سیگنال‌ها و سیستم‌ها

درس ۱۴

تبدیل فوریه‌ی پیوسته–زمان (۲)

The Continuous-Time Fourier Transform (2)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

خاصیت کانوولوشن تبدیل فوریه‌ی پیوسته–زمان
The Convolution Property of the CTFT
بازبینی پاسخ فرکانسی و سیستم‌های خطی تغییرناپذیر با زمان
Frequency Response and LTI Systems Revisited
خاصیت ضرب و رابطه‌ی پارسوال
Multiplication Property and Parseval’s Relation

۱

خاصیت کانوولوشن تبدیل فوریه‌ی پیوسته-زمان

The CT Fourier Transform Pair

$$x(t) \longleftrightarrow X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$ – FT

(Analysis Equation)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$ – Inverse FT

(Synthesis Equation)

Last lecture: some properties

Today: further exploration

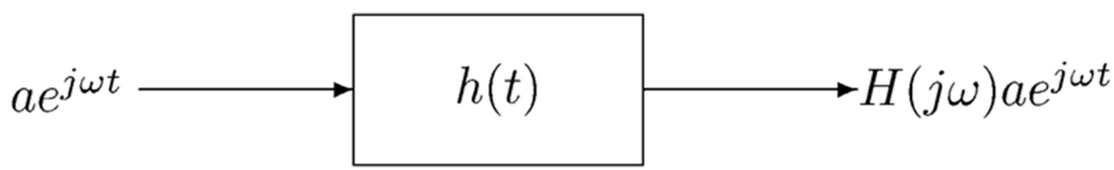
Convolution Property

$$y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$$\text{where } h(t) \longleftrightarrow H(j\omega)$$

A consequence of the eigenfunction property:

$$x(t) = \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi}X(j\omega)d\omega\right)}_{\text{coefficient } a} e^{j\omega t}$$



$$y(t) \;=\; \int_{-\infty}^{\infty} \underbrace{\left(H(j\omega)\frac{1}{2\pi}X(j\omega)d\omega\right)}_{H(j\omega)\cdot a} e^{j\omega t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{H(j\omega)X(j\omega)}_{Y(j\omega)} e^{j\omega t} d\omega$$

Synthesis equation for $y(t)$

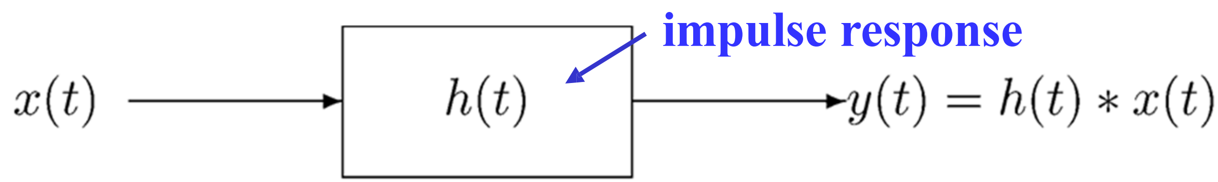
سیگنال‌ها و سیستم‌ها

تبدیل فوریه‌ی پیوسته-زمان (۲)

۲

بازبینی پاسخ فرکانسی و سیستم‌های خطی تغییرناپذیر با زمان

The Frequency Response Revisited

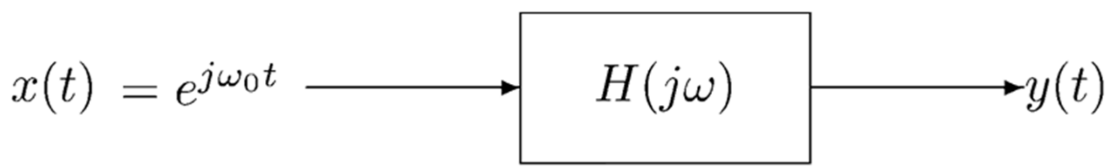


$$Y(j\omega) = H(j\omega)X(j\omega)$$

frequency response

The frequency response of a CT LTI system is simply the Fourier transform of its impulse response

Example #1:



Recall

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$Y(j\omega) = H(j\omega)X(j\omega) = H(j\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(j\omega_0)\delta(\omega - \omega_0)$$

$$\Downarrow \text{ inverse FT}$$

$$y(t) = H(j\omega_0)e^{j\omega_0 t}$$

Example #2: A differentiator

$$y(t) = \frac{dx(t)}{dt}$$ **- an LTI system**

Differentiation property: $Y(j\omega) = j\omega X(j\omega)$

$$\Downarrow$$

$$H(j\omega) = j\omega$$

1) Amplifies high frequencies (enhances sharp edges)

2) $+\pi/2$ phase shift $(j = e^{j\pi/2})$

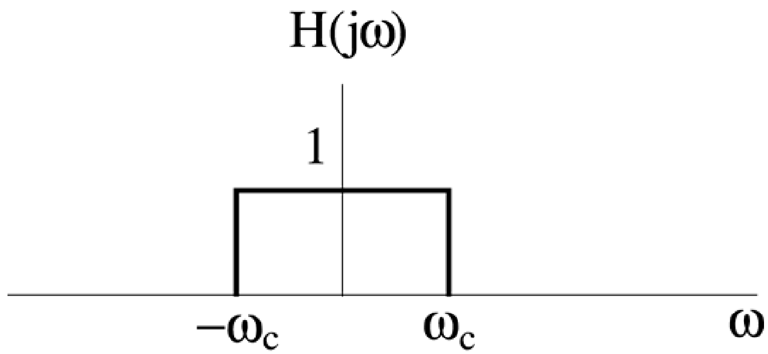
Larger at high ω_o

phase shift

$$\frac{d}{dt}\sin\omega_0 t = \omega_0\cos\omega_0 t = \omega_0\sin(\omega_0 t + \tfrac{\pi}{2})$$

$$\frac{d}{dt}\cos\omega_0 t = -\omega_0\sin\omega_0 t = \omega_0\cos(\omega_0 t + \tfrac{\pi}{2})$$

Example #3: Impulse Response of an Ideal Lowpass Filter



$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega$$
$$= \frac{\sin \omega_c t}{\pi t}$$
$$= \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

Define: $\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

Questions:

1) Is this a causal system? No.

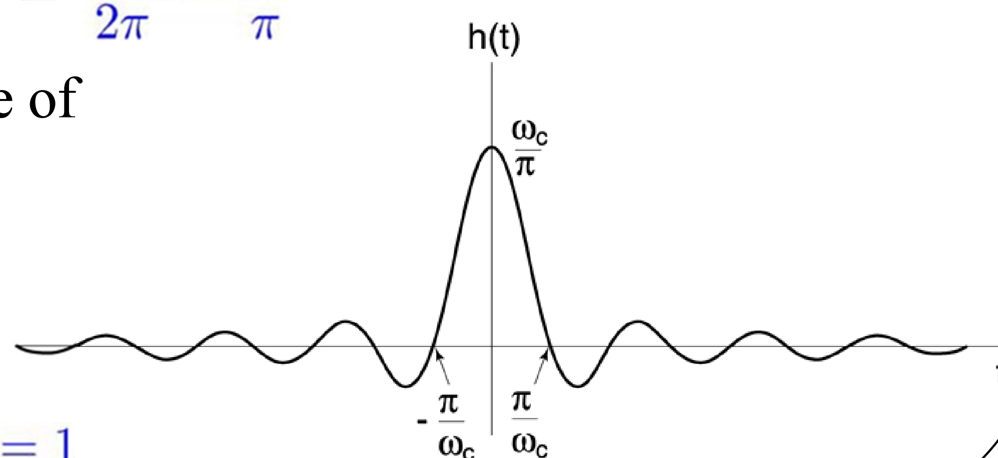
2) What is $h(0)$?

$$h(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)\,d\omega = \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

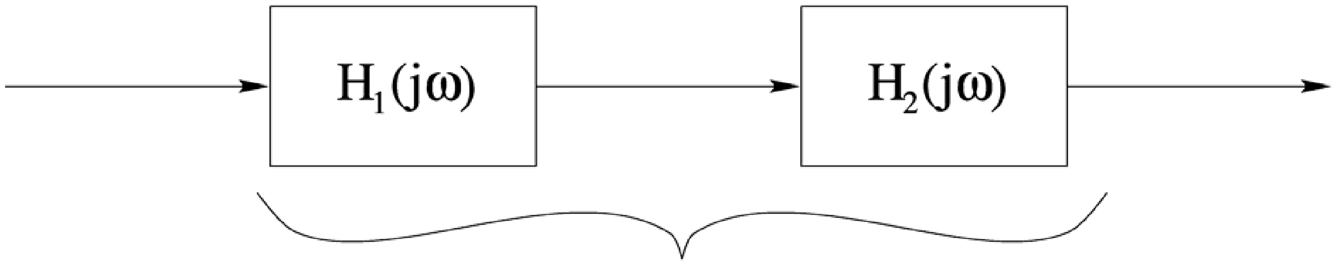
3) What is the steady-state value of the step response, i.e. $s(\infty)$?

$$s(t) = \int_{-\infty}^{t} h(t)\,dt$$

$$s(\infty) = \int_{-\infty}^{\infty} h(t)\,dt = H(j0) = 1$$

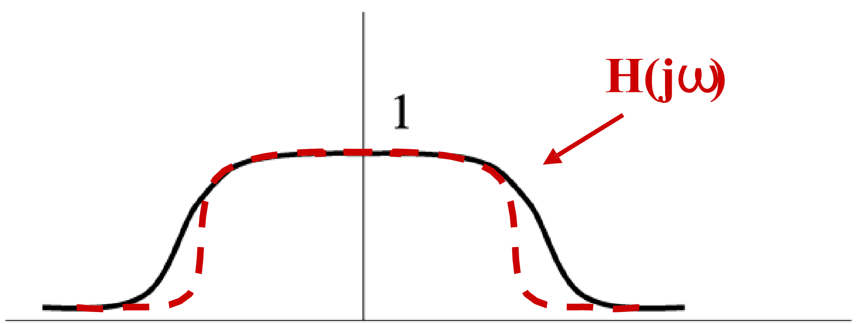


Example #4: Cascading filtering operations



$$H(j\omega) = H_1(j\omega)\, H_2(j\omega)$$

e.g. $H_1(j\omega) = H_2(j\omega)$

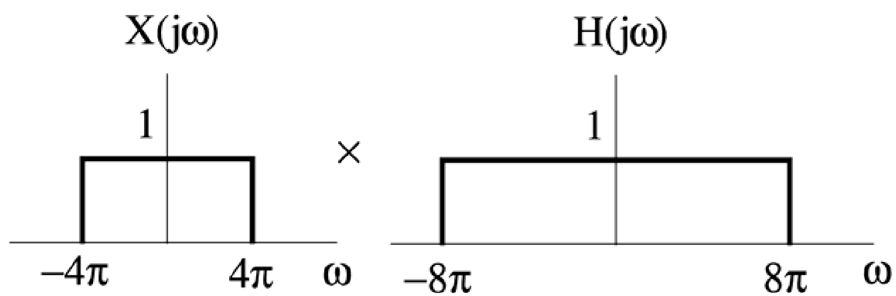


$\Longrightarrow$ $H(j\omega) = H_1^2(j\omega)$ has a sharper frequency selectivity

Example #5:

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} \quad = \quad ?$$

$$\Downarrow$$



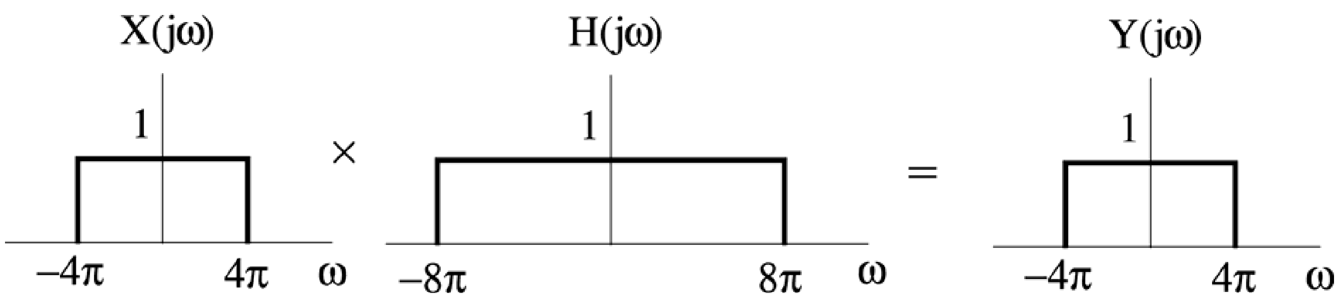
Example #5:

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = \ ?$$

$$\Downarrow$$

$$Y(j\omega) = X(j\omega)$$

$$\Rightarrow y(t) = x(t)$$



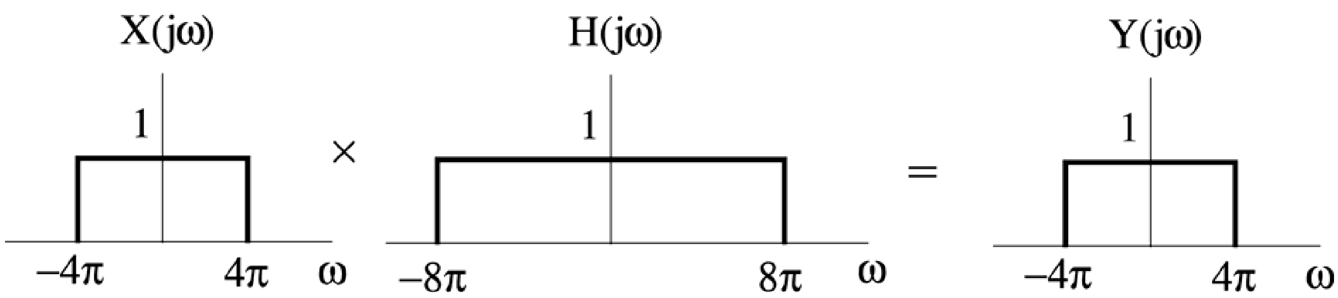
Example #5:

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = \quad ?$$

$$Y(j\omega) = X(j\omega)$$

$$\Rightarrow y(t) = x(t)$$

$$\Downarrow$$



Example #6:

$$e^{-at^2} * e^{-bt^2} = \quad ?$$

$$\Downarrow$$

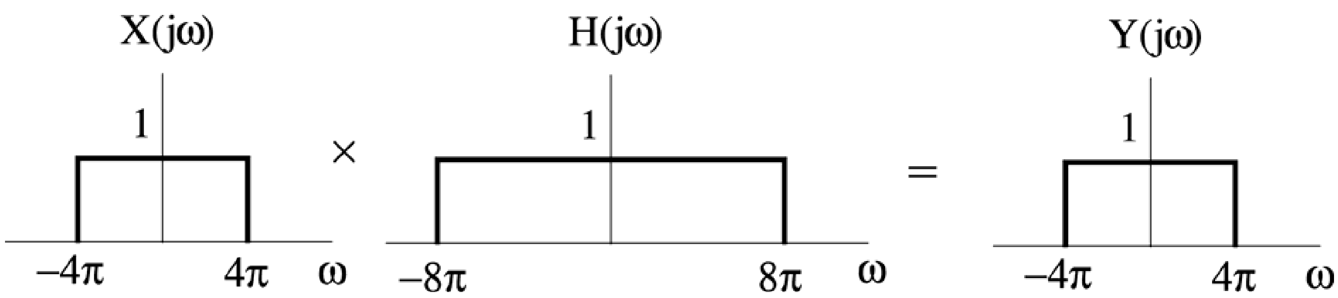
Example #5:

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = \ ?$$

$$Y(j\omega) = X(j\omega)$$

$$\Rightarrow y(t) = x(t)$$

$$\Downarrow$$



Example #6:

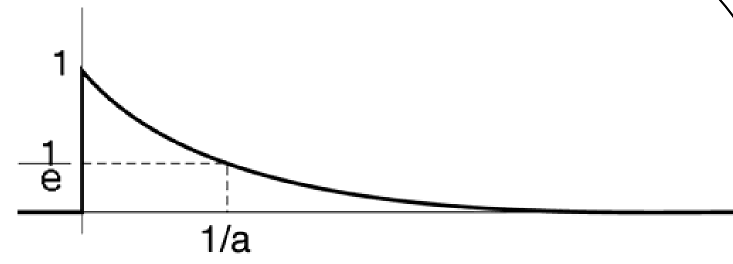
$$e^{-at^2} * e^{-bt^2} = \ ? \qquad \sqrt{\frac{\pi}{a+b}} \cdot e^{-\left(\frac{ab}{a+b}\right)t^2}$$

$$\Downarrow$$

$$\sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \times \sqrt{\frac{\pi}{b}} e^{-\frac{-\omega^2}{4b}} = \frac{\pi}{\sqrt{ab}} e^{-\frac{\omega^2}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Gaussian × Gaussian = Gaussian ⇒ Gaussian ∗ Gaussian = Gaussian

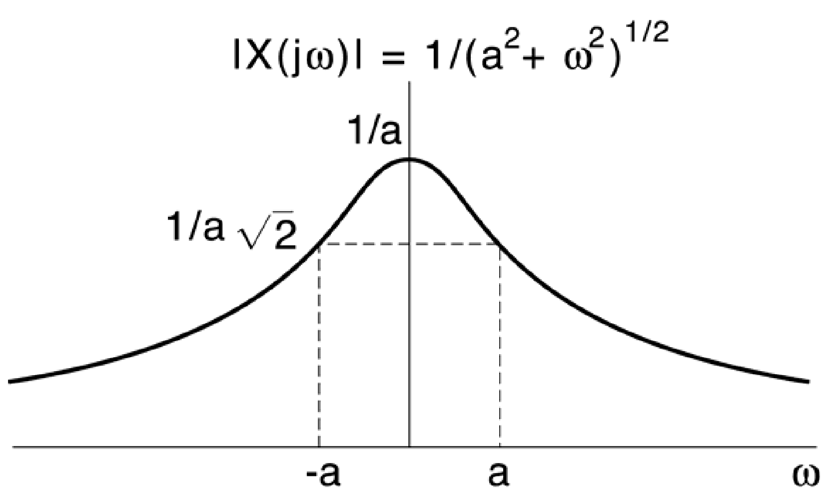
Example #2 from last lecture

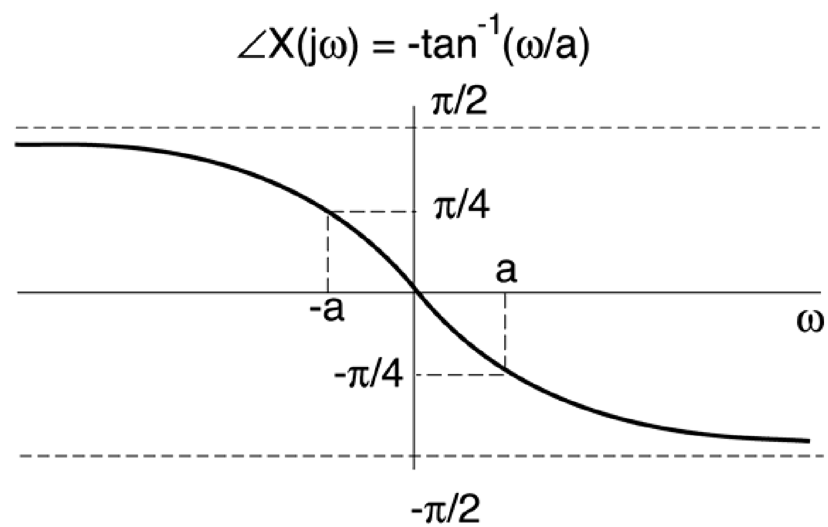


$$x(t) = e^{-at}u(t) \quad , \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)}} dt$$

$$= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$





Example #7:

$$h(t) = e^{-t}u(t), \quad x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t)$$

$$\Downarrow$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+j\omega)} \cdot \frac{1}{(2+j\omega)}$$

- a rational function of $j\omega$, ratio of polynomials of $j\omega$

$$\Downarrow \text{ Partial fraction expansion}$$

$$Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$$\Downarrow \text{ inverse FT}$$

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

Example #8: LTI Systems Described by LCCDE's

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k \cdot (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k \cdot (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \right]}_{H(j\omega)} X(j\omega)$$

1) Rational, can use PFE to get $h(t)$
2) If $X(j\omega)$ is rational
   e.g. $x(t) = \sum c_l e^{-at} u(t)$
   then $Y(j\omega)$ is also rational

تبدیل فوریه‌ی پیوسته–زمان (۲)

۳

خاصیت ضرب و رابطه‌ی پارسوال

Parseval's Relation

$$\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\substack{\text{Total energy} \\ \text{in the time-domain}}} = \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}_{\substack{\text{Total energy} \\ \text{in the frequency-domain}}}$$

$\frac{1}{2\pi}|X(j\omega)|^2$ - Spectral density

Multiplication Property

FT is highly symmetric,

$$x(t) \overset{\mathcal{F}^{-1}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega, \; X(j\omega) \overset{\mathcal{F}}{=} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

We already know that: $x(t) * y(t) \longleftrightarrow X(j\omega) \cdot Y(j\omega)$

Then it isn't a surprise that:

$$x(t) \cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) Y(j(\omega - \theta)) d\theta$$

Convolution in ω

— A consequence of *Duality*

Examples of the Multiplication Property

$$r(t) = s(t) \cdot p(t) \quad \longleftrightarrow \quad R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

$$\text{For } p(t) = \cos \omega_0 t \quad \longleftrightarrow \quad P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

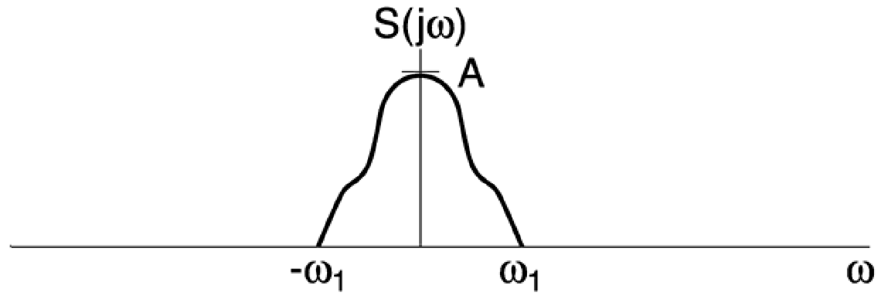
$$\Downarrow$$

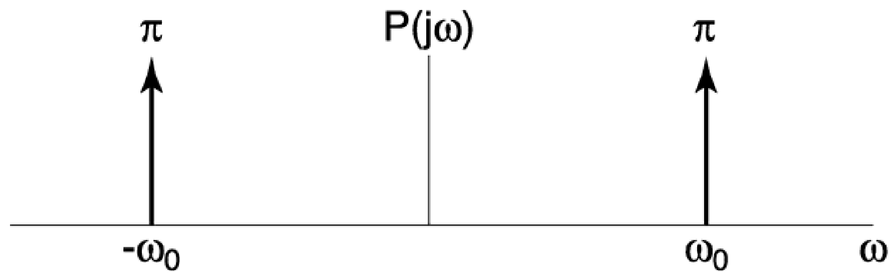
For any *s(t)* ...

$$R(j\omega) = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$$

Example (continued)

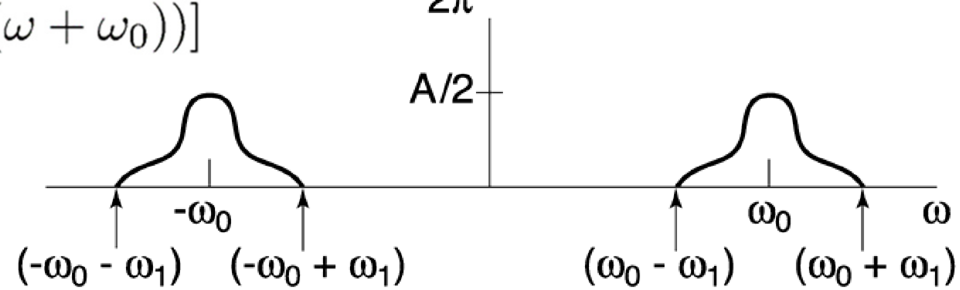
$r(t) = s(t) \cdot \cos(\omega_0 t)$
Amplitude modulation (AM)





$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_0)) + S(j(\omega + \omega_0))]$$

$$R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$



Drawn assuming:

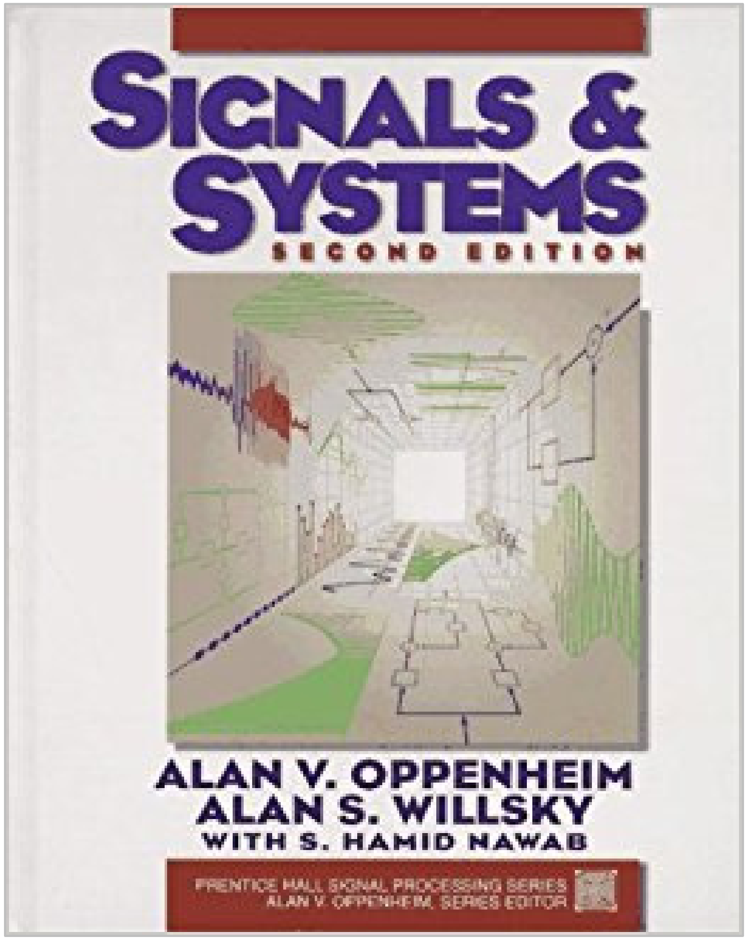
$$\omega_0 - \omega_1 > 0$$

i.e. $\omega_0 > \omega_1$

۴

منابع

منبع اصلی



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Chapter 4