

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ



# سیگنال‌ها و سیستم‌ها

درس ۱۰

## بازنمایی سری فوریه سیگنال‌های متناوب (۲)

### Fourier Series Representation of Periodic Signals (2)

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## طرح درس

COURSE OUTLINE

یادآوری سری فوریه پیوسته-زمان، خواص و مثال‌ها

CT Fourier series reprise, properties, and examples

سری فوریه‌ی گسسته-زمان

DT Fourier series

مثال‌های سری فوریه‌ی گسسته-زمان و تفاوت‌های آن با سری فوریه‌ی پیوسته زمان

DT Fourier series examples and differences with CTFS

بازنمایی سری فوریه سیگنال‌های متناوب (۲)

۱

یادآوری  
سری فوریه  
پیوسته-  
زمان،  
خواص و  
مثال‌ها

## سری فوریه‌ی پیوسته-زمان

$$a_k = \frac{1}{T} \int_T x(t) e^{-j \frac{2\pi}{T} kt} dt$$

(“analysis” equation)

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi}{T} kt}$$

(“synthesis” equation)

## CT Fourier Series Pairs

$$\left( \omega_0 = \frac{2\pi}{T} \right)$$

Review:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

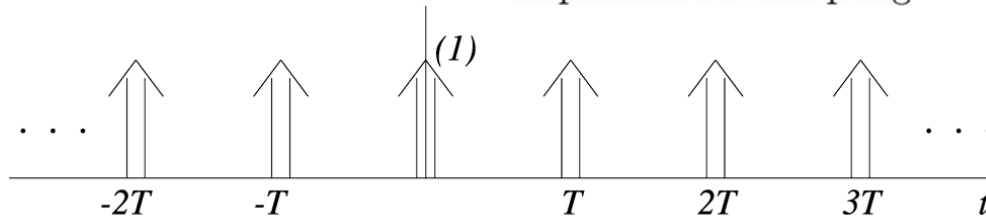
 **Skip it in future  
for shorthand**

$$x(t) \xleftrightarrow{FS} a_k$$

## Another (important!) example: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad - \quad \text{Sampling function}$$

important for sampling



$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \quad \text{for all } k ! \end{aligned}$$

↓

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

— All components have:

- (1) the same amplitude,  
&
- (2) the same phase.

## (A few of the) Properties of CT Fourier Series

- Linearity  $x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k \Rightarrow \alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$
- Conjugate Symmetry

$$x(t) \text{ is real} \Rightarrow a_{-k} = a_k^*$$

$\Downarrow$

$$\begin{aligned} a_k &= \operatorname{Re}\{a_k\} + j\operatorname{Im}\{a_k\} \\ &= |a_k|e^{j\angle a_k} \end{aligned}$$

$\operatorname{Re}\{a_k\}$  is even,  $\operatorname{Im}\{a_k\}$  is odd

or

$|a_k|$  is even,  $\angle a_k$  is odd

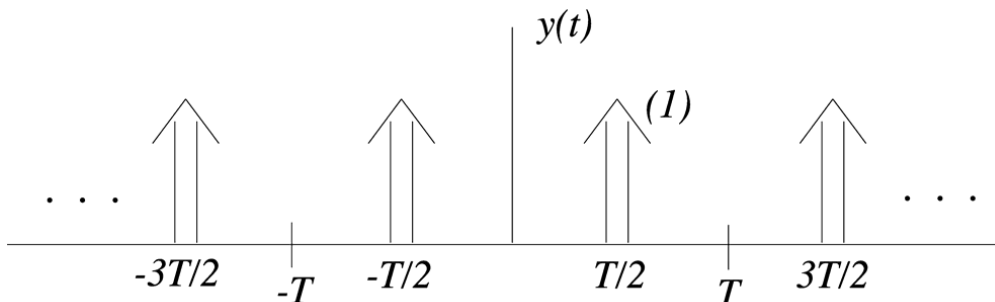
- Time shift
- $$\begin{aligned} x(t) &\leftrightarrow a_k \\ x(t - t_0) &\leftrightarrow a_k e^{-jk\omega_0 t_0} = a_k e^{-jk2\pi t_0/T} \end{aligned}$$

Introduces a linear phase shift  $\propto t_0$

**Example: Shift by half period**

$$y(t) = x(t - T/2) \leftrightarrow a_k e^{-jk\pi} = (-1)^k a_k$$

$$\text{using } e^{-jk\omega_0 T/2} = e^{-jk\pi}$$



$$y(t) \leftrightarrow (-1)^k a_k \left( a_k = \frac{1}{T} = \text{F.C. of } \sum_{n=-\infty}^{\infty} \delta(t - nT) \right)$$

$$\parallel$$

$$\frac{(-1)^k}{T}$$



- **Parseval's Relation**

$$\underbrace{\frac{1}{T} \int_T |x(t)|^2 dt}_{\text{Average signal power}} = \sum_{k=-\infty}^{\infty} \underbrace{|a_k|^2}_{\text{Power in the } k_{th} \text{ harmonic}}$$

Energy is the same whether measured in the time-domain or the frequency-domain

- **Multiplication Property**

$$x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k \quad (\text{Both } x(t) \text{ and } y(t) \text{ are periodic with the same period } T)$$

$$\Downarrow$$

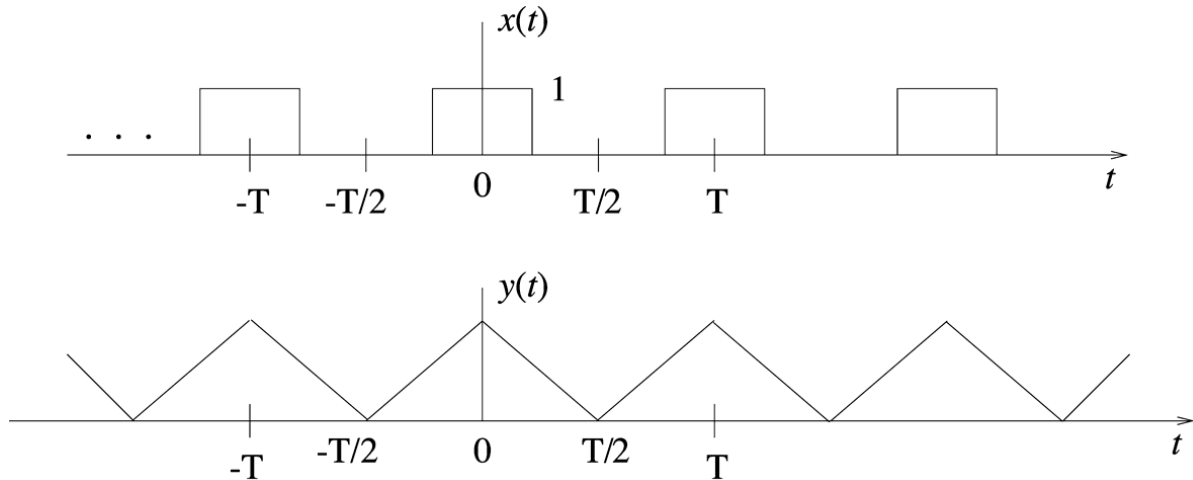
$$x(t) \cdot y(t) \leftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$$

Proof:

$$\underbrace{\sum_l a_l e^{jl\omega_0 t}}_{x(t)} \cdot \underbrace{\sum_m b_m e^{jm\omega_0 t}}_{y(t)} = \sum_{l,m} a_l b_m e^{j(l+m)\omega_0 t} \xrightarrow{l+m=k} \sum_k \left[ \underbrace{\sum_l a_l b_{k-l}}_{c_k} \right] e^{jk\omega_0 t}$$

## Periodic Convolution

$x(t), y(t)$  periodic with period  $T$



$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau \quad - \text{ not very meaningful}$$

E.g. If both  $x(t)$  and  $y(t)$  are positive, then

$$x(t) * y(t) = \infty$$

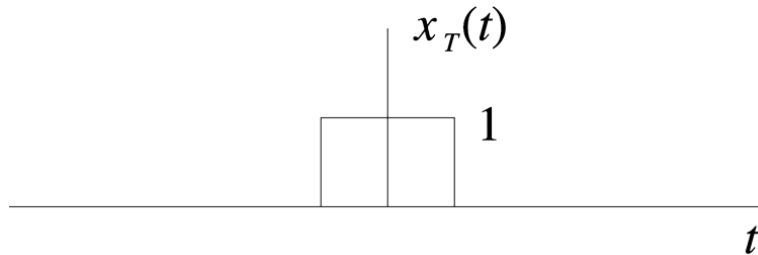
## Periodic Convolution (continued)

Periodic convolution: Integrate over *any* one period (e.g.  $-T/2$  to  $T/2$ )

$$z(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} x_T(\tau)y(t-\tau)d\tau$$

where

$$x_T(t) = \begin{cases} x(t) & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$



## Periodic Convolution (continued) Facts

- 1)  $z(t)$  is periodic with period  $T$  (why?)

From Lecture #2,  $x(t) = x(t + T) \rightarrow y(t) = y(t + T)$  for LTI systems.

In the convolution, treat  $y(t)$  as the input and  $x_T(t)$  as  $h(t)$

- 2) Doesn't matter what period over which we choose to integrate:

$$z(t) = \int_T x(\tau)y(t - \tau)d\tau = x(t) \otimes y(t)$$

**Periodic**

- 3) **convolution  
in time**

$$x(t) \leftrightarrow a_k, y(t) \leftrightarrow b_k, z(t) \leftrightarrow c_k$$

$$c_k = \frac{1}{T} \int_T z(t)e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T \left( \int_T x(\tau)y(t - \tau)d\tau \right) e^{-jk\omega_0 t} dt$$

$$= \int_T \underbrace{\left( \frac{1}{T} \int_T y(t - \tau)e^{-jk\omega_0(t - \tau)} dt \right)}_{b_k} x(\tau)e^{-jk\omega_0 \tau} d\tau$$

$$= \int_T b_k x(\tau)e^{-jk\omega_0 \tau} d\tau = T a_k b_k$$

**Multiplication  
in frequency!**

بازنمایی سری فوریه سیگنال‌های متناوب (۲)

۲

سری  
فوریه‌ی  
گسسته –  
زمان

## Fourier Series Representation of DT Periodic Signals

- $x[n]$  - periodic with fundamental period  $N$ , fundamental frequency

$$x[n + N] = x[n] \quad \text{and} \quad \omega_0 = \frac{2\pi}{N}$$

- Only  $e^{j\omega n}$  which are periodic with period  $N$  will appear in the *FS*

$$\omega N = k2\pi \Leftrightarrow \omega = k\omega_0 \quad , \quad k = 0, \pm 1, \pm 2, \dots$$

- There are only  $N$  distinct signals of this form

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} e^{j\overbrace{N\omega_0 n}^{2\pi n}} = e^{jk\omega_0 n}$$

- So we *could* just use  $e^{j0\omega_0 n}, e^{j1\omega_0 n}, e^{j2\omega_0 n}, \dots, e^{j(N-1)\omega_0 n}$
- However, it is often useful to allow the choice of  $N$  consecutive values of  $k$  to be *arbitrary*.



## DT Fourier Series Representation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$\sum_{k=\langle N \rangle}$  = Sum over *any*  $N$  consecutive values of  $k$

— This is a *finite* series

$\{a_k\}$  - Fourier (series) coefficients

Questions:

- 1) What DT periodic signals have such a representation?
- 2) How do we find  $a_k$ ?

## Answer to Question #1:

Any DT periodic signal has a Fourier series representation

$$\begin{aligned}x[n] &= \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \\&\quad \Downarrow \\x[0] &= \sum_{k=\langle N \rangle} a_k \\x[1] &= \sum_{k=\langle N \rangle} a_k e^{jk\omega_0} \\x[2] &= \sum_{k=\langle N \rangle} a_k e^{j2k\omega_0} \\&\quad \vdots \\x[N-1] &= \sum_{k=\langle N \rangle} a_k e^{j(N-1)k\omega_0}\end{aligned}$$

$N$  equations for  $N$  unknowns,  $a_0, a_1, \dots, a_{N-1}$



## A More Direct Way to Solve for $a_k$

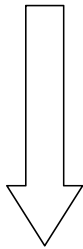
Finite geometric series

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & , \alpha = 1 \\ \frac{1 - \alpha^N}{1 - \alpha} & , \alpha \neq 1 \end{cases}$$

$$\Downarrow \quad \alpha = e^{jk\omega_0}$$

$$\begin{aligned} \sum_{n=\langle N \rangle} e^{jk\omega_0 n} &= \sum_{n=0}^{N-1} (e^{jk\omega_0})^n = \sum_{n=0}^{N-1} \left( e^{jk2\pi/N} \right)^n \\ &= \begin{cases} N & , k = 0, \pm N, \pm 2N, \dots \\ \frac{1 - e^{jk(2\pi/N)N}}{1 - e^{jk\omega_0}} = 0 & , \text{otherwise} \end{cases} \end{aligned}$$

**So, from** 
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$



multiply both sides by  $e^{-jm\omega_0 n}$

and then  $\sum_{n=\langle N \rangle}$

$$\begin{aligned} \sum_{n=\langle N \rangle} x[n] e^{-jm\omega_0 n} &= \sum_{n=\langle N \rangle} \left( \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \right) e^{-jm\omega_0 n} \\ &= \sum_{k=\langle N \rangle} a_k \underbrace{\left( \sum_{n=\langle N \rangle} e^{j(k-m)\omega_0 n} \right)}_{=N\delta[k-m] - \text{orthogonality}} \\ &= Na_m \end{aligned}$$



## DT Fourier Series Pair

$$\left( \omega_0 = \frac{2\pi}{N} \right)$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \quad (\text{Synthesis equation})$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \quad (\text{Analysis equation})$$

Note: It is convenient to think of  $a_k$  as being defined for *all* integers  $k$ . So:

- 1)  $a_{k+N} = a_k$  — Special property of DT Fourier Coefficients.
- 2) We only use  $N$  consecutive values of  $a_k$  in the synthesis equation. (Since  $x[n]$  is periodic, it is specified by  $N$  numbers, either in the time or frequency domain)

# ۳

مثال‌های  
سری فوریه‌ی  
گسسته-زمان  
و تفاوت‌های  
آن با سری  
فوریه‌ی  
پیوسته زمان

### Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

– periodic with period  $N = 16 \Rightarrow \omega_0 = \pi/8$

$$x[n] = \frac{1}{2} [e^{j\omega_0 n} + e^{-j\omega_0 n}] + \frac{1}{2} [e^{j\pi/4} e^{j2\omega_0 n} + e^{-j\pi/4} e^{-j2\omega_0 n}]$$

$$a_0 = 0$$

$$a_1 = 1/2$$

$$a_{-1} = 1/2$$

$$a_2 = e^{j\pi/4}/2$$

$$a_{-2} = e^{-j\pi/4}/2$$

$$a_3 = 0$$

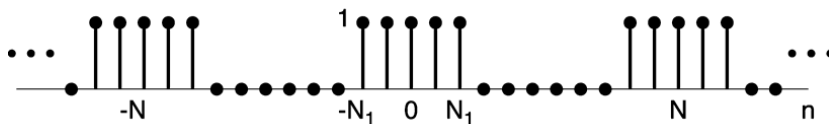
$$a_{-3} = 0$$

$$\vdots$$
$$\Downarrow$$

$$a_{15} = a_{-1+16} = a_{-1} = 1/2$$

$$a_{66} = a_{2+4 \times 16} = a_2 = e^{j\pi/4}/2$$

## Example #2: DT Square Wave



$$a_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] = \frac{2N_1 + 1}{N} = a_N = a_{-N} = a_{6N} = \dots$$

For  $k \neq$  multiple of  $N$ :

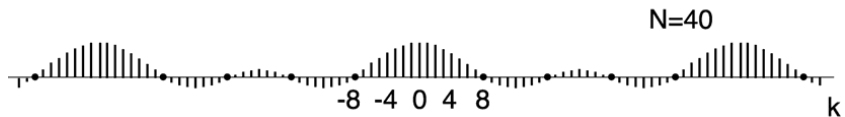
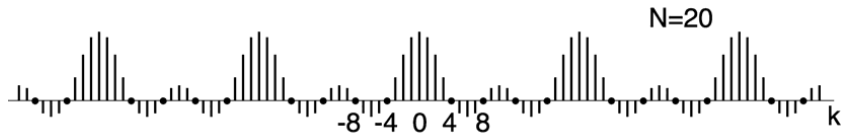
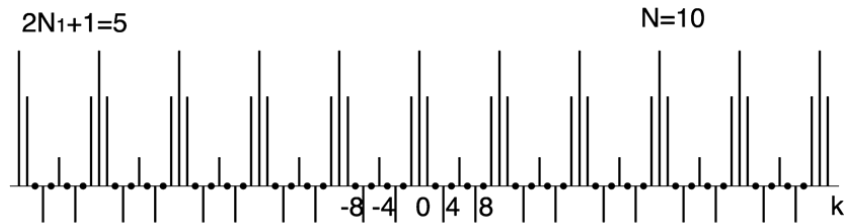
Using  $n = m - N_1$

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\omega_0 (m - N_1)} \\ &= \frac{1}{N} e^{jk\omega_0 N_1} \sum_{m=0}^{2N_1} (e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1} \frac{1 - e^{-jk\omega_0 (2N_1 + 1)}}{1 - e^{jk\omega_0}} \\ &= \frac{1}{N} \frac{\sin[k(N_1 + 1/2)\omega_0]}{\sin(k\omega_0/2)} = \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)} \end{aligned}$$

$\Downarrow$

## Example #2: DT Square wave (continued)

$$a_k = \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}$$



Convergence Issues for DT Fourier Series:

*Not an issue, since all series are finite sums.*

Properties of DT Fourier Series:      Lots, just as with CT Fourier Series

Example:

$$\begin{aligned}x[n] &\leftrightarrow a_k \\e^{jM\omega_0 n} x[n] &\leftrightarrow b_k = ?\end{aligned}$$

$$x[n]e^{jM\omega_0 n} = \sum_{r=\langle N \rangle} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$

$$\stackrel{k=r+M}{=} \sum_{k=\langle N \rangle} a_{k-M} e^{jk\omega_0 n}$$

$\Downarrow$

$$b_k = a_{k-M}$$

Frequency shift

$$jk\omega_0 \rightarrow j(k-M)\omega_0$$

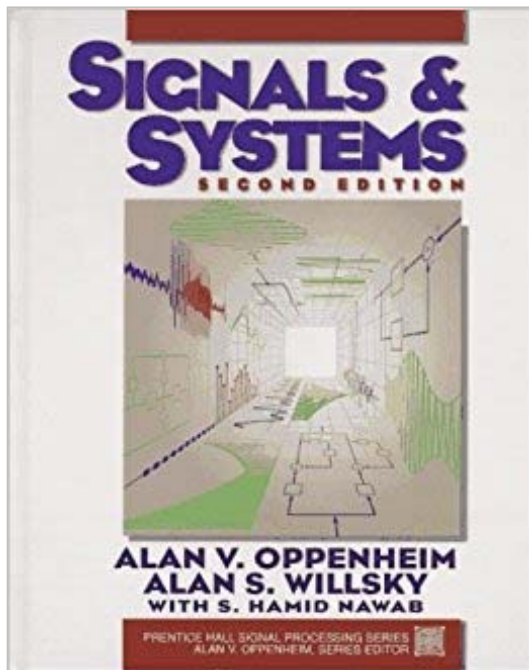


بازنمایی سری فوریه سیگنال‌های متناوب (۲)

۴

منابع

## منبع اصلی



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**Chapter 3**