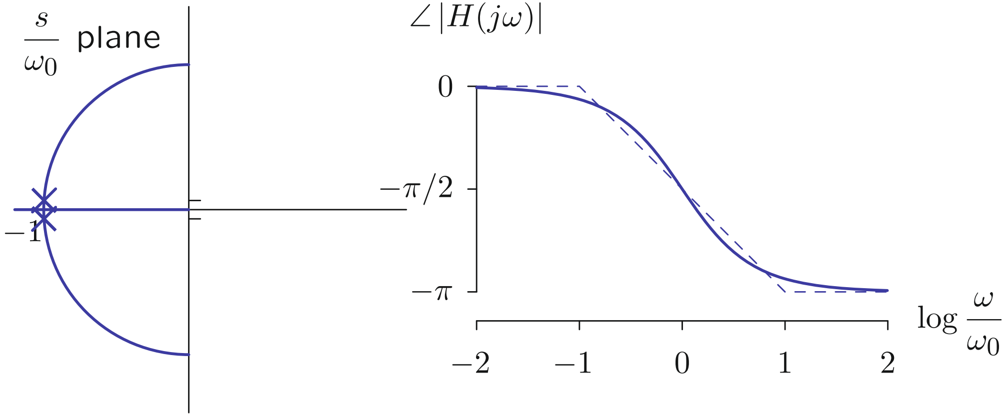
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۳ از ۷)

FREQUENCY RESPONSE OF A HIGH-Q SYSTEM

As Q increases, the phase changes more abruptly with ω.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



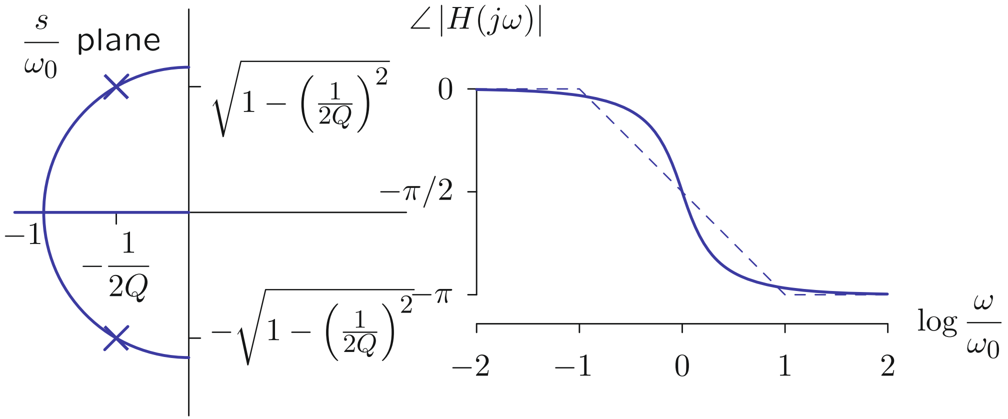
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۴ از ۷)

Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



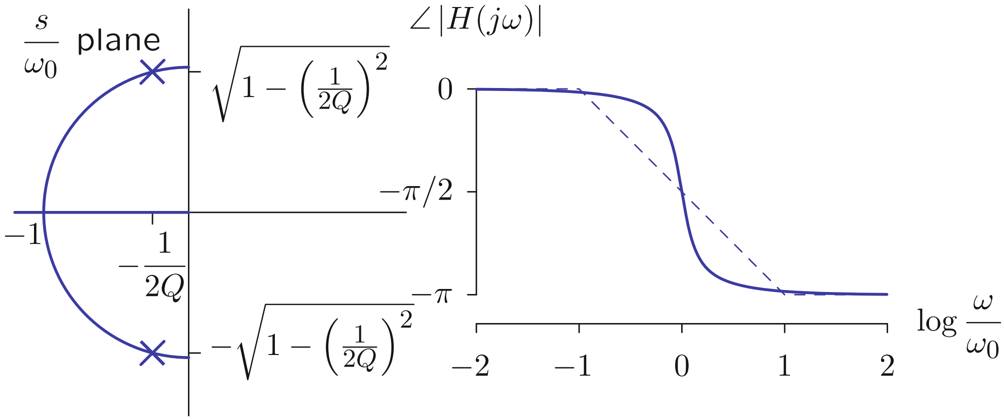
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۵ از ۷)

Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$H(s)=\frac{1}{1+\frac{1}{Q}\frac{s}{\omega_0}+\left(\frac{s}{\omega_0}\right)^2}$$



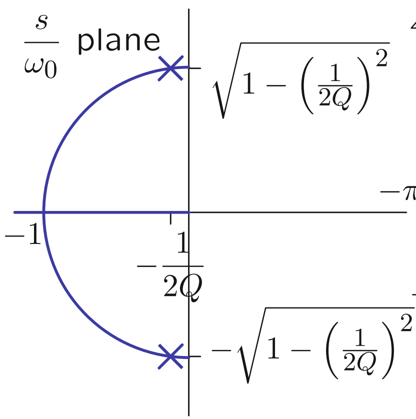
پاسخ فرکانسی یک سیستم با Q بالا

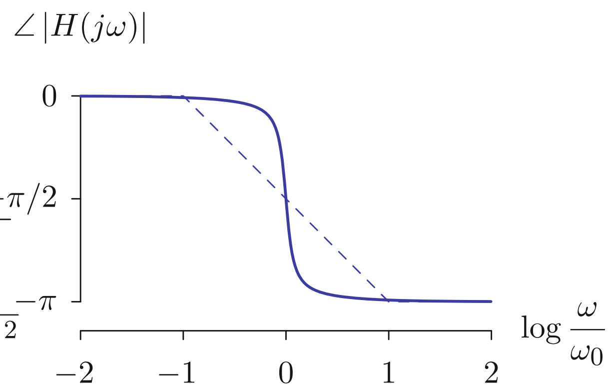
مثال (۶ از ۷)

FREQUENCY RESPONSE OF A HIGH-Q SYSTEM

As Q increases, the phase changes more abruptly with ω.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$





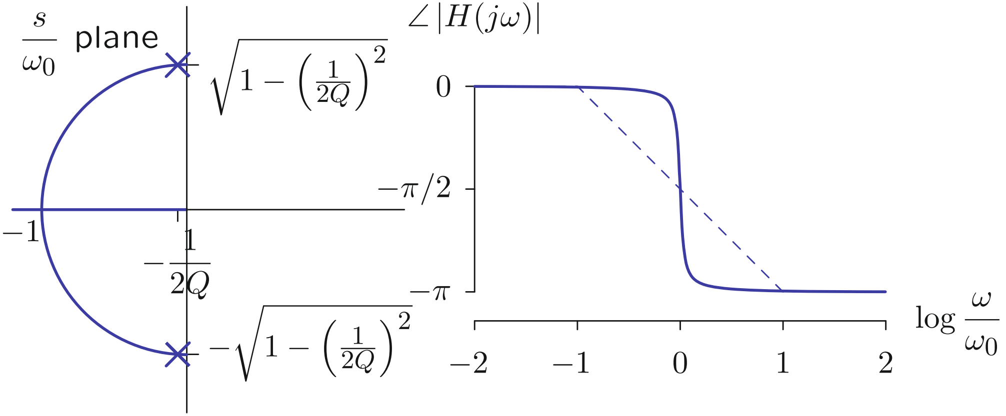
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۷ از ۷)

Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



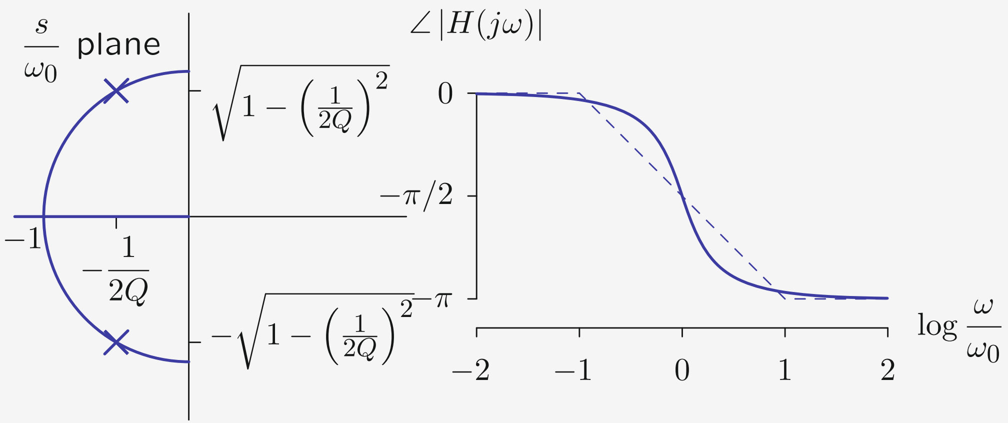
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۱ از ۲)

FREQUENCY RESPONSE OF A HIGH-Q SYSTEM

Estimate change in phase that occurs over the 3dB bandwidth.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



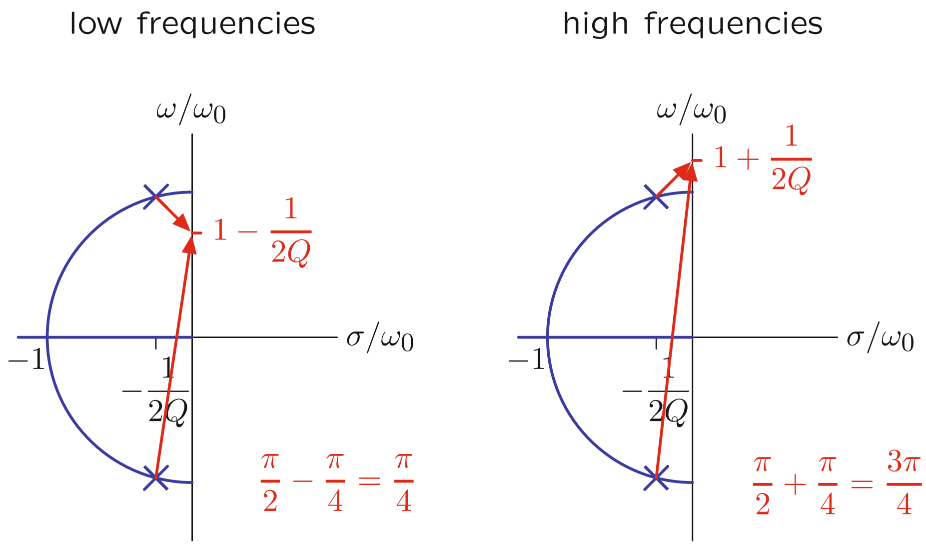
پاسخ فرکانسی یک سیستم با Q بالا

مثال (۲ از ۲)

Frequency Response of a High-Q System

Estimate change in phase that occurs over the 3dB bandwidth.

Analyze with vectors.



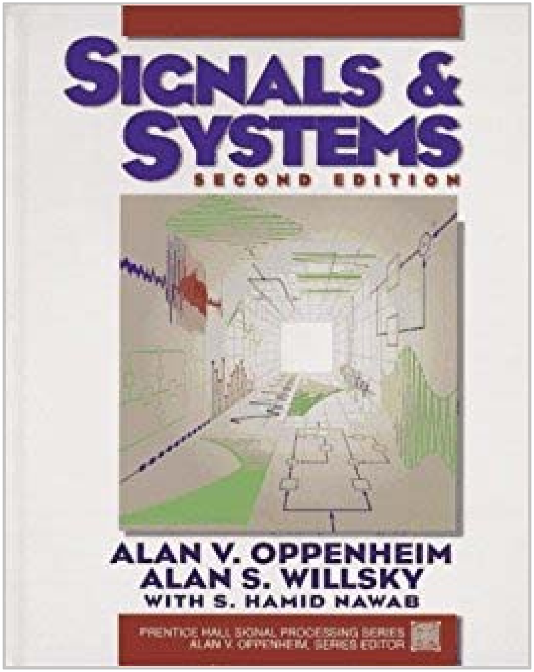
Change in phase approximately $\frac{\pi}{2}$.

مشخصه‌های زمانی و فرکانسی (۲)

۲

منابع

منبع اصلی



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Chapter 6