سیگنال‌ها و سیستم‌ها

درس ۱۵

تبدیل فوریه‌ی گسسته–زمان (۱)

The Discrete-Time Fourier Transform (1)

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طرح درس

COURSE OUTLINE

تبدیل فوریه‌ی گسسته–زمان
Discrete-Time Fourier Transform (DTFT)
مثال‌هایی از تبدیل فوریه‌ی گسسته–زمان
Examples of the DT Fourier Transform
خصوصیات تبدیل فوریه‌ی گسسته–زمان
Properties of the DT Fourier Transform
خاصیت کانوولوشن، ایجاب‌ها و کاربردهای آن
The Convolution Property and its Implications and Uses

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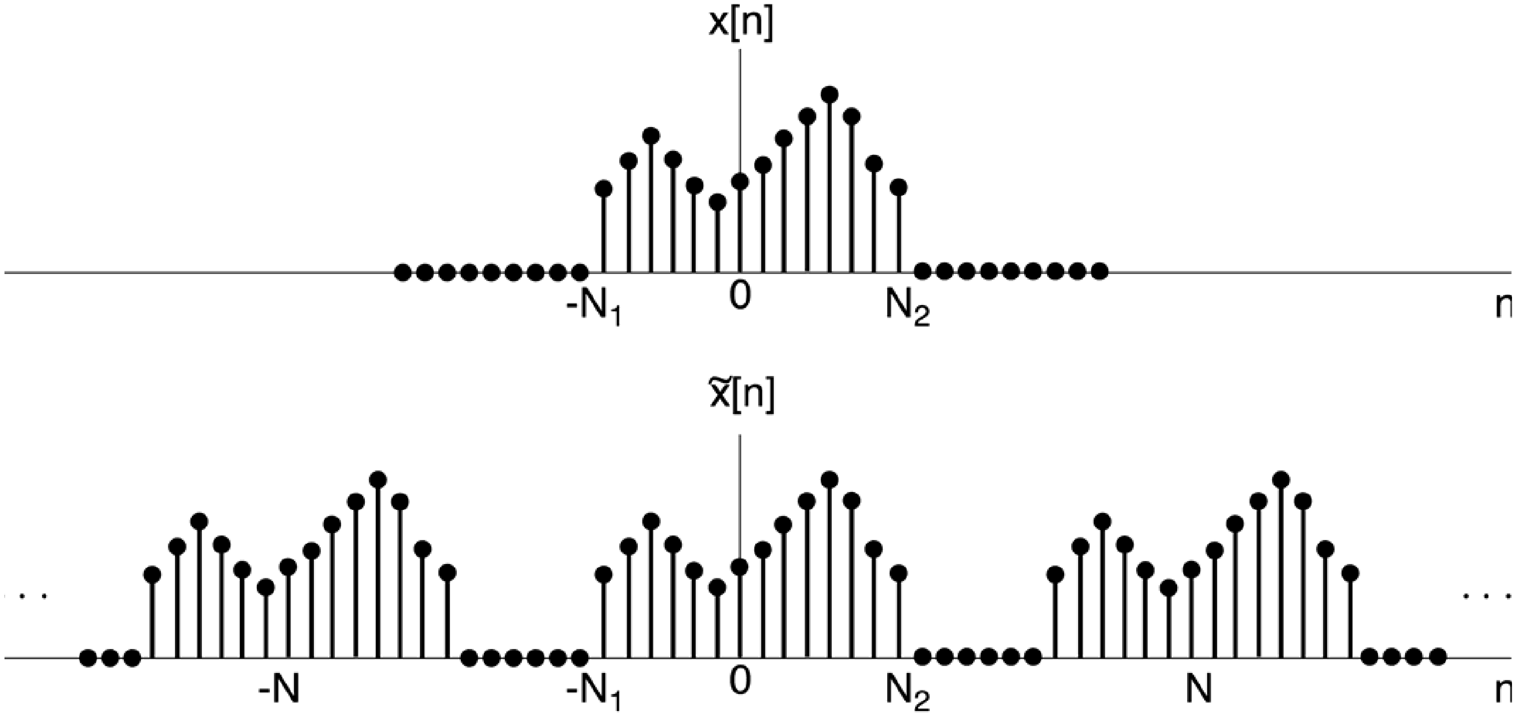
تبدیل فوریه‌ی گسسته-زمان

The Discrete-Time Fourier Transform

Derivation: (Analogous to CTFT except $e^{j\omega n} = e^{j(\omega+2\pi)n}$)

- $x[n]$ - aperiodic and (for simplicity) of finite duration
- N is large enough so that $x[n] = 0$ if $|n| \geq N/2$
- $\tilde{x}[n] = x[n]$ for $|n| \leq N/2$ and periodic with period N



$$\tilde{x}[n] = x[n] \text{ for } \textit{any } n \text{ as } N \to \infty$$

DTFT Derivation (Continued)

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \; \omega_0 = \frac{2\pi}{N}$$

DTFS synthesis eq.

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$

DTFS analysis eq.

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

Define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$-$ periodic in ω with period 2π

$$\Downarrow$$

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

DTFT Derivation (Home Stretch)

$$\tilde{x}[n] = \sum_{k=<N>} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \quad (*)$$

As $N \to \infty$: $\tilde{x}[n] \to x[n]$ for every n

$$\omega_0 \to 0, \sum \omega_0 \to \int d\omega$$

The sum in $(*)$ $\to$ an integral

$\Downarrow$ The DTFT Pair

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad \text{Synthesis equation}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad \text{Analysis equation}$$

Any 2π interval in ω

DT Fourier Transform Pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

– Analysis Equation
– FT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

– Synthesis Equation
– Inverse FT

Convergence Issues

Synthesis Equation: None, since integrating over a finite interval

Analysis Equation: Need conditions analogous to CTFT, e.g.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$ — Finite energy

or

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$ — Absolutely summable

تبدیل فوریه‌ی گسسته-زمان (۱)

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مثال‌هایی از تبدیل فوریه‌ی گسسته-زمان

Examples

Parallel with the CT examples in Lecture #8

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

2) $x[n] = \delta[n - n_0]$ - shifted unit sample

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

– Same amplitude (=1) as above, but with a *linear* phase $-\omega n_0$

More Examples

3) $x[n] = a^n u[n], |a| < 1$ - Exponentially decaying function

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \underbrace{\left(ae^{-j\omega}\right)}_{|ae^{-j\omega}|<1}{}^n$$

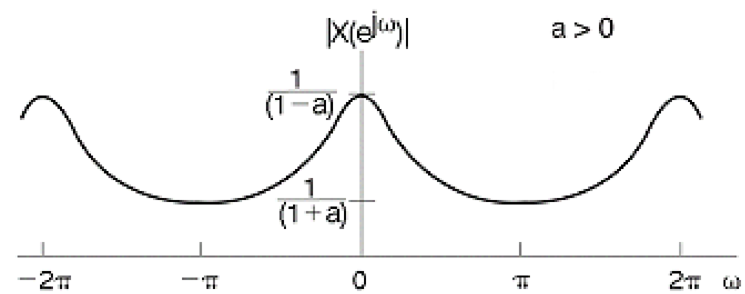
Infinite sum formula

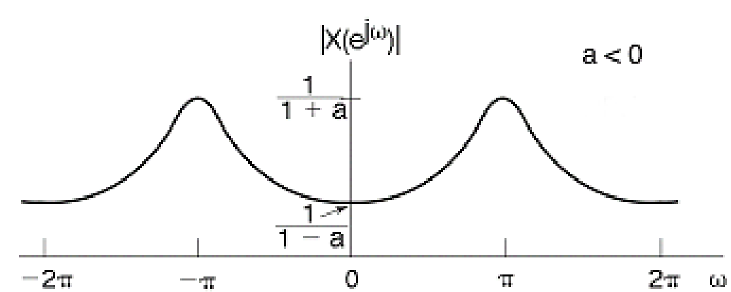
$$= \frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - a\cos\omega) + ja\sin\omega}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\omega = 0: \; X(e^{j\omega}) = \frac{1}{\sqrt{1 - 2a + a^2}} = \frac{1}{1 - a}$$

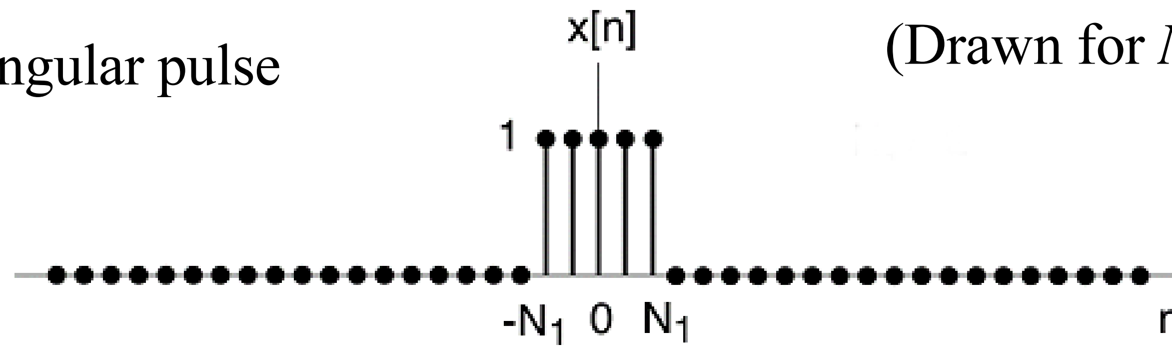
$$\omega = \pi: \; X(e^{j\omega}) = \frac{1}{\sqrt{1 + 2a + a^2}} = \frac{1}{1 + a}$$



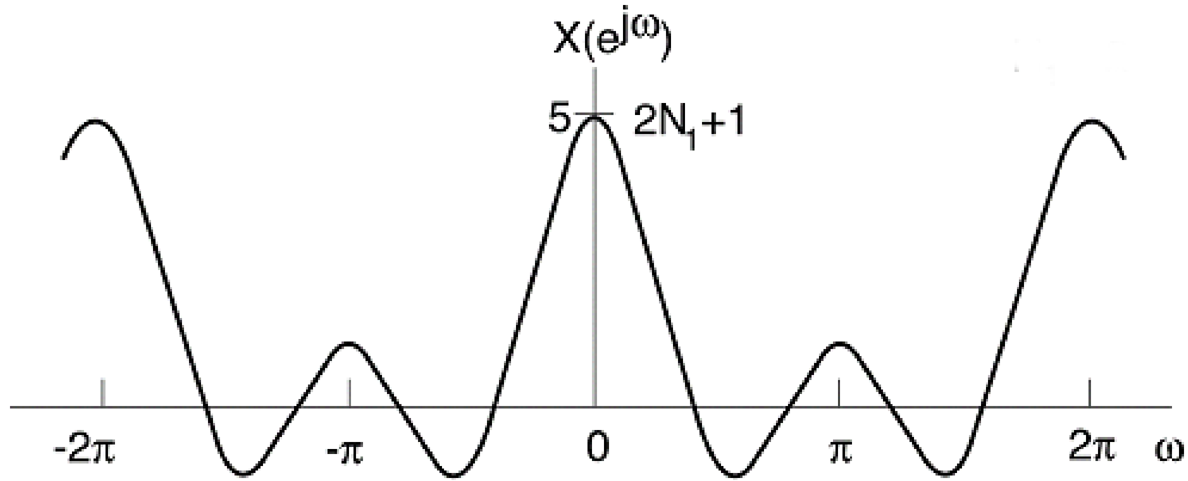


Still More

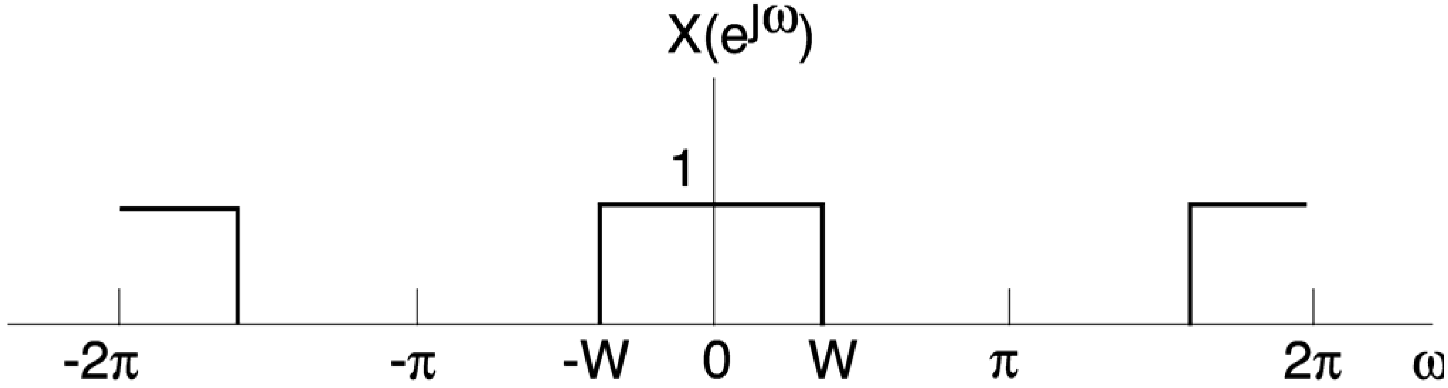
4) DT Rectangular pulse (Drawn for $N_1 = 2$)



$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin\omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)} = X(e^{j(\omega-2\pi)})$$

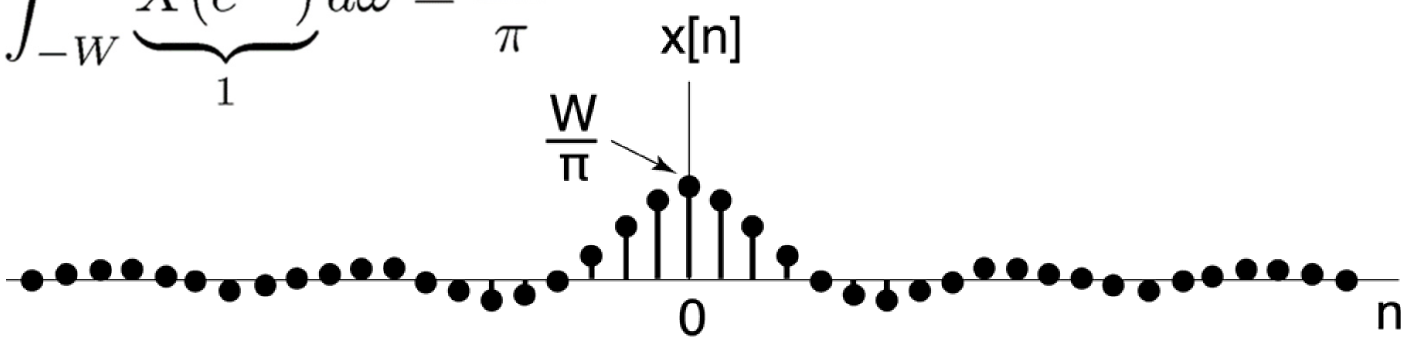


5)



$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega = \frac{\sin Wn}{\pi n}$$

$$x[0] = \frac{1}{2\pi}\int_{-W}^{W} \underbrace{X(e^{j\omega})}_{1}\,d\omega = \frac{W}{\pi}$$



DTFTs of Sums of Complex Exponentials

Recall CT result: $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$

What about DT: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$

a) We expect an impulse (of area 2π) at $\omega = \omega_o$
b) But $X(e^{j\omega})$ must be periodic with period 2π
   In fact

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$$

Note: The integration in the synthesis equation is over 2π period, only need $X(e^{j\omega})$ in *one* 2π period. Thus,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_0 n}$$

DTFT of Periodic Signals $x[n] = x[n+N]$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \ \omega_0 = \frac{2\pi}{N}$$

DTFS synthesis eq.

From the last page: $e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$

$$X(e^{j\omega}) = \sum_{k=<N>} a_k \left[2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m) \right]$$

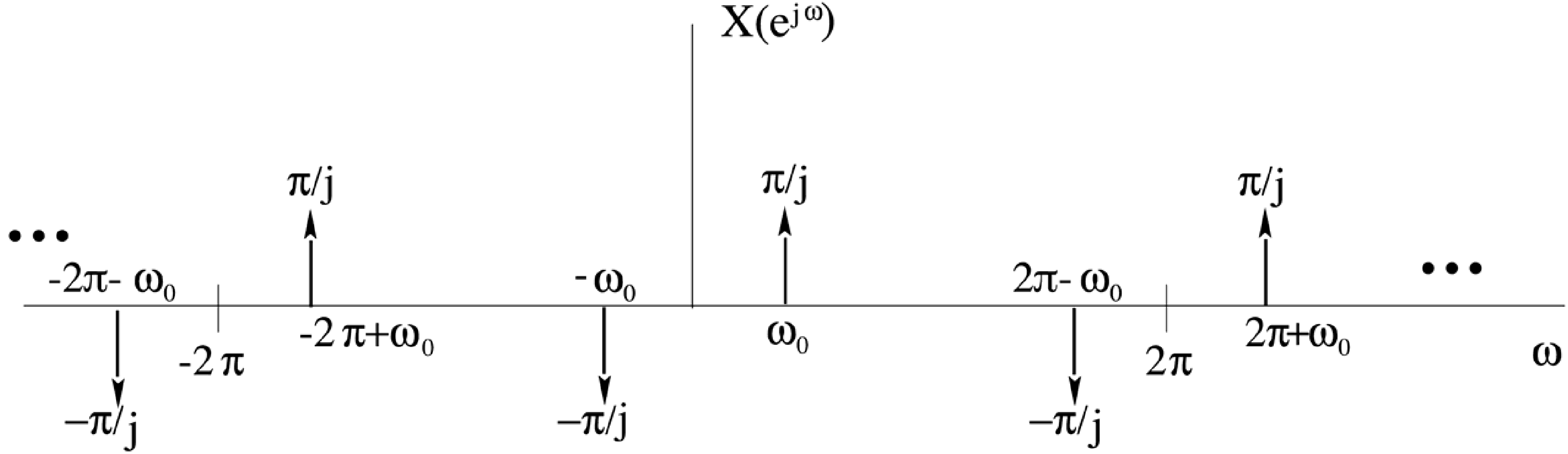
Linearity of DTFT

$$= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Example #1: DT sine function

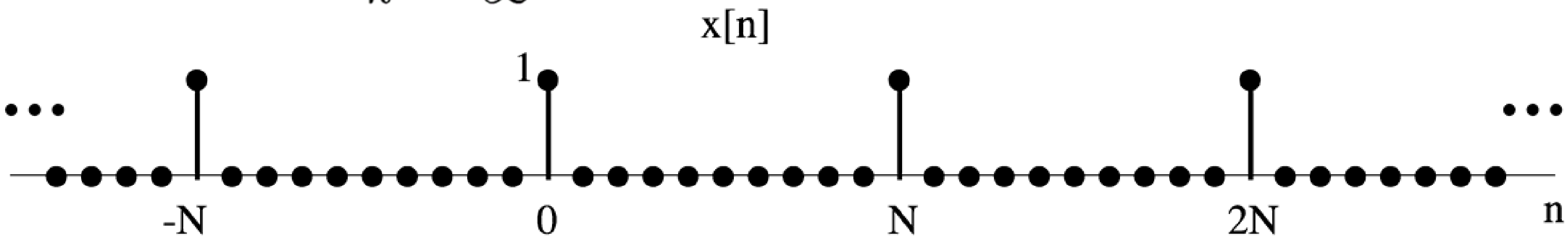
$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi m)$$



Example #2: DT periodic impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \qquad \omega_0 = 2\pi/N$$

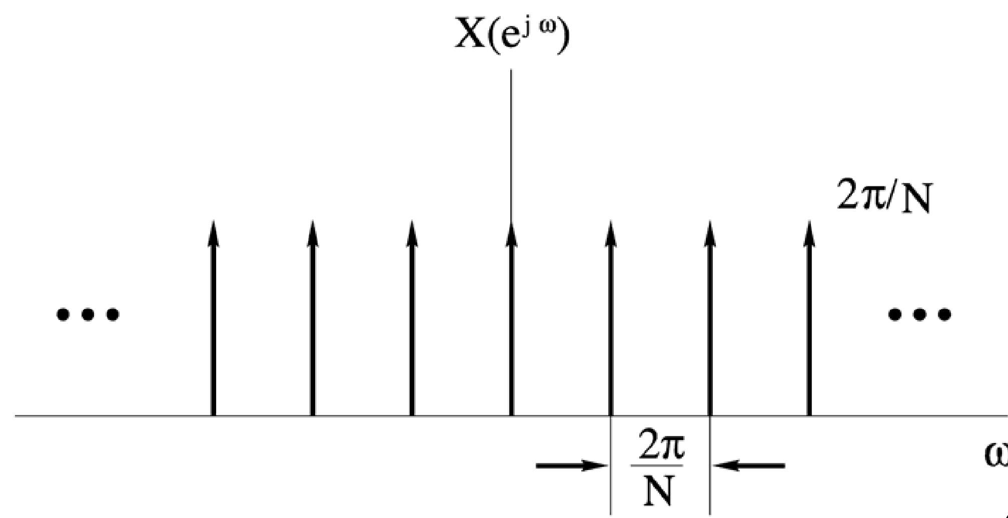


$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{=\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Downarrow$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



— Also periodic impulse train – in the frequency domain!

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خصوصیات تبدیل فوریه‌ی گسسته-زمان

Properties of the DT Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{– Analysis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \qquad \text{– Synthesis equation}$$

1) Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$ — Different from CTFT

2) Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

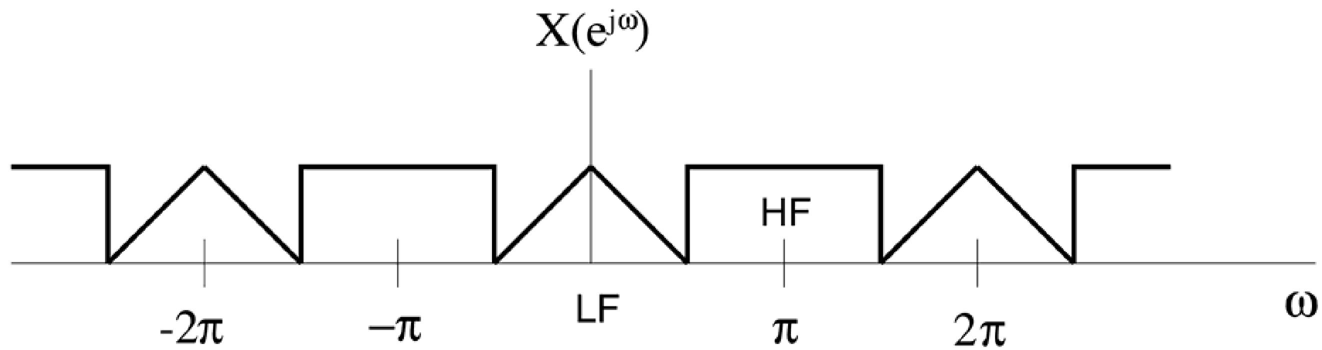
More Properties

3) Time Shifting: $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

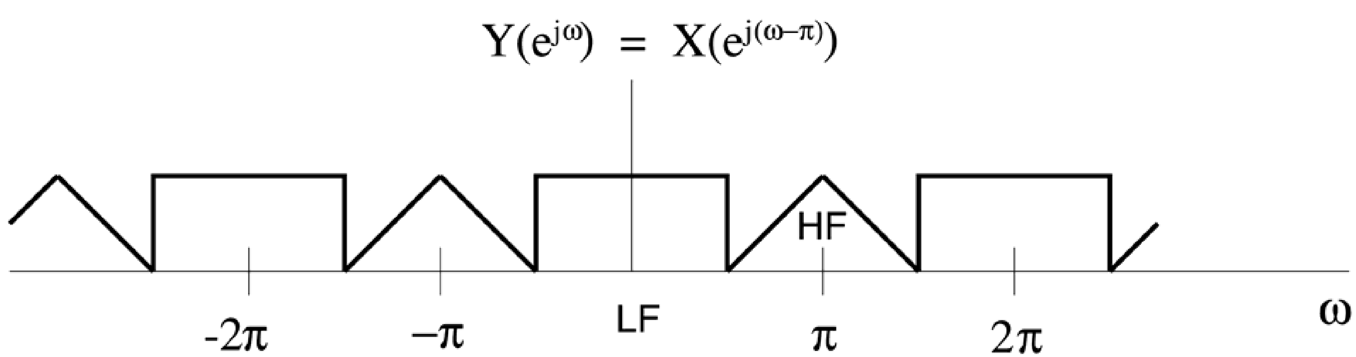
4) Frequency Shifting: $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

— Important implications in DT because of periodicity

Example



$$\omega_0 = \pi \,,\; y[n] = e^{j\pi n} x[n] = (-1)^n x[n]$$



Still More Properties

5) Time Reversal:

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

6) Conjugate Symmetry:

$$x[n] \text{ real} \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$\Downarrow$$

$$|X(e^{j\omega})| \text{ and } \Re e\left\{X(e^{j\omega})\right\} \text{ are even functions}$$
$$\angle X(e^{j\omega}) \text{ and } \Im m\left\{X(e^{j\omega})\right\} \text{ are odd functions}$$

and

$$x[n] \text{ real and even} \Leftrightarrow X(e^{j\omega}) \text{ real and even}$$
$$x[n] \text{ real and odd} \Leftrightarrow X(e^{j\omega}) \text{ purely imaginary and odd}$$

Yet Still More Properties

7) Time Expansion

Recall CT property: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$ Time scale in CT is infinitely fine

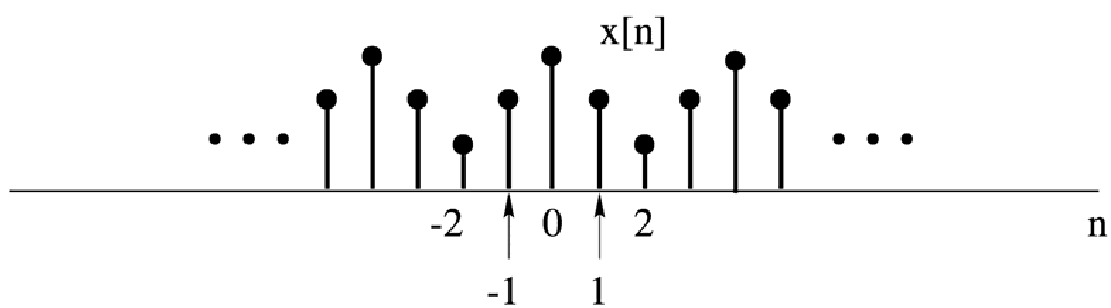
But in DT: $x[n/2]$ makes no sense

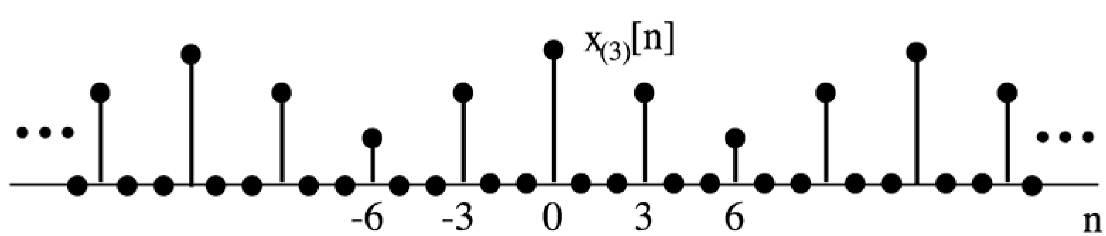
$x[2n]$ misses odd values of $x[n]$

But we can "slow" a DT signal down by inserting zeros:

k — an integer ≥ 1

$x_{(k)}[n]$ — insert $(k-1)$ zeros between successive values





Insert two zeros in this example ($k = 3$)

Time Expansion (continued)

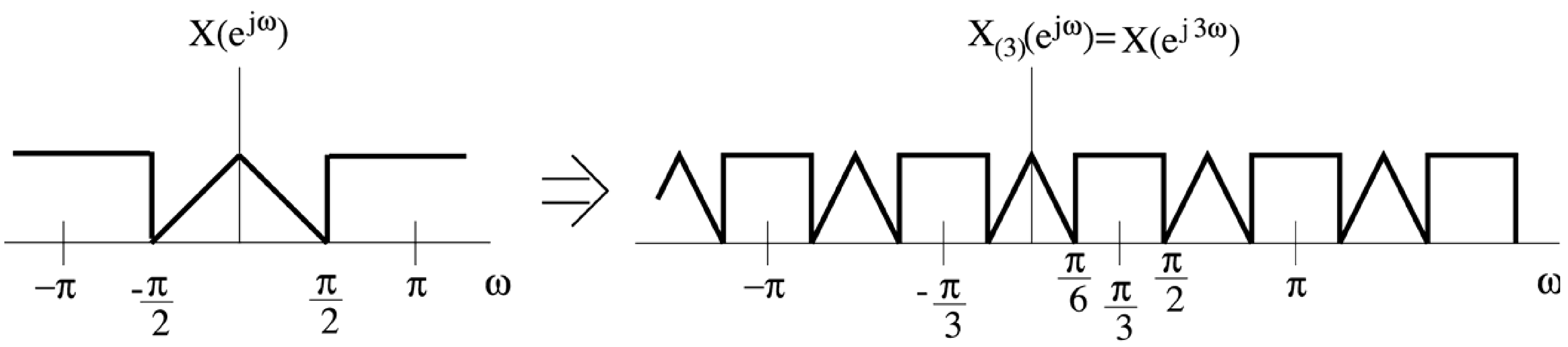
$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is an integer multiple of k} \\ 0 & \text{otherwise} \end{cases}$$

— Stretched by a factor of *k* in time domain

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} \overset{n=mk}{=} \sum_{m=-\infty}^{\infty} x_{(k)}[mk]e^{-j\omega mk}$$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j(k\omega)m} = X(e^{jk\omega})$$

-compressed by a factor of *k* in frequency domain



Is There No End to These Properties?

8) Differentiation in Frequency

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = -j\sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$$

$\Downarrow$ multiply by j on both sides

Multiplication by n $\quad nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})\quad$ Differentiation in frequency

9) Parseval's Relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{Total energy in time domain}} = \underbrace{\frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega}_{\text{Total energy in frequency domain}}$$

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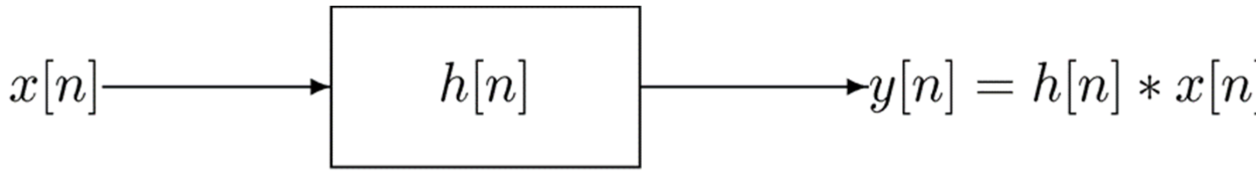
خاصیت کانوولوشن، ایجاب‌ها و کاربردهای آن

The Convolution Property



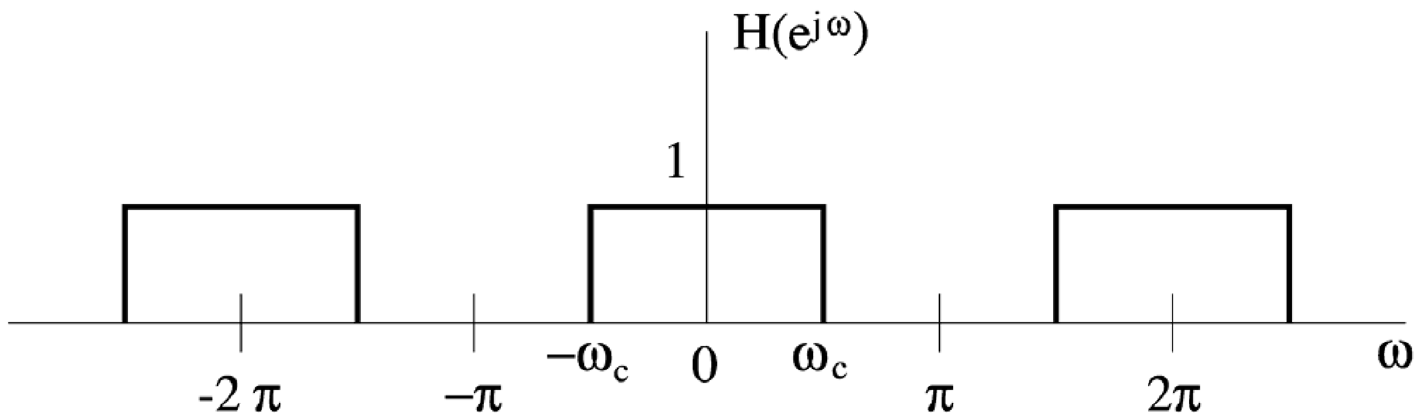
$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$\Rightarrow$ Frequency response $H(e^{j\omega})$ = DTFT of the unit sample response

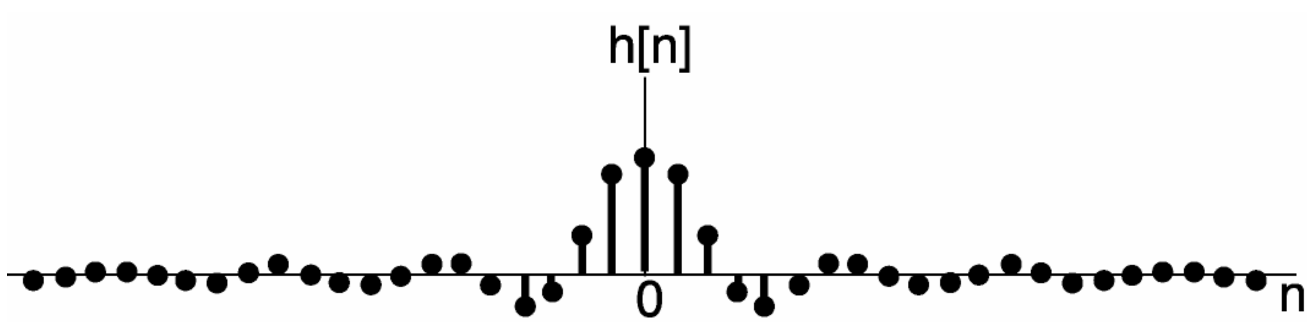
Example #1:

$$
\begin{aligned}
x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) &= 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
Y(e^{j\omega}) &= H(e^{j\omega}) 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
&= 2\pi \sum_{k=-\infty}^{\infty} H(e^{j(\omega_0 + 2\pi k)}) \delta(\omega - \omega_0 - 2\pi k) \\
&\overset{H \text{ Periodic}}{=} H(e^{j\omega_0}) 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
&\Downarrow \\
y[n] &= H(e^{j\omega_0}) e^{j\omega_0 n}
\end{aligned}
$$

Example #2: Ideal Lowpass Filter



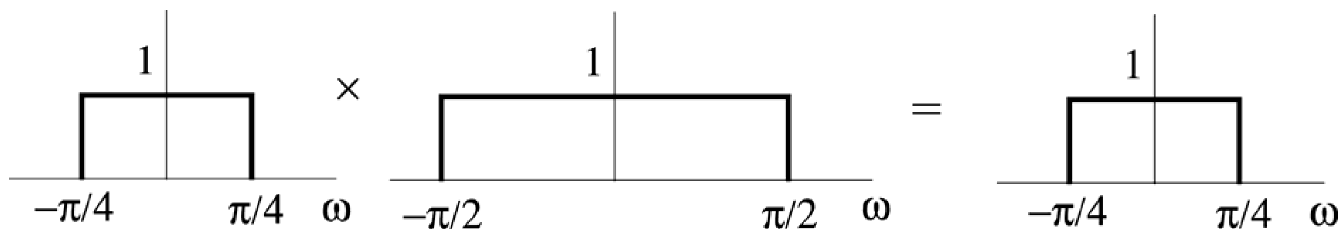
$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{\sin \omega_c n}{\pi n}$$



Example #3:

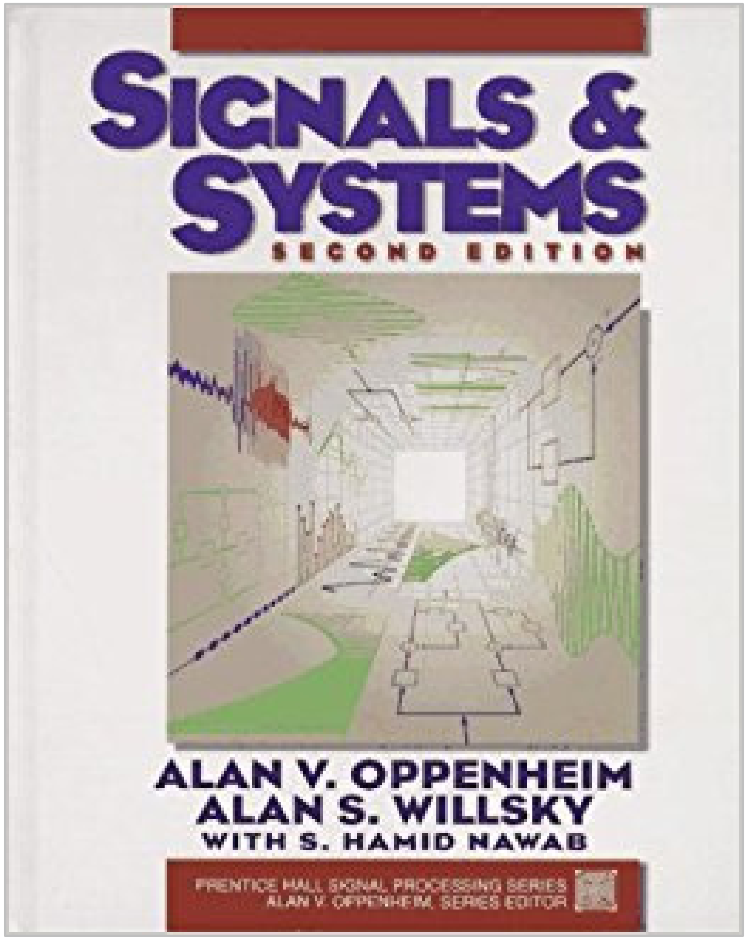
$$\frac{\sin(\pi n/4)}{\pi n} * \frac{\sin(\pi n/2)}{\pi n} \quad = \quad \frac{\sin(\pi n/4)}{\pi n}$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$



منابع

منبع اصلی



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Chapter 5