سیگنال‌ها و سیستم‌ها

درس ۱۸

مشخصه‌های زمانی و فرکانسی (۱)

Time and Frequency Characterization (1)

کاظم فولادی قلعه
دانشکده مهندسی، پردیس فارابی
دانشگاه تهران

http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

تبدیل‌های اندازه/ فاز و پاسخ‌های فرکانسی
Magnitude/Phase of Transforms and Frequency Responses
فاز خطی و غیرخطی
Linear and Nonlinear Phase
فیلترهای فرکانس-گزین ایده‌آل و غیر ایده‌آل
Ideal and Nonideal Frequency-Selective Filters
پاسخ‌های فرکانسی گویای پیوسته-زمان و گسسته-زمان
CT & DT Rational Frequency Responses
سیستم‌های مرتبه-اول و مرتبه-دوم گسسته-زمان
DT First- and Second-Order Systems

مشخصه‌های زمانی و فرکانسی (۱)

۱

تبدیل‌های اندازه/ فاز و پاسخ‌های فرکانسی

Magnitude and Phase of FT, and Parseval Relation

CT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$$

Parseval Relation:

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}\frac{1}{2\pi}\underbrace{|X(j\omega)|^2}_{\text{Energy density in }\omega} d\omega$$

DT:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

Parseval Relation:

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \int_{2\pi}\frac{1}{2\pi}|X(e^{j\omega})|^2 d\omega$$

Effects of Phase

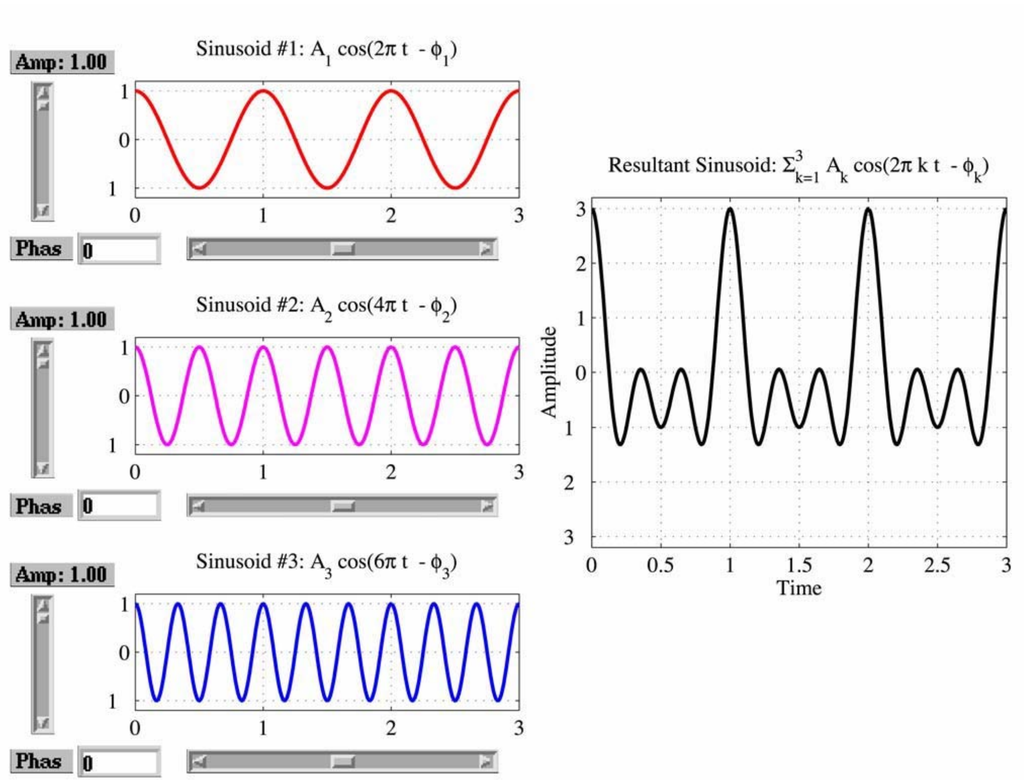
- *Not* on signal energy distribution as a function of frequency

- *Can* have dramatic effect on signal shape/character
  - Constructive/Destructive interference

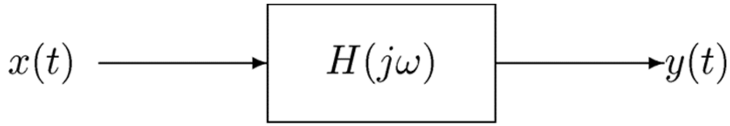
- Is that important?
  - Depends on the signal and the context

Demo: 1) Effect of phase on Fourier Series

2) Effect of phase on image processing



Log-Magnitude and Phase



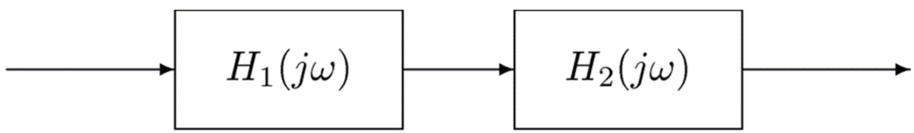
$$\Downarrow$$

$$|Y(j\omega)| = |H(j\omega)| \cdot |X(j\omega)|$$

or $$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

and $$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Cascading:



$$\left.\begin{array}{c} \log|H(j\omega)| = \log|H_1(j\omega)| + \log|H_2(j\omega)| \\ \angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega) \end{array}\right\} \text{ Easy to add}$$

Plotting Log-Magnitude and Phase

a) For real-valued signals and systems

$$\begin{aligned} |H(-j\omega)| &= |H(j\omega)| \\ \angle H(-j\omega) &= -\angle H(j\omega) \end{aligned} \Bigg\} \Rightarrow$$

Plot for ω ≥ 0, often with a *logarithmic* scale for frequency in CT

b) In DT, need only plot for $0 \leq \omega \leq \pi$ (with *linear* scale)

c) For historical reasons, log-magnitude is usually plotted in units of *decibels* (dB):

$$\left(1\ bel = 10\ decibels = \frac{\text{output power}}{\text{input power}} = 10\right)$$

power ↙ magnitude ↙

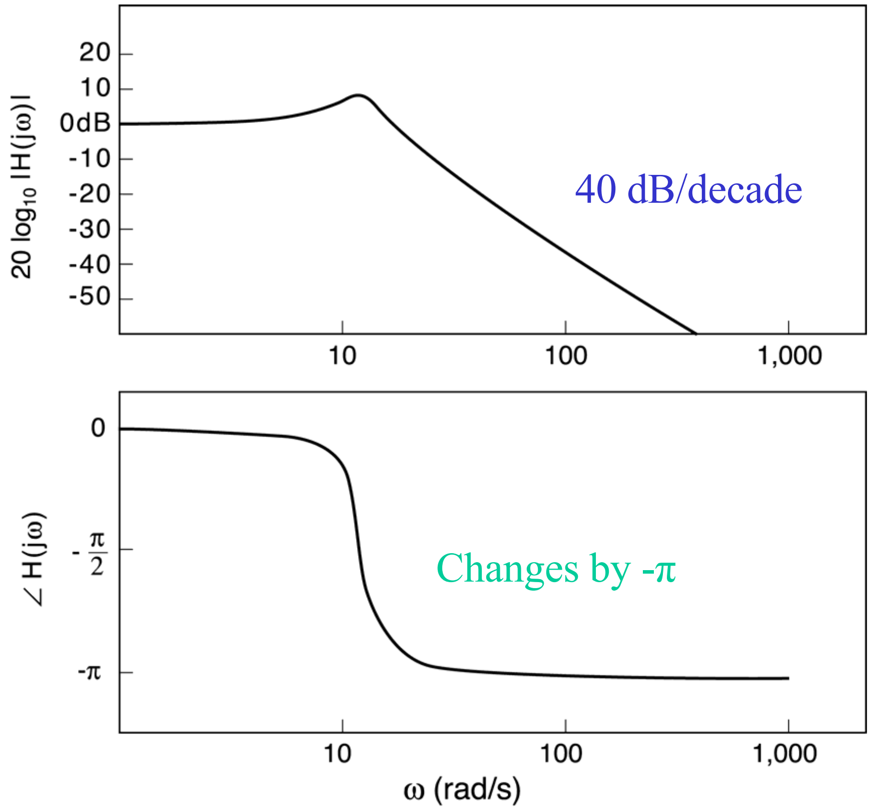
$$10\log_{10}|H(j\omega)|^2 = 20\log_{10}|H(j\omega)|$$

$$\begin{aligned} |H(j\omega)| = 1 &\longrightarrow 0\,\text{dB} \\ |H(j\omega)| = \sqrt{2} &\longrightarrow \sim 3\,\text{dB} \\ |H(j\omega)| = 2 &\longrightarrow \sim 6\,\text{dB} \\ |H(j\omega)| = 10 &\longrightarrow 20\,\text{dB} \\ |H(j\omega)| = 100 &\longrightarrow 40\,\text{dB} \end{aligned}$$

So… 20 dB or 2 *bels*:
= 10 amplitude gain
= 100 power gain

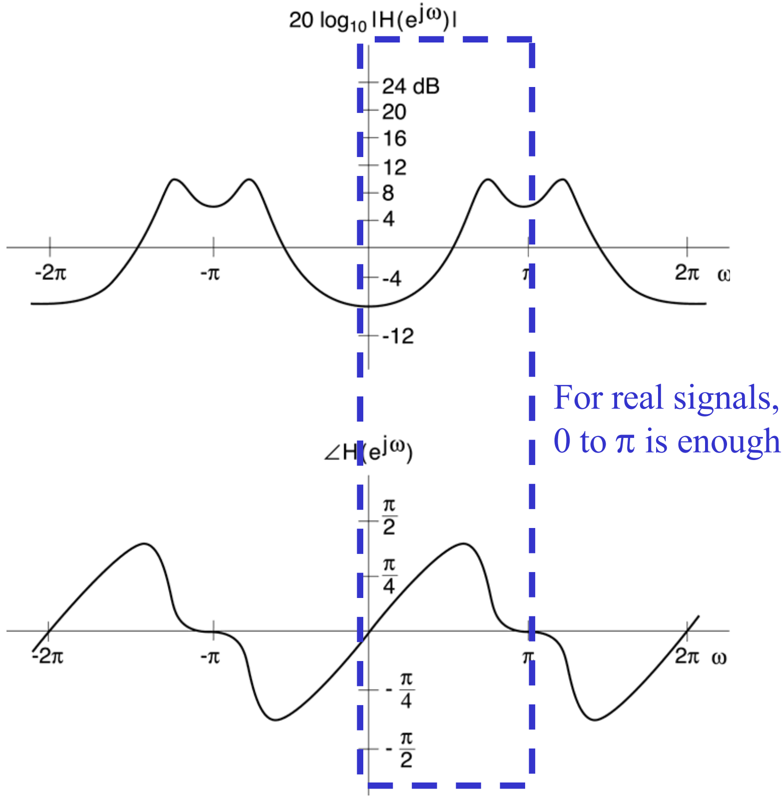
A Typical Bode plot for a second-order CT system

$$20\log|H(j\omega)| \quad \text{and} \quad \angle H(j\omega) \quad \text{vs.} \quad \log\omega$$



A typical plot of the magnitude and phase of a second- order DT frequency response

$20\log|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ vs. ω



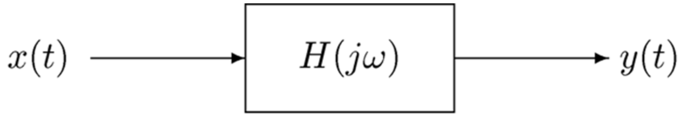
مشخصه‌های زمانی و فرکانسی (۱)

۲

فاز
خطی
و غیرخطی

Linear Phase

CT



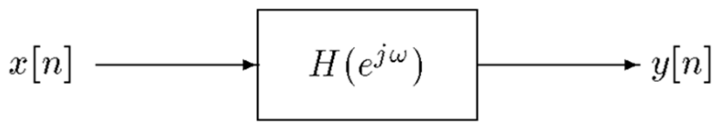
$$H(j\omega) = e^{-j\omega\alpha} \quad \Rightarrow \quad |H(j\omega)| = 1, \quad \angle H(j\omega) = -\alpha\omega \ (\text{Linear in } \omega)$$

$$Y(j\omega) = e^{-j\omega\alpha} X(j\omega) \overset{\text{time-shift}}{\longleftrightarrow} y(t) = x(t-\alpha)$$

Result: Linear phase ⇔ simply a rigid shift in time, *no distortion*

Nonlinear phase ⇔ distortion as well as shift

DT



$$y[n] = x[n-n_0] \longleftrightarrow Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

$$H(e^{j\omega}) = e^{-j\omega n_0} \quad \Rightarrow \quad |H(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = -n_0\omega$$

Question: What about $H(e^{j\omega}) = e^{-j\omega\alpha}$, $\alpha \neq$ integer ?

All-Pass Systems

$$\Rightarrow |H(j\omega)| = |H(e^{j\omega})| = 1$$

CT

$$H(j\omega) = e^{-j\alpha\omega} \quad \text{– Linear phase}$$

$$H(j\omega) = \frac{\alpha - j\omega}{\alpha + j\omega} \quad \text{– Nonlinear phase}$$

$$|H(j\omega)| = \sqrt{\frac{\alpha^2 + \omega^2}{\alpha^2 + \omega^2}} = 1$$

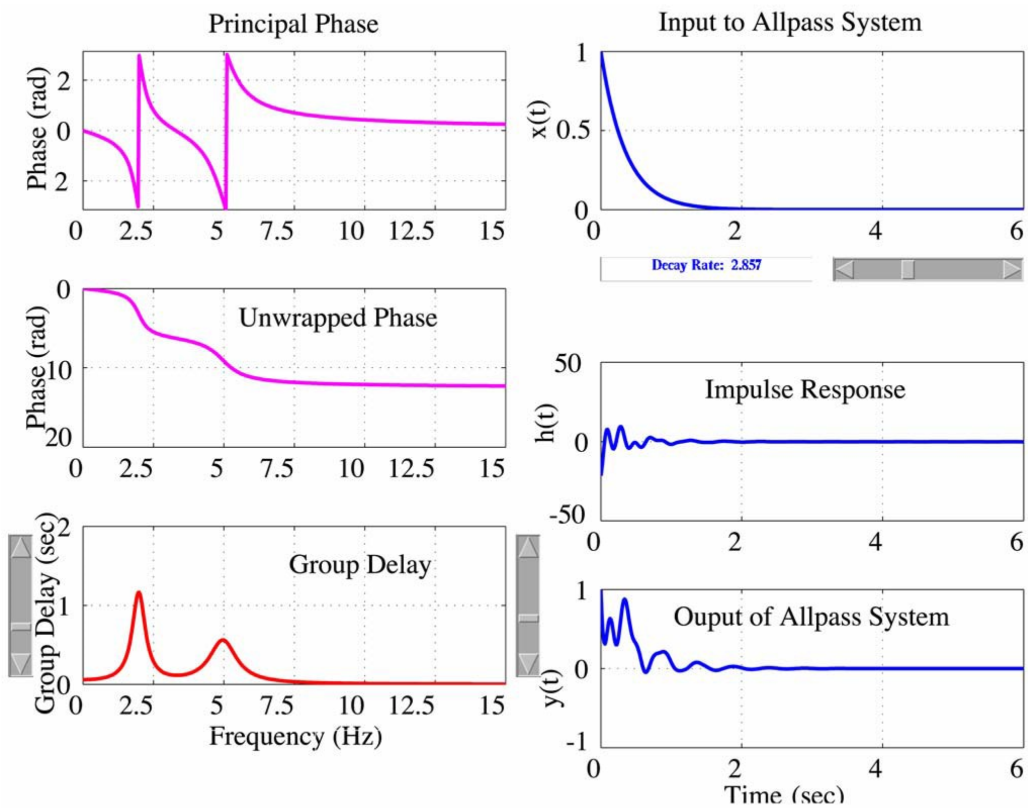
DT

$$H(e^{j\omega}) = e^{-j\omega n_0} \quad \text{– Linear phase}$$

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{j\omega}}{1 - \frac{1}{2}e^{-j\omega}} \quad \text{– Nonlinear phase}$$

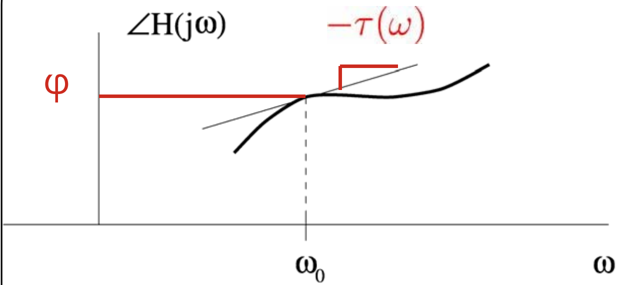
$$|H(e^{j\omega})| = \sqrt{\frac{(1 - 1/2 \cdot \cos\omega)^2 + (1/2 \cdot \sin\omega)^2}{(1 - 1/2 \cdot \cos\omega)^2 + (1/2 \cdot \sin\omega)^2}} = 1$$

Demo: Impulse response and output of an all-pass system with nonlinear phase



How do we think about signal delay when the phase is nonlinear?

Group Delay



When the signal is narrow-band and concentrated near ω_0, $\angle H(j\omega) \sim$ linear with ω near ω_0, then $-\frac{d\angle H(j\omega)}{d\omega}$ instead of $-\frac{\angle H(j\omega)}{\omega}$ reflects the time delay.

For frequencies "near" ω_0

$$\angle H(j\omega) \approx \angle H(j\omega_0) - \tau(\omega_0)(\omega - \omega_0) = \phi - \tau(\omega_0) \cdot \omega$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\} = \text{ Group Delay}$$

$$\Downarrow$$

For ω near ω_0

$$H(j\omega) \approx |H(j\omega_0)| e^{j\phi} e^{-j\tau(\omega_0)\omega}$$

$$\Rightarrow \quad e^{j\omega t} \longrightarrow \sim |H(j\omega)| e^{j\phi} e^{j\omega(t-\tau(\omega_0))}$$

مشخصه‌های زمانی و فرکانسی (۱)

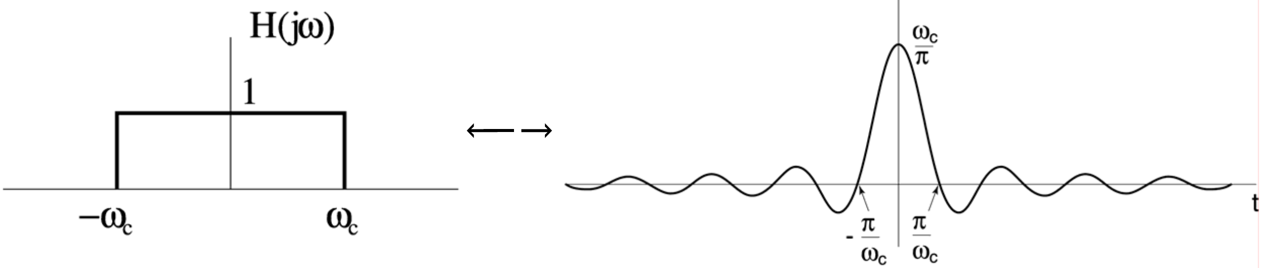
۳

فیلترهای فرکانس–گزین ایده‌آل و غیر ایده‌آل

Ideal Lowpass Filter

CT

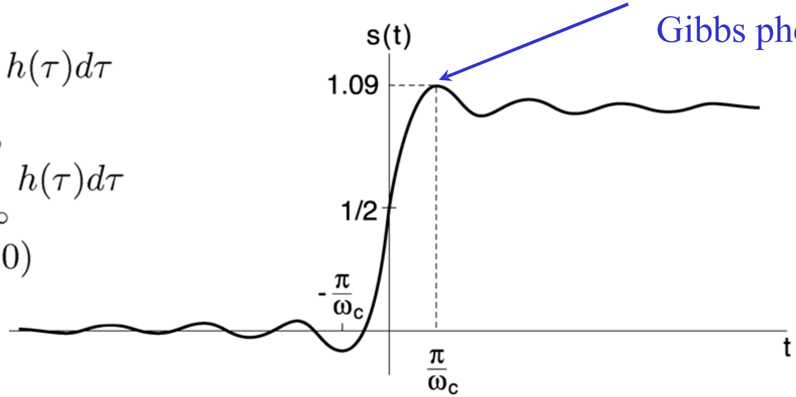
$$h(t) = \frac{\sin \omega_c t}{\pi t}$$



- Noncausal $h(t < 0) \neq 0$
- Oscillatory Response — e.g. step response

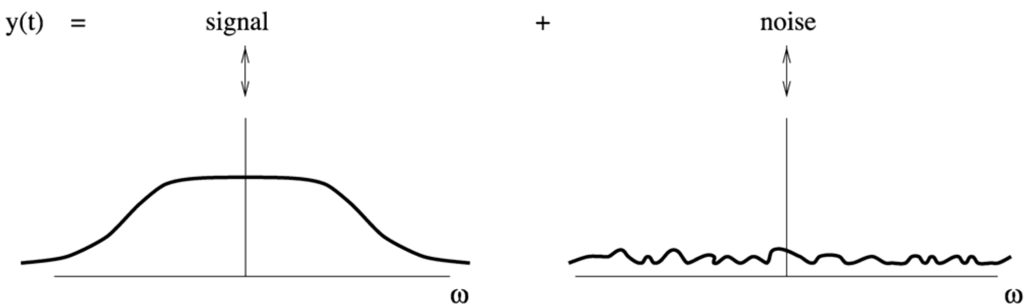
$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$s(\infty) = \int_{-\infty}^{\infty} h(\tau)d\tau = H(j0) = 1$$

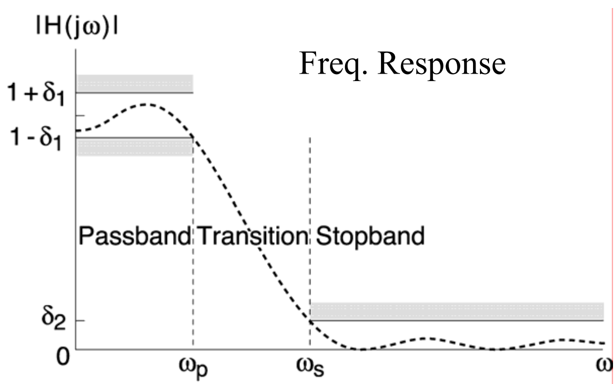


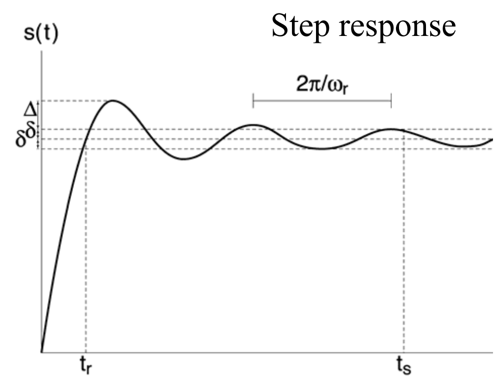
Nonideal Lowpass Filter

- Sometimes we don't *want* a sharp cutoff, e.g.



- Often have specifications in time and frequency domain ⇒ Trade-offs





مشخصه‌های زمانی و فرکانسی (۱)

۴

پاسخ‌های فرکانسی گویای پیوسته-زمان و گسسته-زمان

CT Rational Frequency Responses

CT: If the system is described by LCCDEs, then

$$\frac{d^k}{dt^k} \longleftrightarrow (j\omega)^k$$

$$H(j\omega) = \frac{\sum_k b_k (j\omega)^k}{\sum_k a_k (j\omega)^k} = \prod_i H_i(j\omega)$$

$$H_i(j\omega) = \text{ First- or Second-order factors}$$

Prototypical Systems

$$H_1(j\omega) = \frac{1}{j\omega\tau + 1}$$

— First-order system, has only *one* energy storing element, *e.g. L or C*

$$H_2(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + 1} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

— Second-order system, has *two* energy storing elements, *e.g. L and C*

DT Rational Frequency Responses

If the system is described by LCCDE's (Linear-Constant-Coefficient Difference Equations), then

$$y[n-k] \longleftrightarrow Y(e^{j\omega})e^{-jk\omega}, \quad x[n-k] \longleftrightarrow X(e^{j\omega})e^{-jk\omega}$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{\sum_k b_k e^{-jk\omega}}{\sum_k a_k e^{-jk\omega}} = \frac{\sum_k b_k (e^{-j\omega})^k}{\sum_k a_k (e^{-j\omega})^k} \\ &= \prod_i H_i(e^{j\omega}) \end{aligned}$$

$$H_i(e^{j\omega}) = \text{ First- or Second-order}$$

مشخصه‌های زمانی و فرکانسی (۱)

۵

سیستم‌های مرتبه–اول و مرتبه–دوم گسسته–زمان

DT First-Order Systems

$$y[n] - ay[n-1] = x[n], |a| < 1, \text{ initial rest}$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

Frequency domain

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

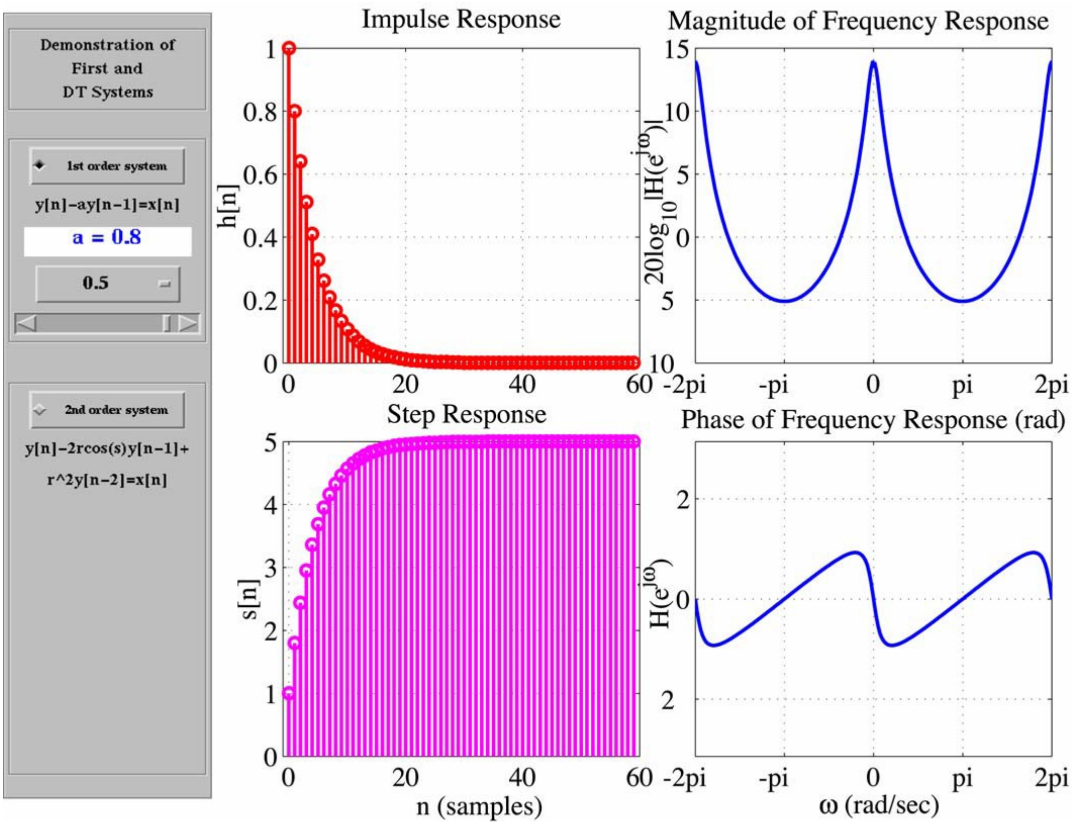
$$\angle H(e^{j\omega}) = -\tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right]$$

Time Domain

$$h[n] = a^n u[n]$$

$$s[n] = h[n] * u[n] = \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} u[n]$$

Demo: Unit-sample, unit-step, and frequency response of DT first-order systems



DT Second-Order System

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n], \quad 0 < r < 1 \text{ and } 0 \le \theta \le \pi$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1-(2r\cos\theta)e^{-j\omega} + r^2 e^{-j2\omega}} \\ &= \frac{1}{1-(re^{j\theta})e^{-j\omega}} \cdot \frac{1}{1-(re^{-j\theta})e^{-j\omega}} \\ &= \frac{A_1}{1-(re^{j\theta})e^{-j\omega}} + \frac{A_2}{1-(re^{-j\theta})e^{-j\omega}} \end{aligned}$$

$\Downarrow$ PFE

$$h[n] = [A_1 r^n e^{jn\theta} + A_2 r^n e^{-jn\theta}]u[n]$$

decaying

$$= \left[\frac{r^n \sin(n+1)\theta}{\sin\theta}\right] u[n]$$

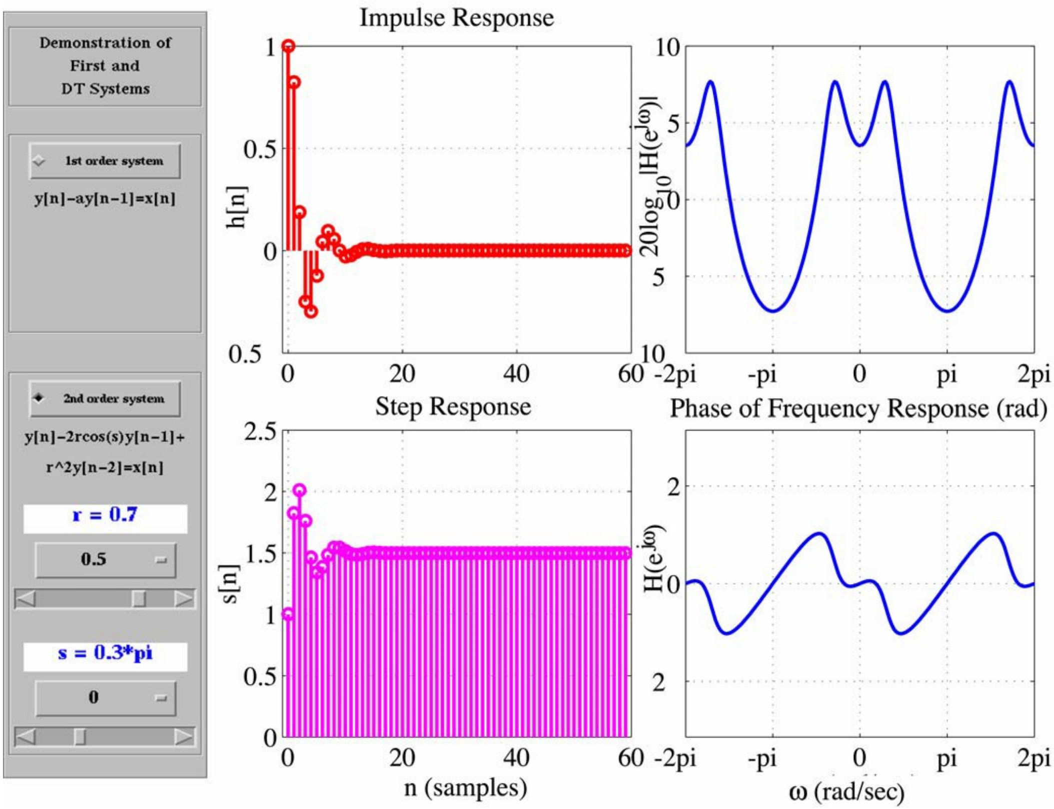
oscillations

where $A_1 = \frac{e^{j\theta}}{2j\sin\theta}$

$A_2 = -\frac{e^{-j\theta}}{2j\sin\theta}$.

$$s[n] = h[n] * u[n]$$

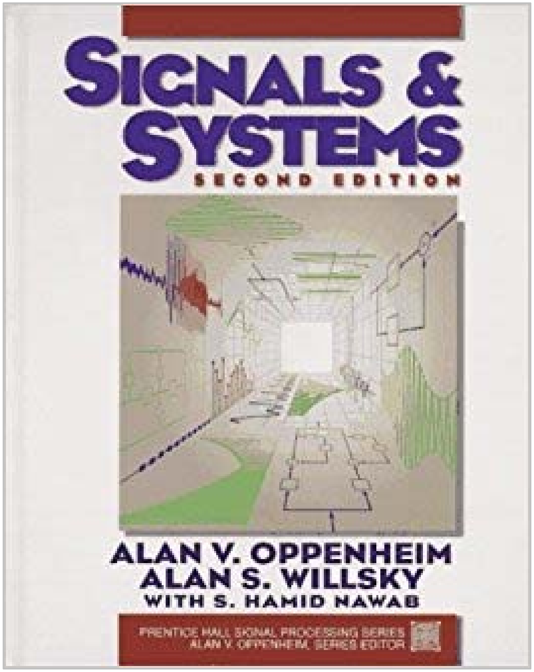
Demo: Unit-sample, unit-step, and frequency response of DT second-order systems



۶

منابع

منبع اصلی



A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
Signals and Systems,
Second Edition, Prentice Hall, 1997.

Chapter 6