سیگنال‌ها و سیستم‌ها

درس ۲۸

تبدیل Z (۱)

The z-Transform (1)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

مقدمه‌ای بر تبدیل z
Introduction to the z-Transform
خصوصیات ناحیه‌ی همگرایی تبدیل z
Properties of the ROC of the z-Transform
تبدیل z معکوس
Inverse z-Transform

Prepared by Kazim Fouladi | Spring 2024 | 6th Edition
کاظم فولادی

تبدیل Z

۱

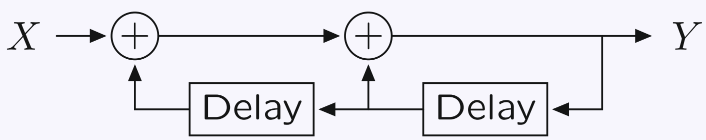
بازنمایی‌های چندگانه‌ی سیستم‌های گسسته-زمان

نقشه‌ی مفهومی سیستم‌های گسسته–زمان

Concept Map of Discrete-Time Systems

سیستم‌های گسسته–زمان را می‌توان با روش‌های مختلفی بازنمایی کرد.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

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نقشه‌ی مفهومی سیستم‌های گسسته–زمان

مثال

Concept Map of Discrete-Time Systems

Example: Fibonacci system

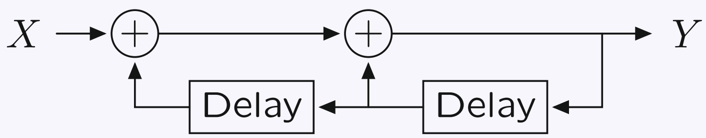
difference equation	$y[n] = x[n] + y[n-1] + y[n-2]$
operator expression	$Y = X + \mathcal{R}Y + \mathcal{R}^2 Y$
system functional	$\frac{Y}{X} = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$
unit-sample response	$h[n]:\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

نقشه‌ی مفهومی سیستم‌های گسسته–زمان

CONCEPT MAP OF DISCRETE-TIME SYSTEMS

رابطه‌ی بین پاسخ نمونه‌ی واحد و معادله‌ی تابعی سیستم:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

series

Unit-Sample Response

$$h[n]: \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

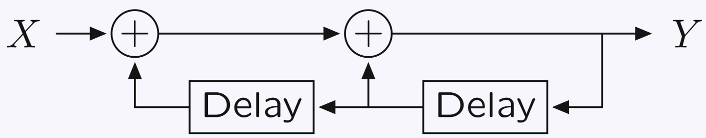
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

نقشه‌ی مفهومی سیستم‌های گسسته–زمان

Concept Map of Discrete-Time Systems

رابطه‌ی بین معادله‌ی تابعی سیستم و تابع سیستم:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

$$\mathcal{R} \rightarrow \frac{1}{z}$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

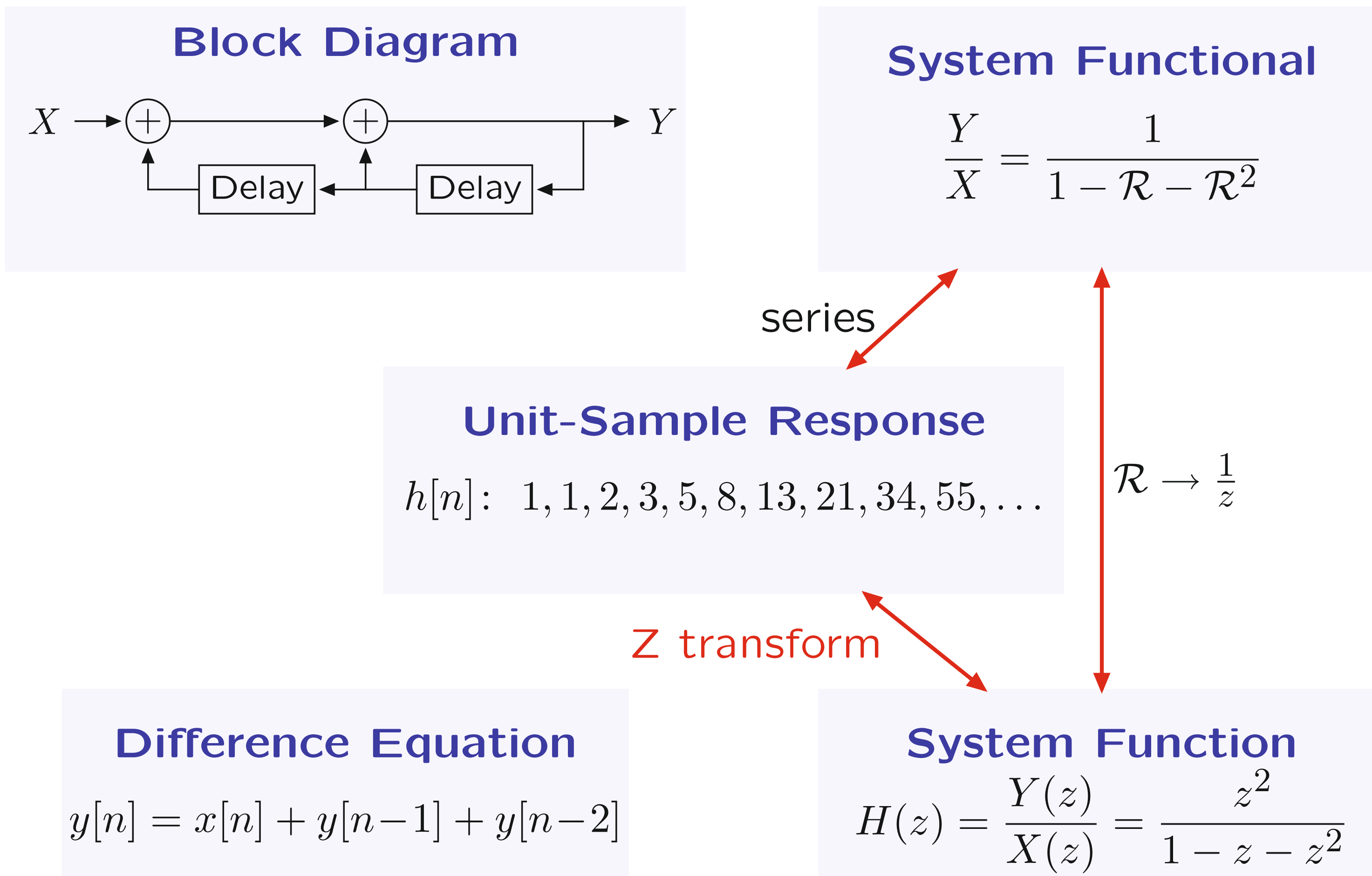
System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

نقشه‌ی مفهومی سیستم‌های گسسته–زمان

Concept Map of Discrete-Time Systems

رابطه‌ی بین پاسخ نمونه واحد و تابع سیستم:

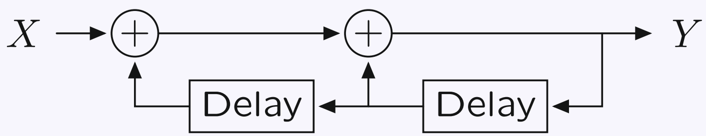


نقشه‌ی مفهومی سیستم‌های گسسته-زمان

CONCEPT MAP OF DISCRETE-TIME SYSTEMS

رابطه‌ی بین بازنمایی‌ها:

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]:\quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

سیگنال‌ها و سیستم‌ها

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مقدمه‌ای بر تبدیل z

تبدیل z

The z-Transform

تبدیل z معادل گسسته–زمان تبدیل لاپلاس است.

تبدیل z تابعی از زمان گسسته n را به تابعی از z نگاشت می‌دهد.

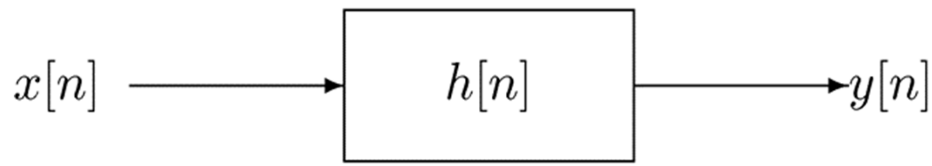
$$X(z) = \sum_{n} x[n] z^{-n}$$

یک‌طرفه *Unilateral*	**دوطرفه** *Bilateral*
$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$	$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

هر دو خواص مهم مشترکی دارند + تفاوت‌ها (مشابه تبدیل لاپلاس)

The z-Transform

Motivation: Analogous to Laplace Transform in CT



We now do *not* restrict ourselves just to $z = e^{j\omega}$

$$x[n] = \underbrace{z^n}_{\substack{\text{Eigenfunction} \\ \text{for DT LTI}}} \longrightarrow y[n] = H(z)z^n$$

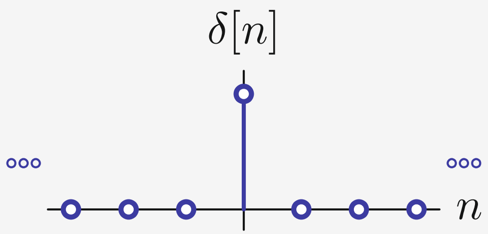
$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad \text{assuming it converges}$$

The (Bilateral) z-Transform

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{x[n]\}$$

The z-Transform

Find the Z transform of the unit-sample signal.



$$x[n] = \delta[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[0] z^0 = 1$$

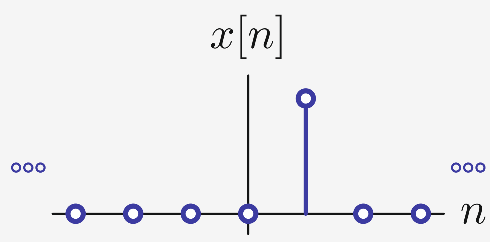
$\mathcal{Z}\{\delta[n]\} = 1$, analogous to $\mathcal{L}\{\delta(t)\} = 1$.

تبدیل z

مثال

THE Z-TRANSFORM

Find the Z transform of a delayed unit-sample signal.



$$x[n] = \delta[n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[1] z^{-1} = z^{-1}$$

The ROC and the Relation Between zT and DTFT

$$z = re^{j\omega} \quad , r = |z|$$

$$\begin{aligned} X(re^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n} \\ &= \mathcal{F}\{x[n]r^{-n}\} \end{aligned}$$

- $$\text{ROC} = \left\{ z = re^{j\omega} \text{ at which } \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \right\}$$

  — depends only on $r = |z|$, just like the ROC in s-plane only depends on $Re(s)$

- Unit circle ($r = 1$) in the ROC $\Rightarrow$ DTFT $X(e^{j\omega})$ exists

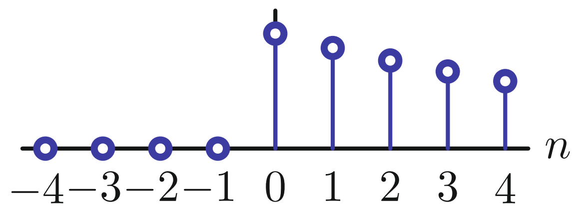
تبدیل z

مثال

THE Z-TRANSFORM

Example: Find the Z transform of the following signal.

$$x[n] = \left(\frac{7}{8}\right)^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1 - \frac{7}{8} z^{-1}} = \frac{z}{z - \frac{7}{8}}$$

provided $\left|\frac{7}{8} z^{-1}\right| < 1$, i.e., $|z| > \frac{7}{8}$.

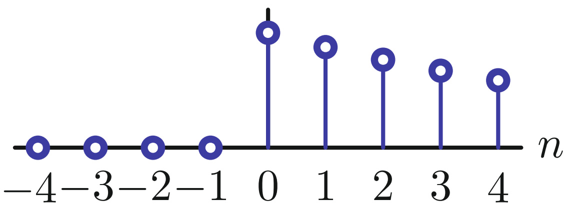
تبدیل z

مثال

THE Z-TRANSFORM

Example: Find the Z transform of the following signal.

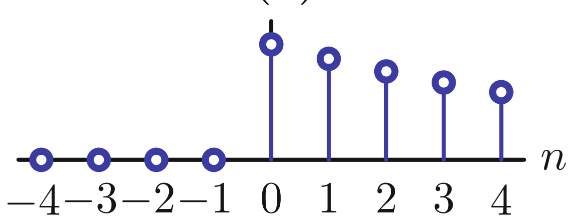
$$x[n] = \left(\frac{7}{8}\right)^n u[n]$$



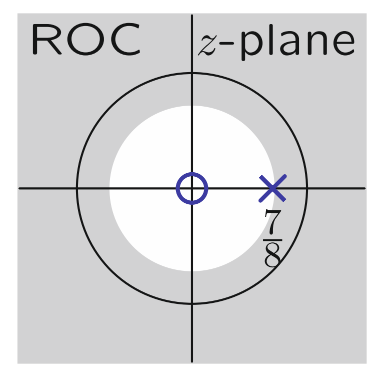
$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1-\frac{7}{8}z^{-1}} = \frac{z}{z-\frac{7}{8}}$$

provided $\left|\frac{7}{8}z^{-1}\right| < 1$, i.e., $|z| > \frac{7}{8}$.

$$x[n] = \left(\frac{7}{8}\right)^n u[n]$$



$$\frac{z}{z-\frac{7}{8}}$$



تبدیل z

مثال

THE Z-TRANSFORM

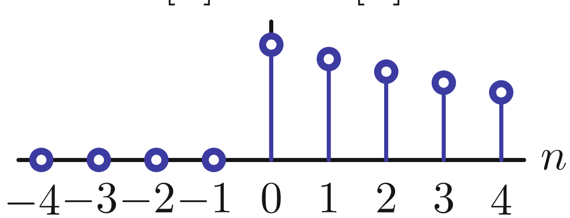
Example: $x[n] = \alpha^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

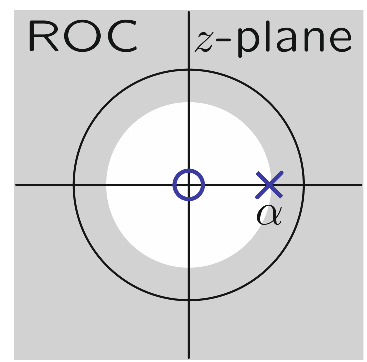
$$= \frac{1}{1 - \alpha z^{-1}}\,; \qquad \left|\alpha z^{-1}\right| < 1$$

$$= \frac{z}{z - \alpha}\,; \qquad |z| > |\alpha|$$

$x[n] = \alpha^n u[n]$



$$\frac{z}{z-\alpha}$$



Example #1

$x[n] = a^n u[n]$ - right-sided

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

This form for PFE and inverse z-transform

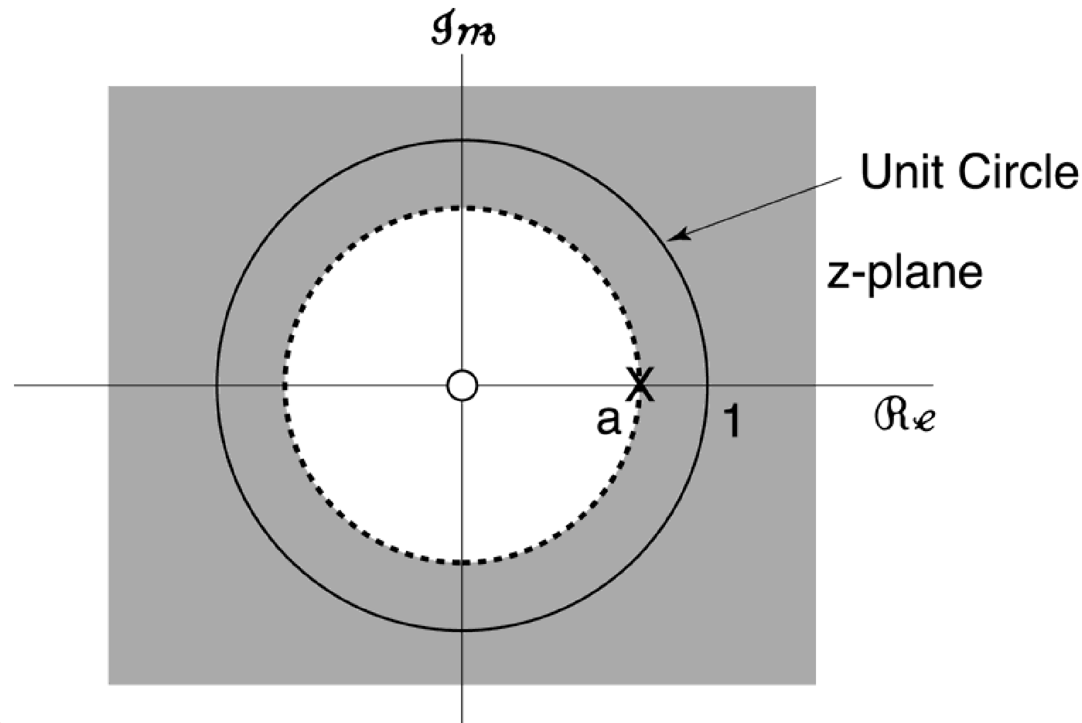
$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

This form to find pole and zero locations

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

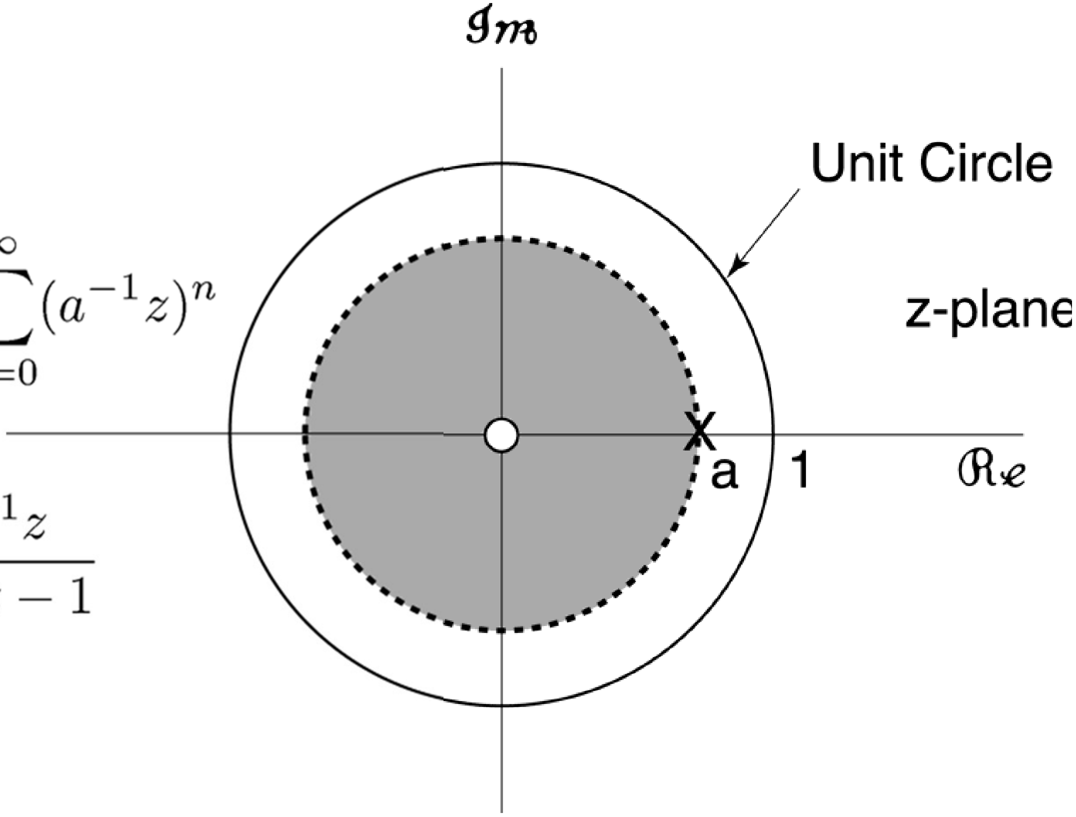
That is, ROC $|z| > |a|$, outside a circle



Example #2:

$x[n] = -a^n u[-n-1]$ - left-sided

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1] z^{-n}\} \\
&= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\
&= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \\
&= 1 - \frac{1}{1-a^{-1}z} = \frac{a^{-1}z}{a^{-1}z-1} \\
&= \frac{z}{z-a},
\end{aligned}
$$



If $|a^{-1}z| < 1$, *i.e.*, $|z| < |a|$

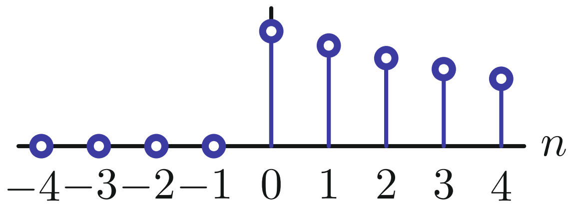
Same $X(z)$ as in **Ex #1**, but different ROC.

تبدیل z

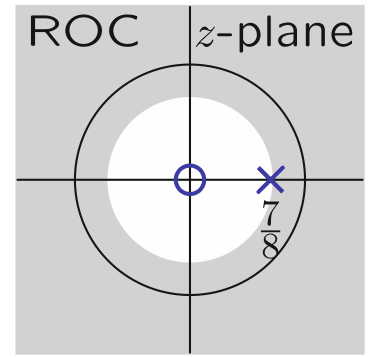
مثال: سیگنال‌های سمت راستی و سمت چپی

THE Z-TRANSFORM

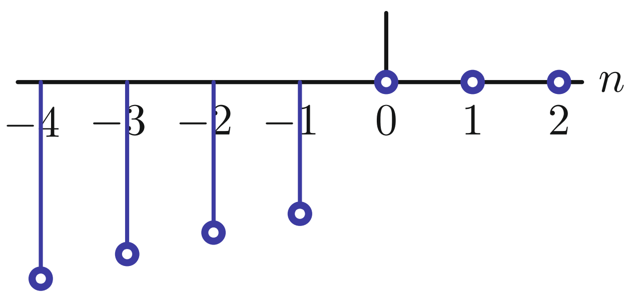
$$x[n] = \left(\frac{7}{8}\right)^n u[n]$$



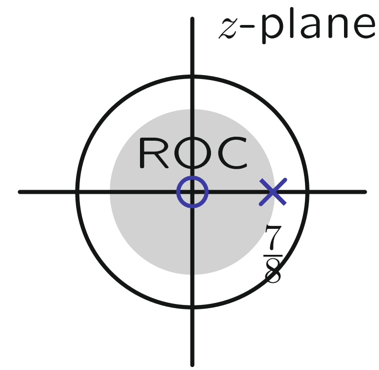
$$\frac{z}{z - \frac{7}{8}}$$



$$y[n] = -\left(\frac{7}{8}\right)^n u[-1-n]$$



$$\frac{z}{z - \frac{7}{8}}$$



Rational z-Transforms

$x[n]$ = linear combination of exponentials for $n > 0$ and for $n < 0$

$$\Downarrow$$

$$X(z) \text{ is rational}$$

$$X(z) = \frac{N(z)}{D(z)}$$

← Polynomials in z

— characterized (except for a gain) by its poles and zeros

The z-Transform

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \mathcal{Z}\{x[n]\}$$

$$\text{ROC} = \left\{ z = re^{j\omega} \text{ at which } \sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty \right\}$$

depends only on $r = |z|$, just like the ROC in s-plane only depends on $Re(s)$

- Last time:
  - Unit circle ($r = 1$) in the ROC $\Rightarrow$ DTFT $X(e^{j\omega})$ exists
  - Rational transforms correspond to signals that are linear combinations of DT exponentials

Some Intuition on the Relation between *z*T and LT

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}$$

Let $t = nT$

$$= \lim_{T\to 0} \sum_{n=-\infty}^{\infty} \underbrace{x(nT)}_{x[n]}(e^{sT})^{-n} \cdot T$$

$$= \lim_{T\to 0} T \sum_{n=-\infty}^{\infty} x[n](e^{sT})^{-n}$$

The (Bilateral) z-Transform

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{x[n]\}$$

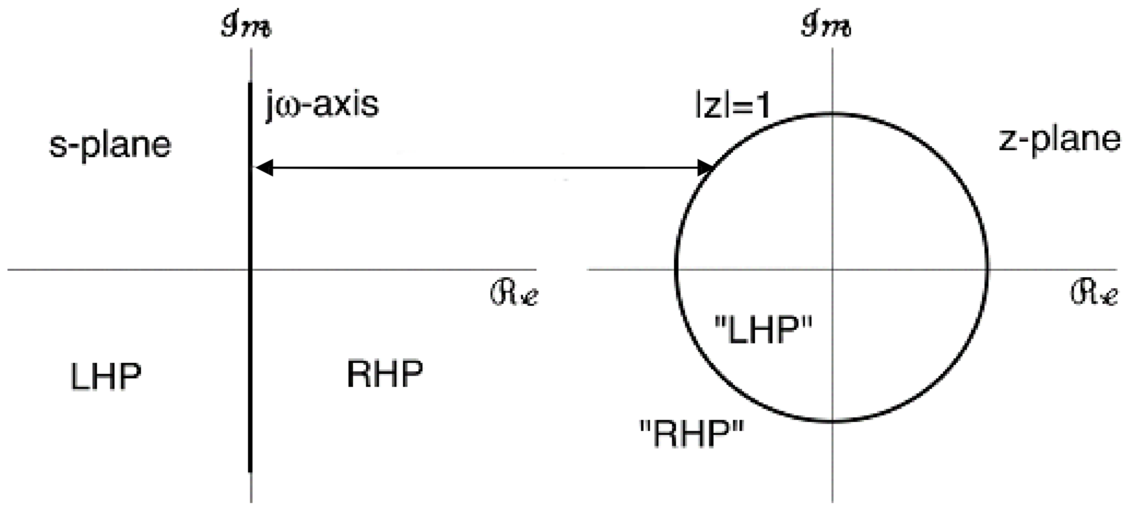
Can think of z-transform as DT version of Laplace transform with

$$z = e^{sT}$$

More intuition on zT-LT, s-plane - z-plane relationship

$$e^{sT} = z$$

$j\omega$ axis in s-plane $(s = j\omega) \Leftrightarrow \; |z| = |e^{j\omega T}| = 1$ - a unit circle in z-plane



- LHP in s-plane, $Re(s) < 0 \;\Rightarrow\; |z| = |\, e^{sT}| < 1$, inside the $|z| = 1$ circle.
  Special case, $Re(s) = -\infty \Leftrightarrow \; |z| = 0$.
- RHP in s-plane, $Re(s) > 0 \;\Rightarrow\; |z| = |e^{sT}| > 1$, outside the $|z| = 1$ circle.
  Special case, $Re(s) = +\infty \Leftrightarrow \; |z| = \infty$.
- A vertical line in s-plane, $Re(s)$ = constant $\Leftrightarrow \; |e^{sT}|$ = constant, a circle in z-plane.

تبدیل Z

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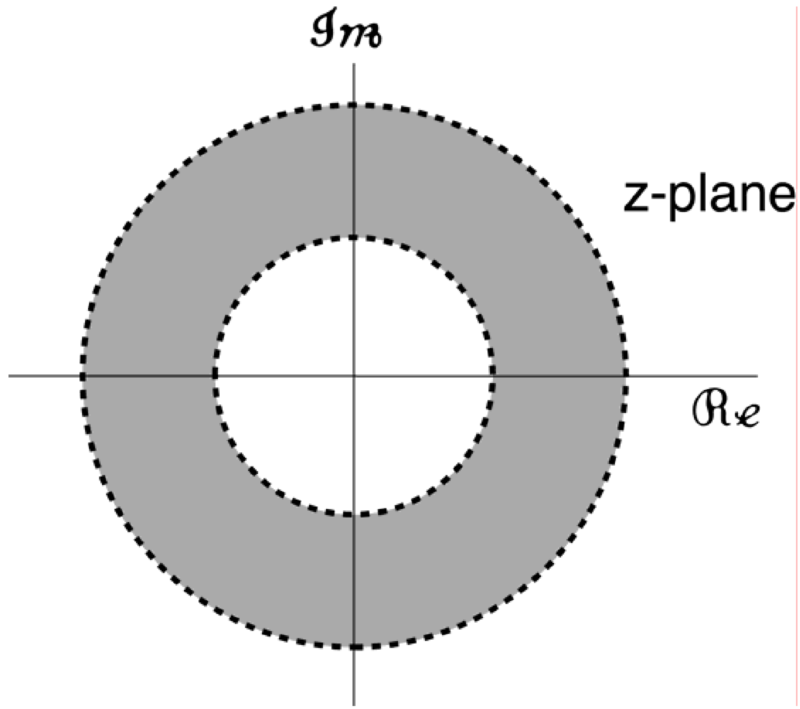
خصوصیات ناحیه‌ی همگرایی تبدیل Z

Properties of the ROCs of *z*-Transforms

(1) The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin (equivalent to a vertical strip in the s-plane)



(2) The ROC does *not* contain any poles (same as in *LT*).

More ROC Properties

(3) If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly at $z = 0$ and/or $z = \infty$.

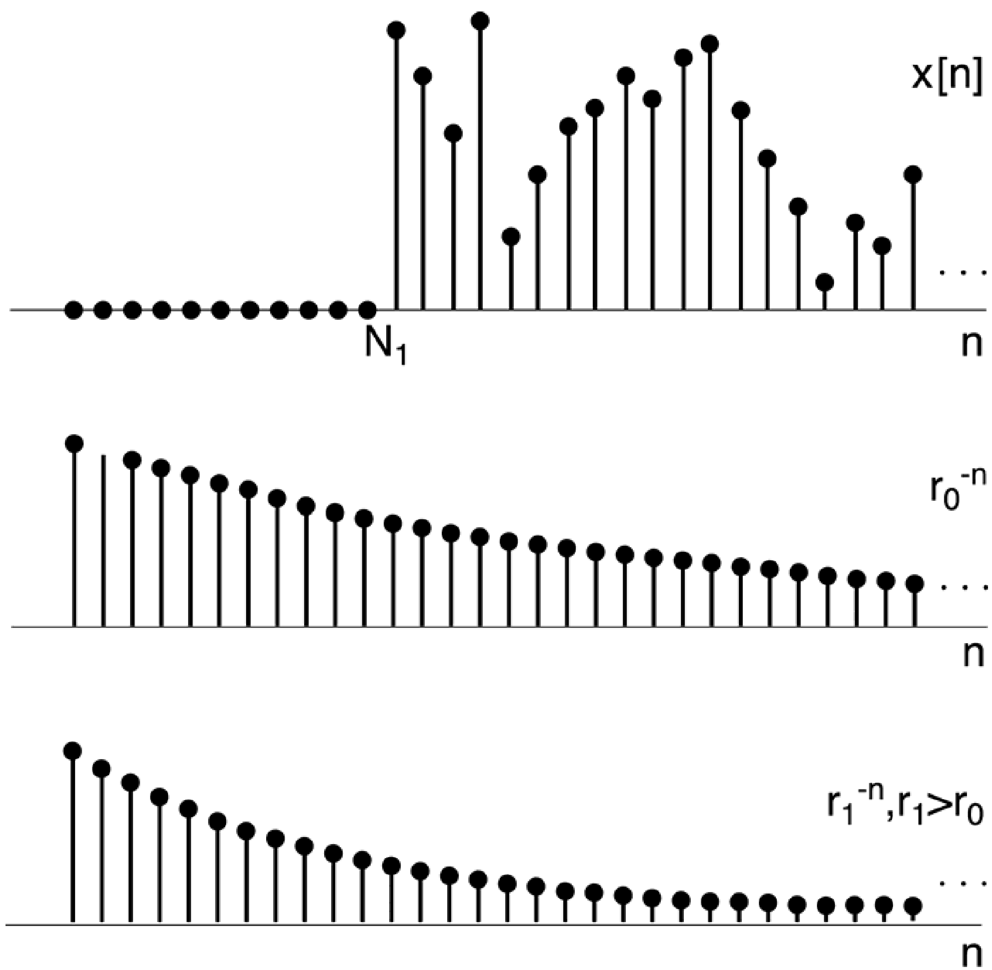
Why?

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

Examples:			CT counterpart		
$\delta[n]$ $\longleftrightarrow$	1	ROC all z	$\delta(t)$ $\longleftrightarrow$	1	ROC all s
$\delta[n-1]$ $\longleftrightarrow$	z^{-1}	ROC $z \neq 0$	$\delta(t-T)$ $\longleftrightarrow$	e^{-sT}	$\Re e\{s\} \neq -\infty$
$\delta[n+1]$ $\longleftrightarrow$	z	ROC $z \neq \infty$	$\delta(t+T)$ $\longleftrightarrow$	e^{sT}	$\Re e\{s\} \neq \infty$

ROC Properties Continued

(4) If $x[n]$ is a right-sided sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $|z| > r_o$ are also in the ROC.



$$\sum_{n=N_1}^{\infty} x[n] r_1^{-n}$$

converges faster than

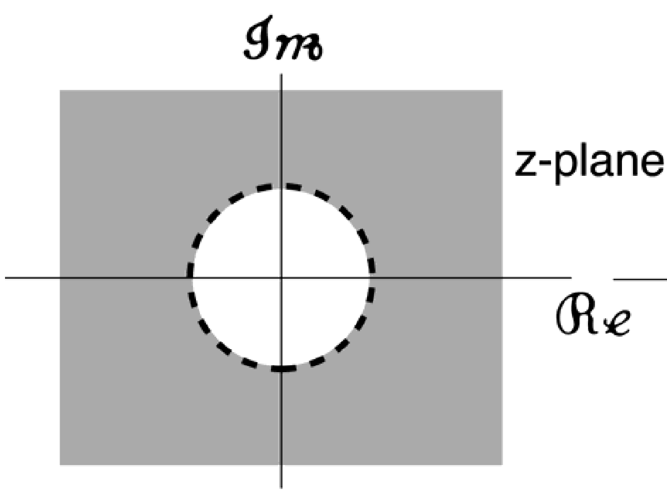
$$\sum_{n=N_1}^{\infty} x[n] r_0^{-n}$$

Side by Side

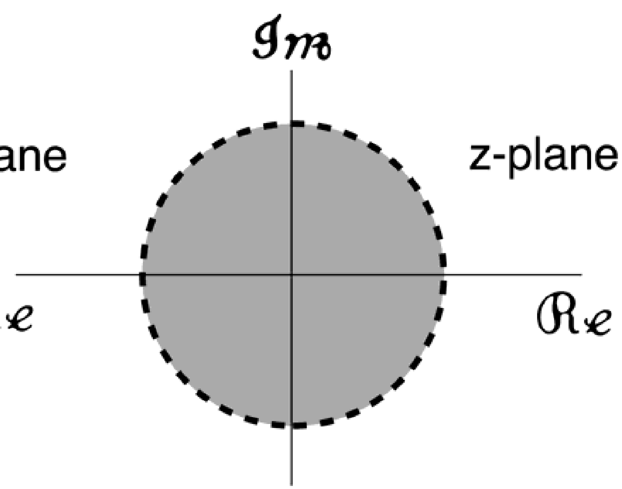
(5) If $x[n]$ is a left-sided sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $0 < |z| < r_o$ are also in the ROC.

(6) If $x[n]$ is two-sided, and if $|z| = r_o$ is in the ROC, then the ROC consists of a ring in the z-plane including the circle $|z| = r_o$.

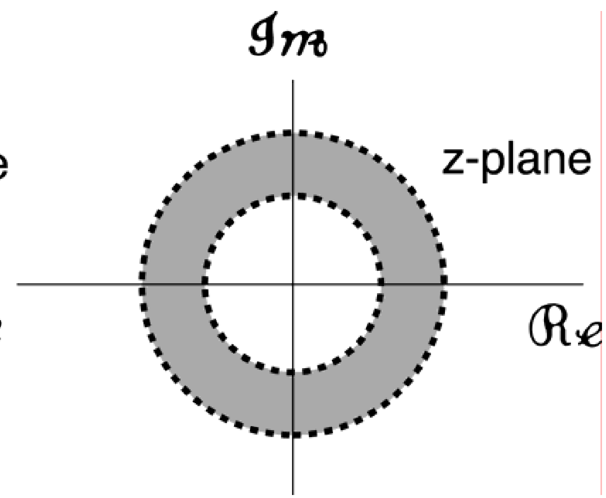
What types of signals do the following ROC correspond to?



right-sided



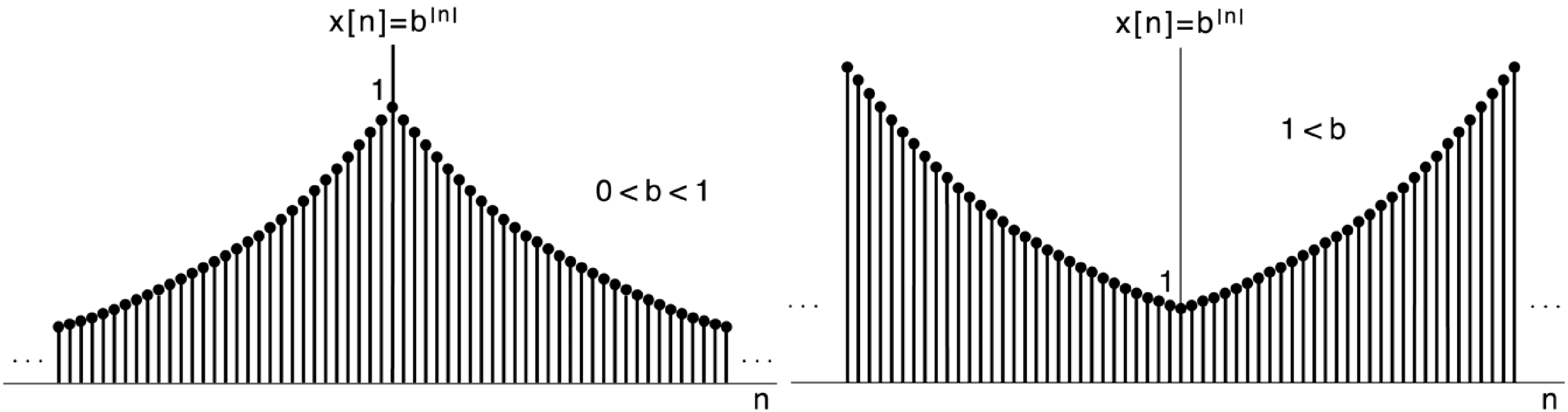
left-sided



two-sided

Example #1

$$x[n] = b^{|n|}, \quad b > 0$$



$$x[n] = b^n u[n] + b^{-n} u[-n-1]$$

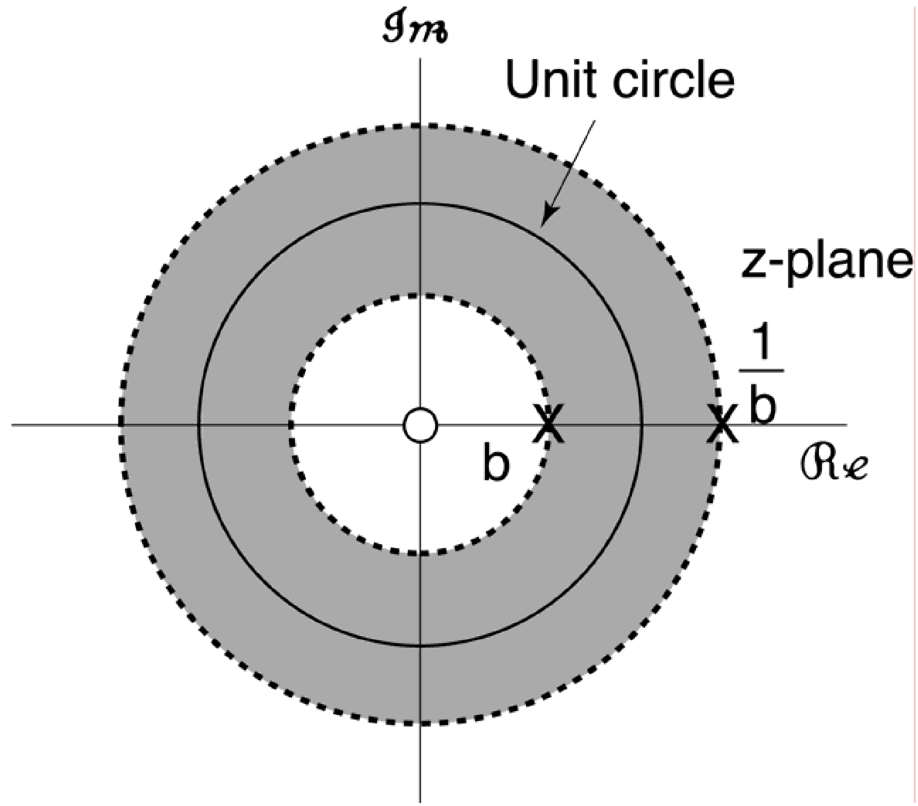
From:

$$b^n u[n] \longleftrightarrow \frac{1}{1 - bz^{-1}}, \quad |z| > b$$

$$b^{-n} u[-n-1] \longleftrightarrow \frac{-1}{1 - b^{-1} z^{-1}}, \quad |z| < \frac{1}{b}$$

Example #1 continued

$$X(z) = \frac{1}{1 - bz^{-1}} + \frac{-1}{1 - b^{-1}z^{-1}} \quad , \quad b < |z| < \frac{1}{b}$$



Clearly, ROC does *not* exist if $b > 1 \Rightarrow$ *No* z-transform for $b^{|n|}$.

سیگنال‌ها و سیستم‌ها

تبدیل z

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تبدیل z معکوس

Inverse z-Transforms

$$X(z) = X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\}, z = re^{j\omega} \in \text{ ROC}$$

$$\Downarrow$$

$$x[n]r^{-n} = \mathcal{F}^{-1}\left\{X(re^{j\omega})\right\} = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$\Downarrow$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\underbrace{r^n e^{j\omega n}}_{z^n}d\omega$$

for fixed r:

$$z = re^{j\omega} \Rightarrow dz = jre^{j\omega}d\omega \Rightarrow d\omega = \frac{1}{j}z^{-1}dz$$

$$\Downarrow$$

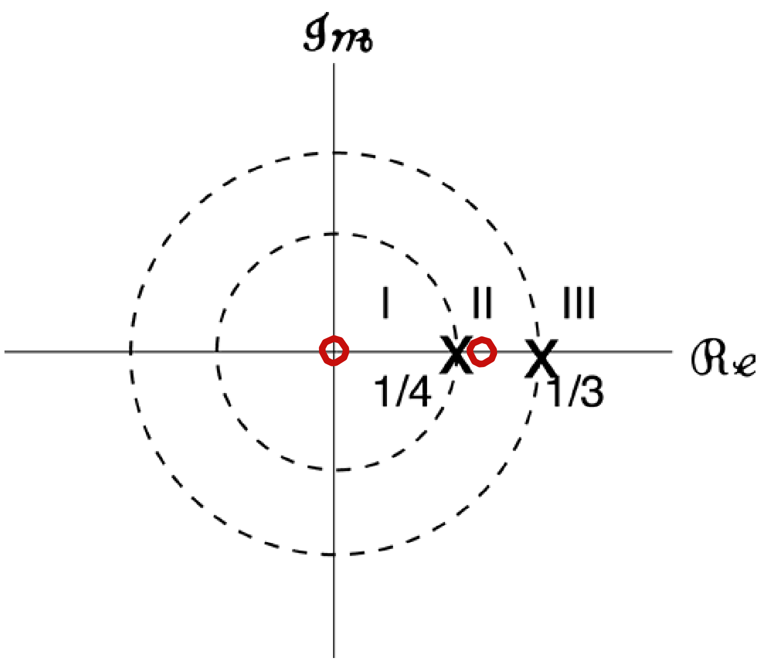
$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Example #2

$$X(z) = \frac{3z^2 - \frac{5}{6}z}{(z-\frac{1}{4})(z-\frac{1}{3})} = \frac{3 - \frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

Partial Fraction Expansion Algebra: $A = 1,\ B = 2$

$$\begin{array}{ccccc} X(z) & = & \frac{1}{1-\frac{1}{4}z^{-1}} & + & \frac{2}{1-\frac{1}{3}z^{-1}} \\ \updownarrow & & \updownarrow & & \updownarrow \\ x[n] & = & x_1[n] & + & x_2[n] \end{array}$$



zeros at $z = 0$ and
$3z - \frac{5}{6} = 0$ or $z = \frac{5}{18}$

Note, particular to z-transforms:

1) When finding poles and zeros, express $X(z)$ as a function of z.

2) When doing inverse z-transform using PFE, express $X(z)$ as a function of z^{-1}.

ROC III: $|z| > \frac{1}{3}$ - right-sided signal

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = 2 \cdot \left(\frac{1}{3}\right)^n u[n]$$

ROC II: $\frac{1}{4} < |z| < \frac{1}{3}$ - two-sided signal

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = -2 \cdot \left(\frac{1}{3}\right)^n u[-n-1]$$

ROC I: $|z| < \frac{1}{4}$ - left-sided signal

$$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1]$$

$$x_2[n] = -2 \cdot \left(\frac{1}{3}\right)^n u[-n-1]$$

Inversion by Identifying Coefficients in the Power Series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$x[n]$ - coefficient of z^{-n}

Example #3: $$X(z) = 3z^3 - z + 2z^{-4}$$

$$x[-3] = 3$$

$$x[-1] = -1$$

$$x[4] = 2$$

$x[n] = 0$ for all other *n*'s

— A finite-duration DT sequence

Example #4:

(a) $$X(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + (az^{-1})^2 + \cdots$$

$$\Downarrow - \text{ convergent for } |az^{-1}| < 1, \text{i.e.,} |z| > |a|$$

$$x[n] = a^n u[n]$$

(b) $$\begin{aligned} X(z) &= \frac{1}{1 - az^{-1}} = -a^{-1}z \left\{ \frac{1}{1 - a^{-1}z} \right\} \\ &= -a^{-1}z(1 + a^{-1}z + (a^{-1}z)^2 + \cdots) \\ &= -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots \end{aligned}$$

$$\Downarrow - \text{ convergent for } |a^{-1}z| < 1, \text{i.e.,} |z| < |a|$$

$$x[n] = -a^n u[-n-1]$$

تبدیل Z

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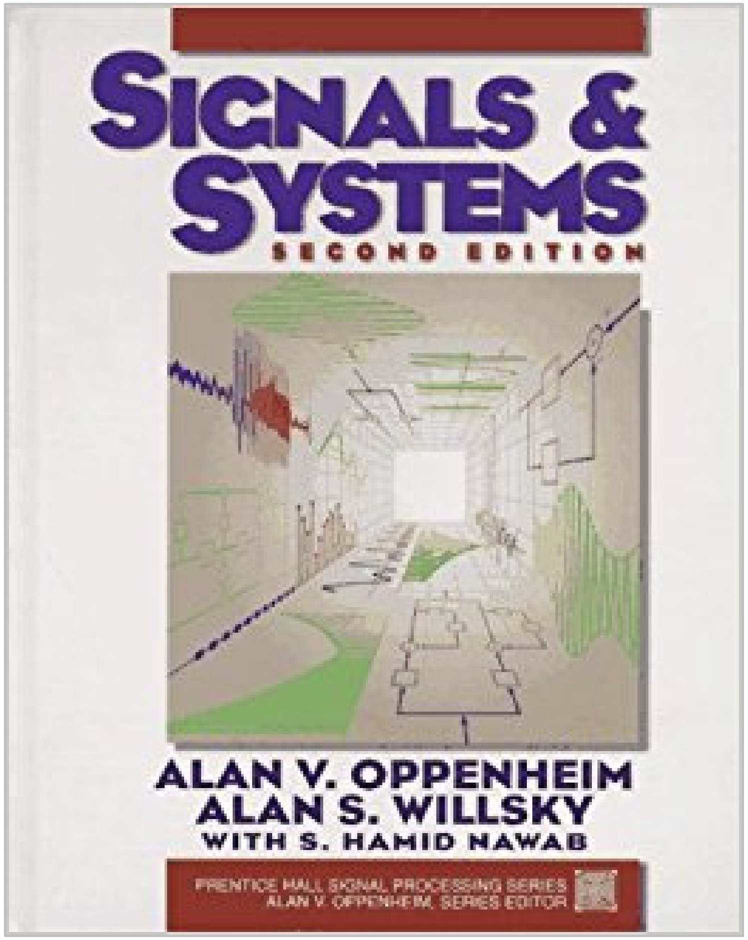
منابع

منبع اصلی



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Chapter 10