**سیگنال‌ها و سیستم‌ها**

درس ۱۳

# تبدیل فوریه‌ی پیوسته–زمان (۱)

## The Continuous-Time Fourier Transform (1)


کاظم فولادی قلعه
دانشکده مهندسی، دانشکدگان فارابی
دانشگاه تهران


http://courses.fouladi.ir/sigsys

# طرح درس

COURSE OUTLINE

| |
|---|
| **استخراج جفت تبدیل فوریه‌ی پیوسته–زمان** |
| Derivation of the CT Fourier Transform pair |
| **مثال‌هایی از تبدیل‌های فوریه** |
| Examples of Fourier Transforms |
| **تبدیل‌های فوریه‌ی سیگنال‌های متناوب** |
| Fourier Transforms of Periodic Signals |
| **خصوصیات تبدیل فوریه‌ی پیوسته–زمان** |
| Properties of the CT Fourier Transform |

# ۱

# استخراج جفت تبدیل فوریه‌ی پیوسته–زمان

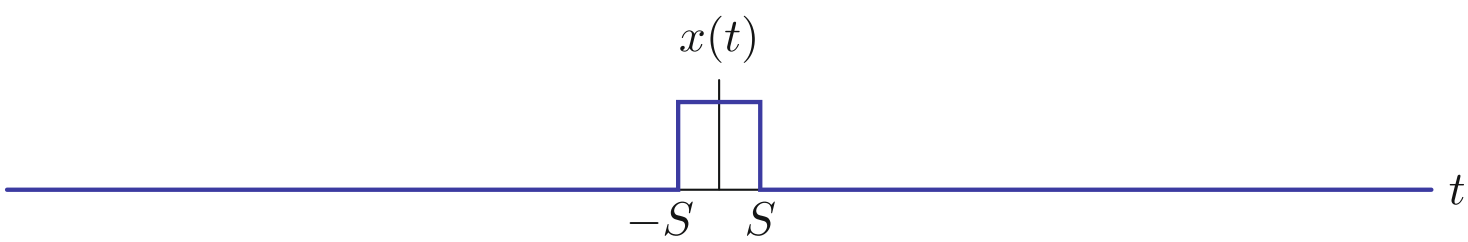
# تبدیل فوریه

## سیگنال نامتناوب

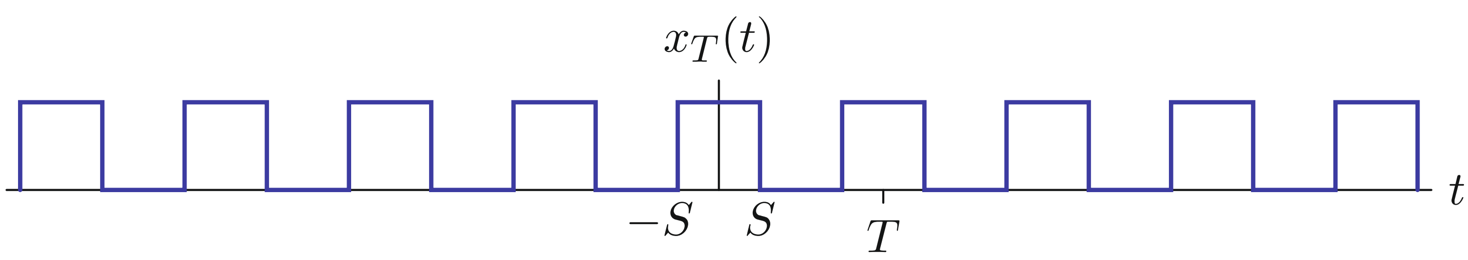
### Fourier Transform

یک سیگنال نامتناوب می‌تواند به‌عنوان یک سیگنال متناوب با دوره‌ی تناوب نامتناهی دیده شود.

Let $x(t)$ represent an aperiodic signal.



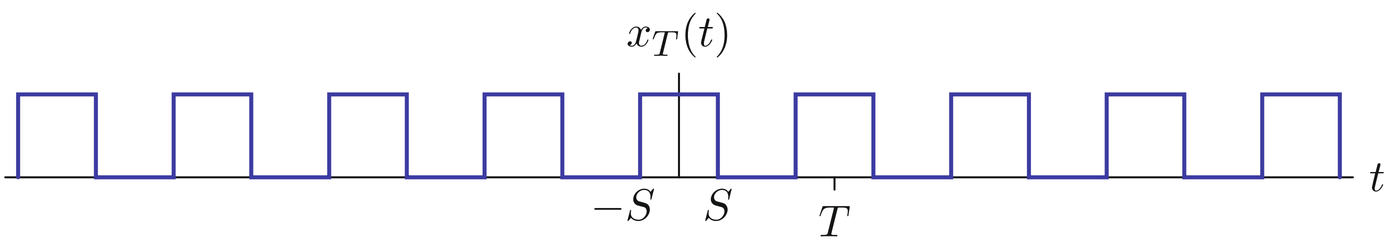

"Periodic extension": $x_T(t) = \sum_{k=-\infty}^{\infty} x(t+kT)$



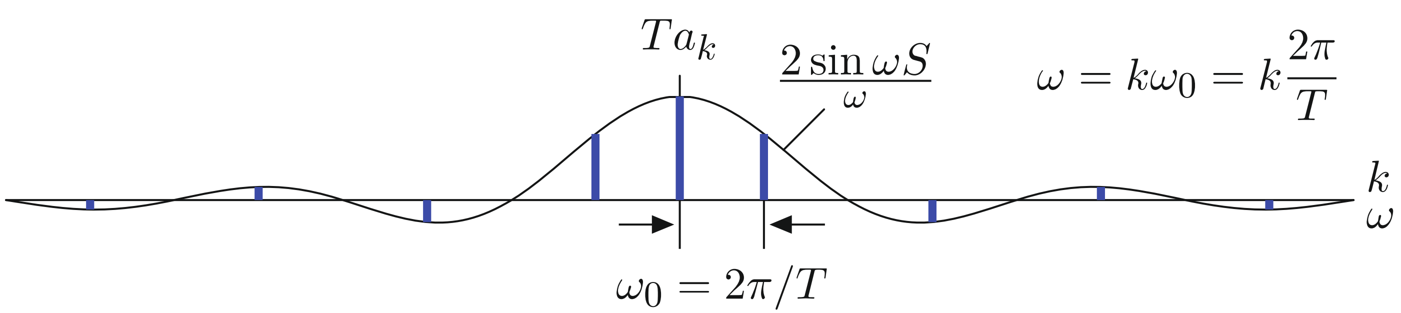

Then $x(t) = \lim_{T\to\infty} x_T(t)$.

## Fourier Transform

Represent $x_T(t)$ by its Fourier series.



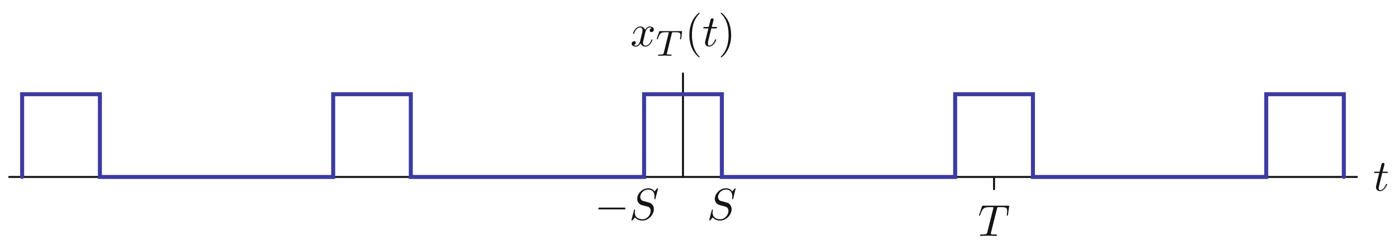

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$



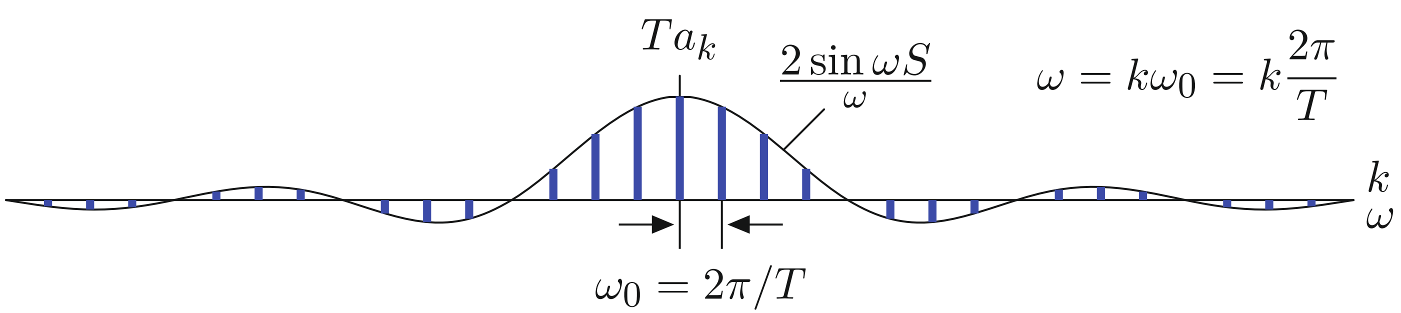
# تبدیل فوریه

## FOURIER TRANSFORM

Doubling period doubles # of harmonics in given frequency interval.



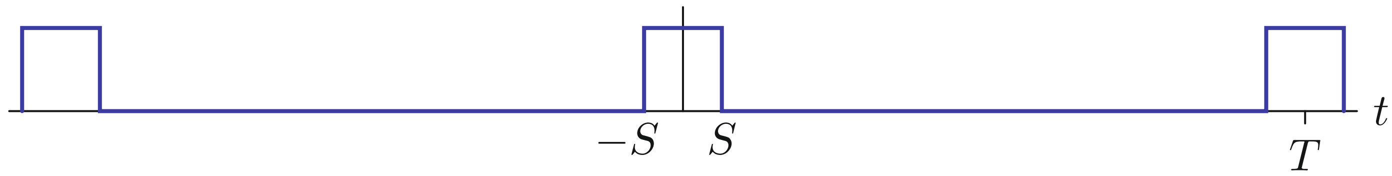

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi kS}{T}}{\pi k} = \frac{2}{T}\frac{\sin\omega S}{\omega}$$



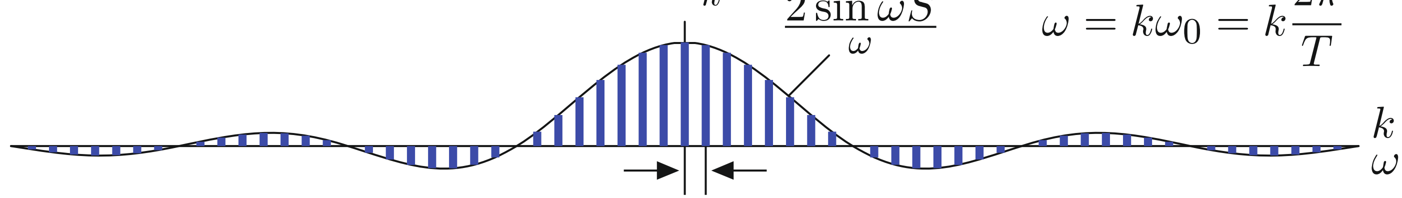
# تبدیل فوریه

## Fourier Transform

As $T \to \infty$, discrete harmonic amplitudes $\to$ a continuum $E(\omega)$.




$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt = \frac{\sin\frac{2\pi k S}{T}}{\pi k} = \frac{2}{T}\,\frac{\sin\omega S}{\omega}$$

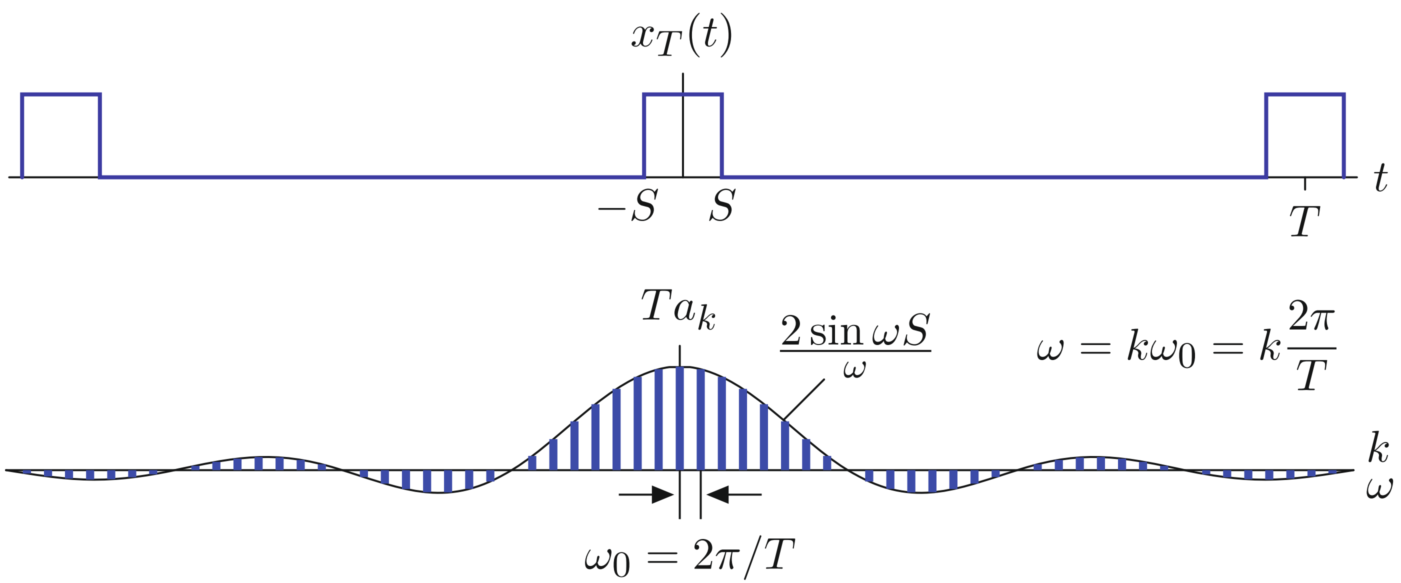



$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin\omega S = E(\omega)$$

# تبدیل فوریه

## FOURIER TRANSFORM

As $T \to \infty$, synthesis sum $\to$ integral.




$$\lim_{T\to\infty} Ta_k = \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \frac{2}{\omega}\sin \omega S = E(\omega)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}E(\omega)}_{a_k} e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi}E(\omega)e^{j\omega t} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\omega)e^{j\omega t}d\omega$$

# Fourier's Derivation of the CT Fourier Transform

- $x(t)$ - an aperiodic signal
  - view it as the limit of a periodic signal as $T \to \infty$

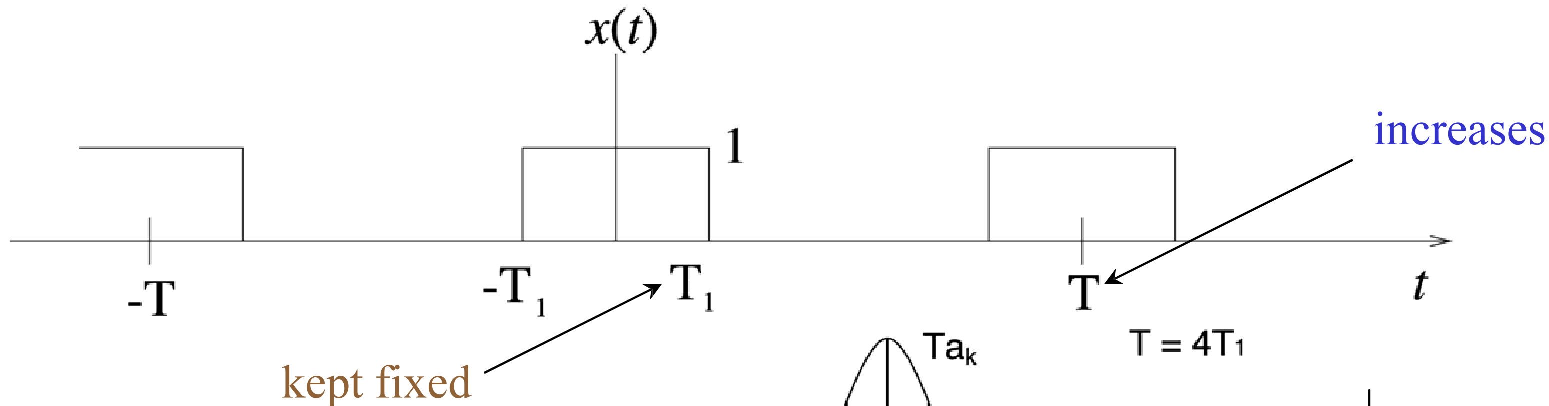
- For a periodic signal, the harmonic components are spaced $\omega_0 = 2\pi/T$ apart ...

- As $T \to \infty$, $\omega_0 \to 0$, and harmonic components are spaced closer and closer in frequency

$$\Downarrow$$

$$\text{Fourier series} \longrightarrow \text{Fourier integral}$$

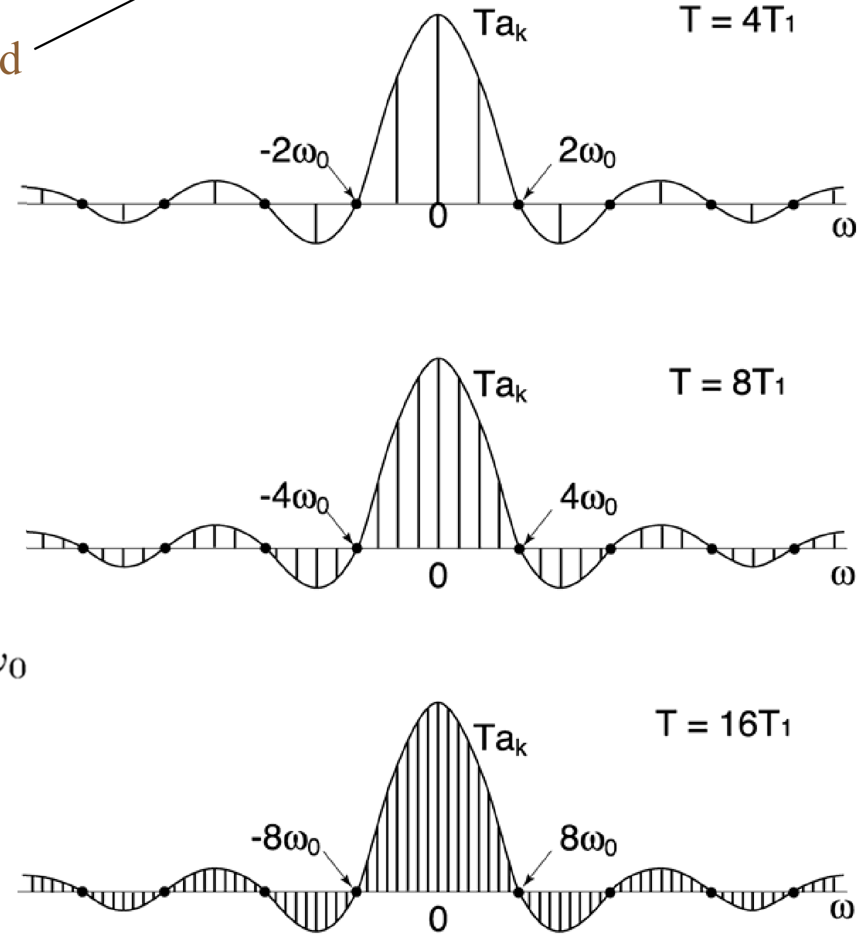
# Motivating Example: Square wave




$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

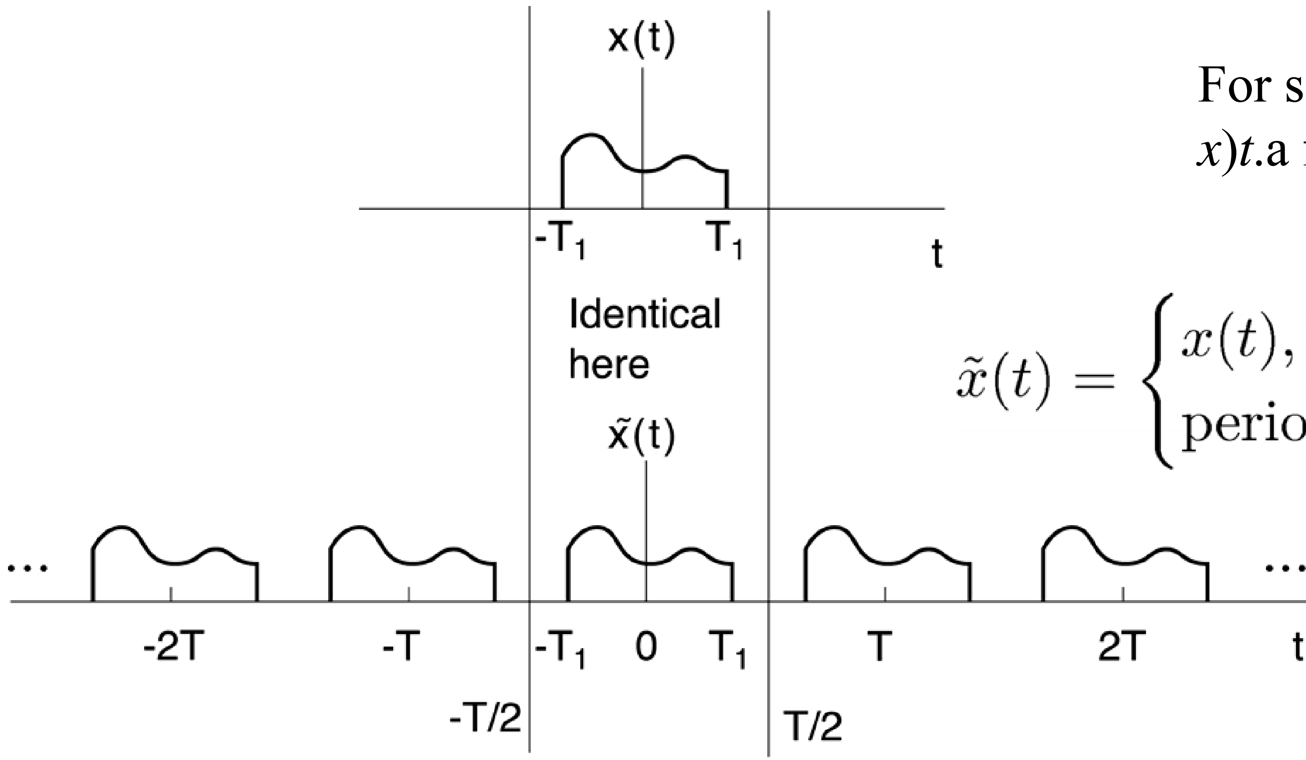
$$\Downarrow$$

$$T a_k = \left. \frac{2\sin \omega T_1}{\omega} \right|_{\omega = k\omega_0}$$




Discrete frequency points become denser in $\omega$ as $T$ increases

## So, on with the derivation ...




For simplicity, assume
*x*)*t*.a finite duration has (

$$\tilde{x}(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ \text{periodic}, & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \to \infty,\ \tilde{x}(t) = x(t) \text{ for all } t$$

# Derivation (continued)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

$\uparrow$

$\tilde{x}(t) = x(t)$ in this interval

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt \qquad (1)$$

If we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

then Eq.(1) $\Rightarrow$

$$a_k = \frac{X(jk\omega_0)}{T}$$

# Derivation (continued)

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t}$$

$$\Downarrow$$

As $T \to \infty$, $\sum \omega_0 \to \int d\omega$, we get the CT Fourier Transform pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Synthesis equation}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Analysis equation}$$

## For what kinds of signals can we do this?

(1) It works also even if $x(t)$ is infinite duration, but satisfies:

a) Finite energy $$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

In this case, there is *zero* energy in the error

$$e(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \text{Then} \quad \int_{-\infty}^{\infty} |e(t)|^2 dt = 0$$

b) Dirichlet conditions

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = x(t)$ at points of continuity

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega =$ midpoint at discontinuity

(iii) Gibb's phenomenon

c) By allowing impulses in $x(t)$ or in $X(j\omega)$, we can represent even *more* signals

E.g. It allows us to consider *FT* for *periodic* signals

تبدیل فوریه‌ی پیوسته-زمان (۱)

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# مثال‌هایی از تبدیل‌های فوریه

**Example #1**

(a) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$ — Synthesis equation for $\delta(t)$

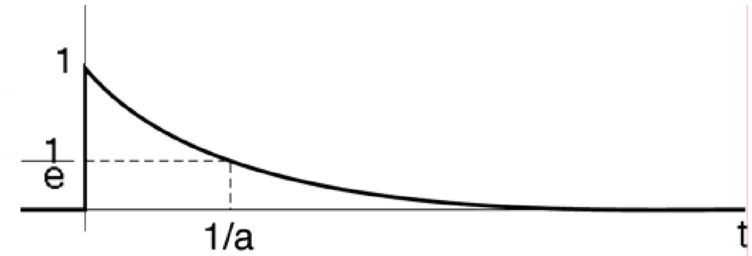
(b) $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_0}$$

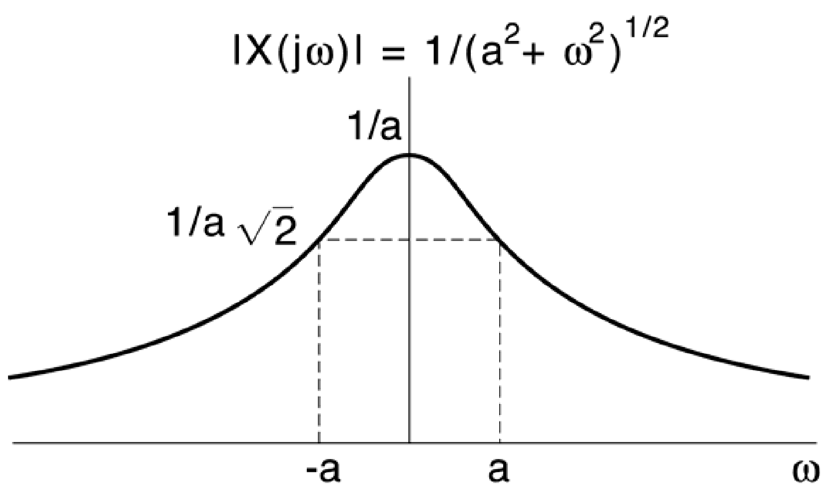
**Example #2: Exponential function**

$$x(t) = e^{-at}u(t), a > 0$$

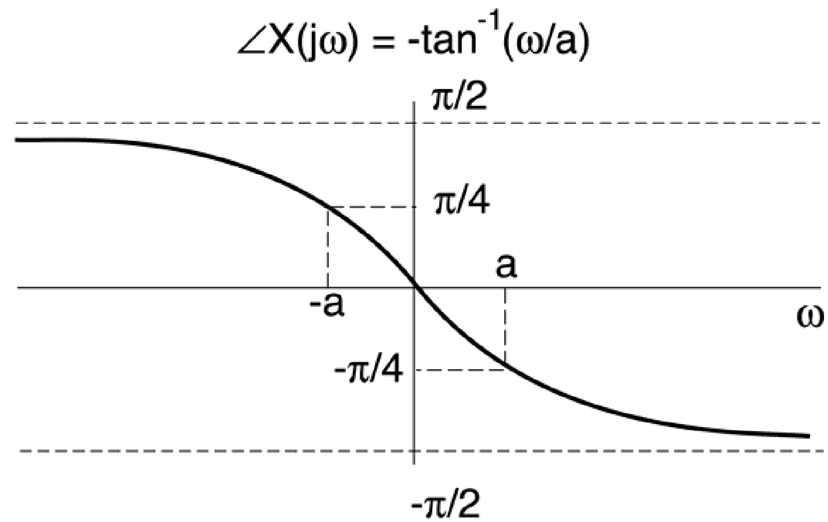



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}}dt$$

$$= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$



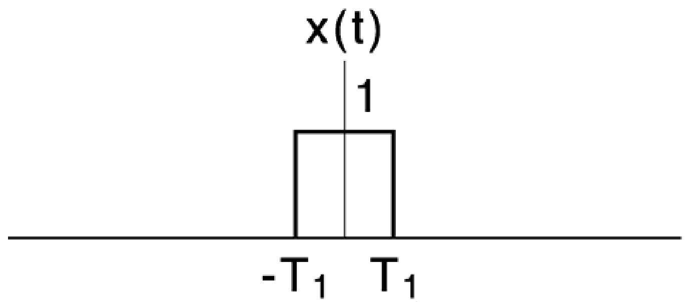

Even symmetry

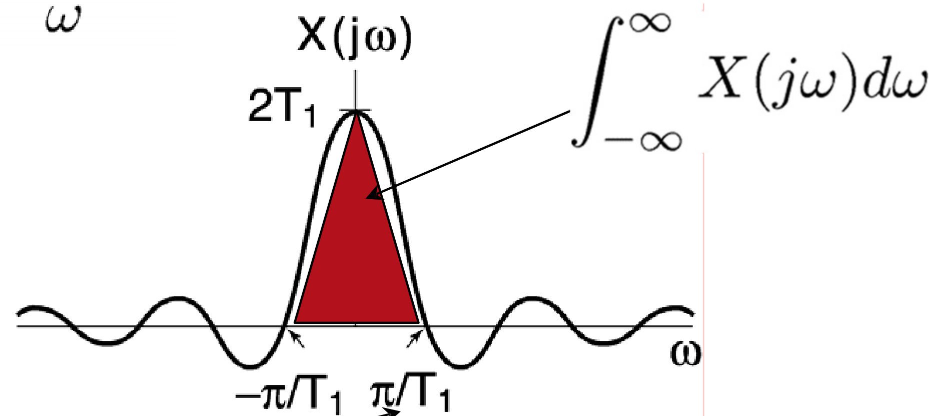



Odd symmetry

**Example #3: A square pulse in the time-domain**

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin \omega T_1}{\omega}$$




Note the inverse relation between the two widths $\Rightarrow$ Uncertainty principle

Useful facts about CTFT's

$$X(0) = \int_{-\infty}^{\infty} x(t)dt$$

Example above: $\int_{-\infty}^{\infty} x(t)dt = 2T_1 = X(0)$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$$

Ex. above: $x(0) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$

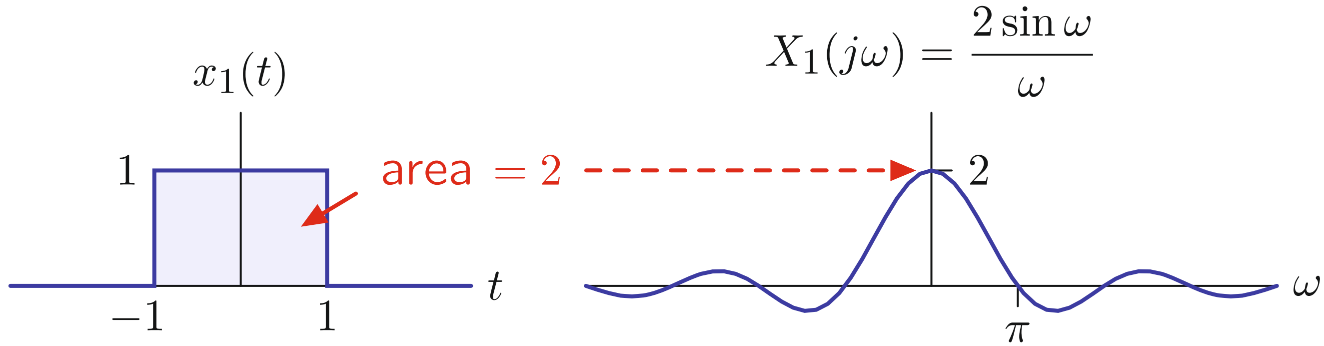
$$= \frac{1}{2\pi} \times (\text{Area of the triangle})$$

# خصوصیات تبدیل فوریه‌ی پیوسته-زمان

## رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۱)

The value of $X(j\omega)$ at $\omega = 0$ is the integral of $x(t)$ over time $t$.

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j0t}dt = \int_{-\infty}^{\infty} x(t)\,dt$$

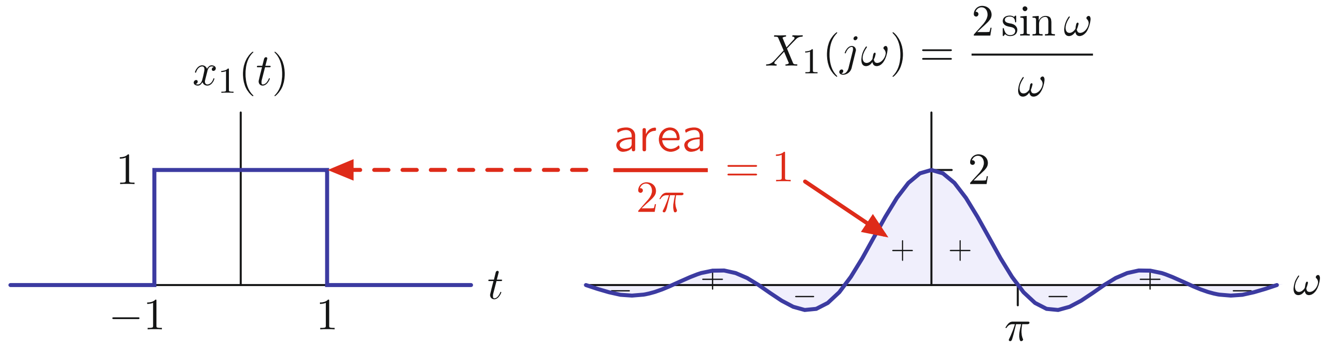


# خصوصیات تبدیل فوریه‌ی پیوسته-زمان

## رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۲)

The value of $x(0)$ is the integral of $X(j\omega)$ divided by $2\pi$.

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$

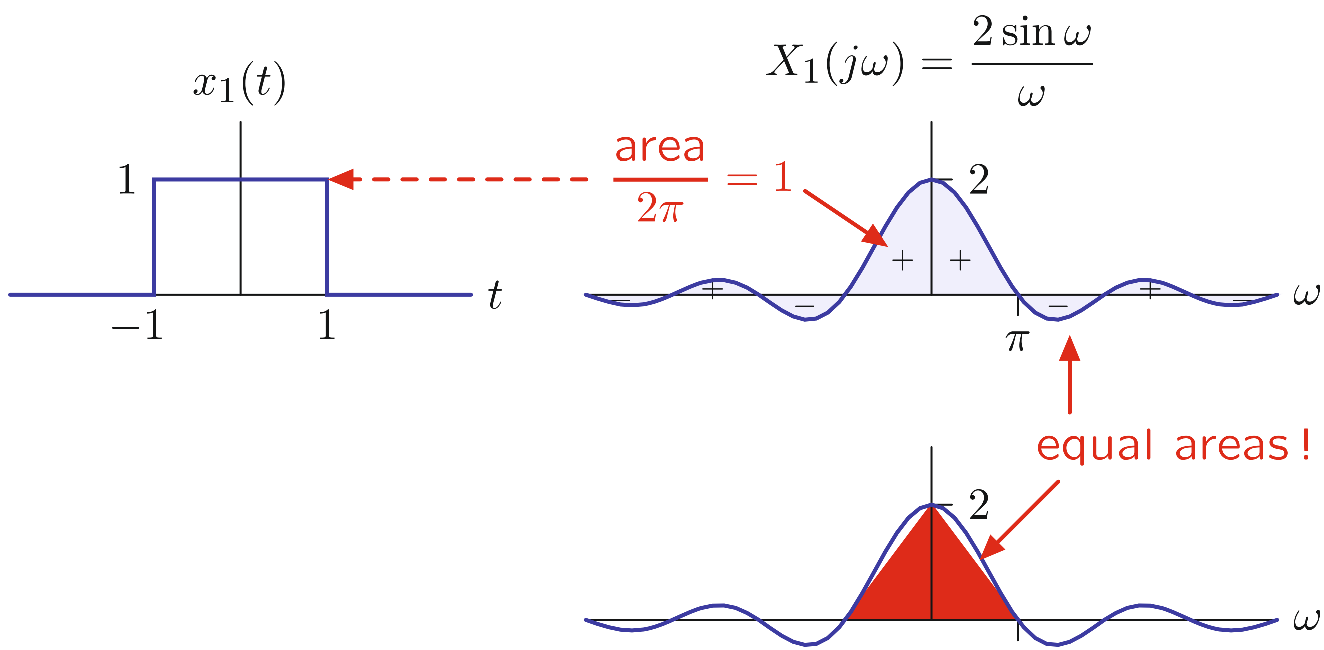


# خصوصیات تبدیل فوریه‌ی پیوسته–زمان

## رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۳)

The value of $x(0)$ is the integral of $X(j\omega)$ divided by $2\pi$.

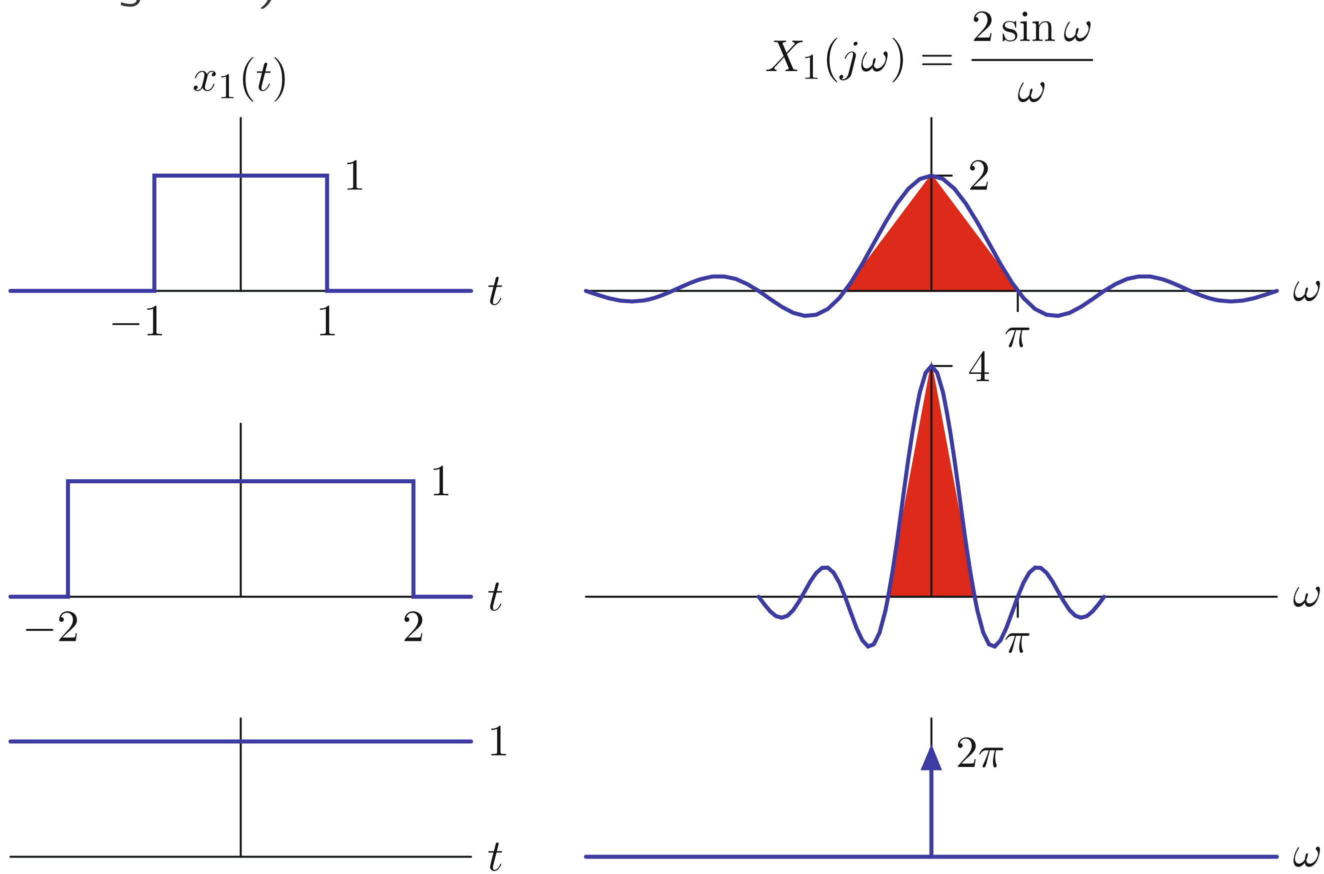
$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega$$



# خصوصیات تبدیل فوریه‌ی پیوسته-زمان

## رابطه‌ی حوزه‌ی زمان و حوزه‌ی فرکانس (۴)

Stretching time compresses frequency and increases amplitude (preserving area).




New way to think about an impulse!

**Example #4:** $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$$X(j\omega)$$

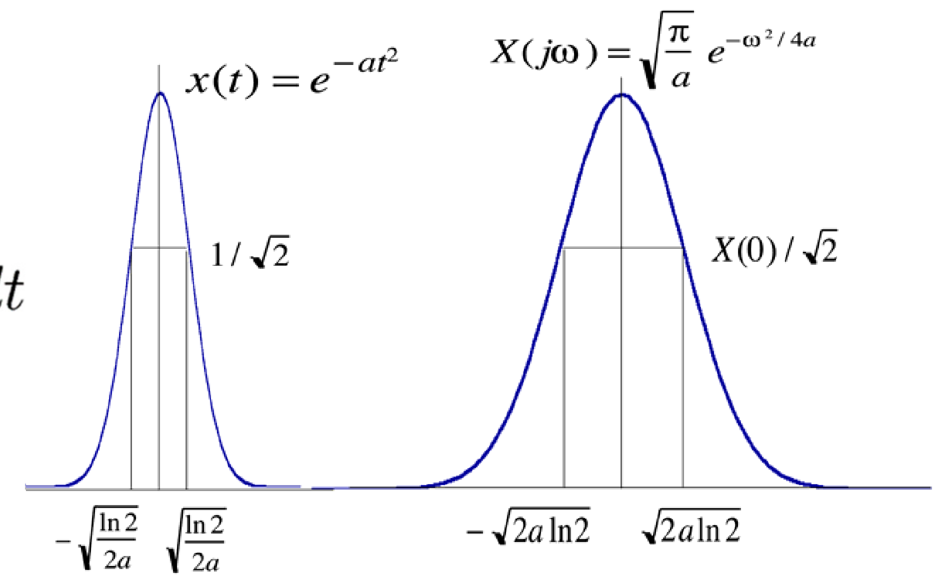
$$= \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2} dt$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t + \frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi}/a} \cdot e^{-\frac{\omega^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$$

Also a Gaussian!




(Pulse width in $t$)•(Pulse width in $\omega$)

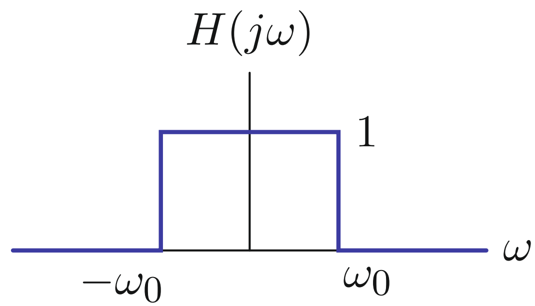
$\Rightarrow \Delta t \bullet \Delta\omega \sim (1/a^{1/2})\bullet(a^{1/2}) = 1$

Uncertainty Principle! Cannot make *both* $\Delta t$ and $\Delta\omega$ arbitrarily small.

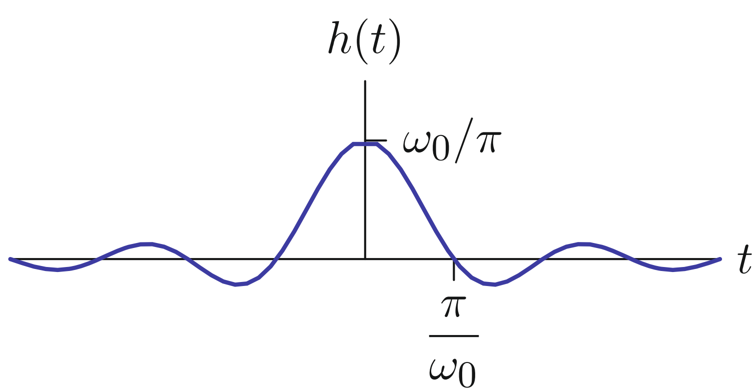
# تبدیل وارون فوریه

## INVERSE FOURIER TRANSFORM

Find the impulse reponse of an "ideal" low pass filter.




$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}d\omega = \frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0 t}{\pi t}$$




This result is not so easily obtained without inverse relation.

# خاصیت دوگانی



## DUALITY

The Fourier transform and its inverse have very similar forms.

$$X(j\omega)= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(Fourier transform)}$$

$$x(t)= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{(“inverse” Fourier transform)}$$

Convert one to the other by
- $t \to \omega$
- $\omega \to -t$
- scale by $2\pi$

# خاصیت دوگانی

تبدیل مستقیم و تبدیل وارون فوریه

## DUALITY

The Fourier transform and its inverse have very similar forms.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Two differences:

- minus sign: flips time axis (or equivalently, frequency axis)
- divide by $2\pi$ (or multiply in the other direction)

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega)$$

$\omega \rightarrow t$ $\quad$ $t \rightarrow \omega$ ; flip ; $\times 2\pi$

$$x_2(t) = g(t) \leftrightarrow X_2(j\omega) = 2\pi f(-\omega)$$

# خاصیت دوگانی

تبدیل مستقیم و تبدیل وارون فوریه

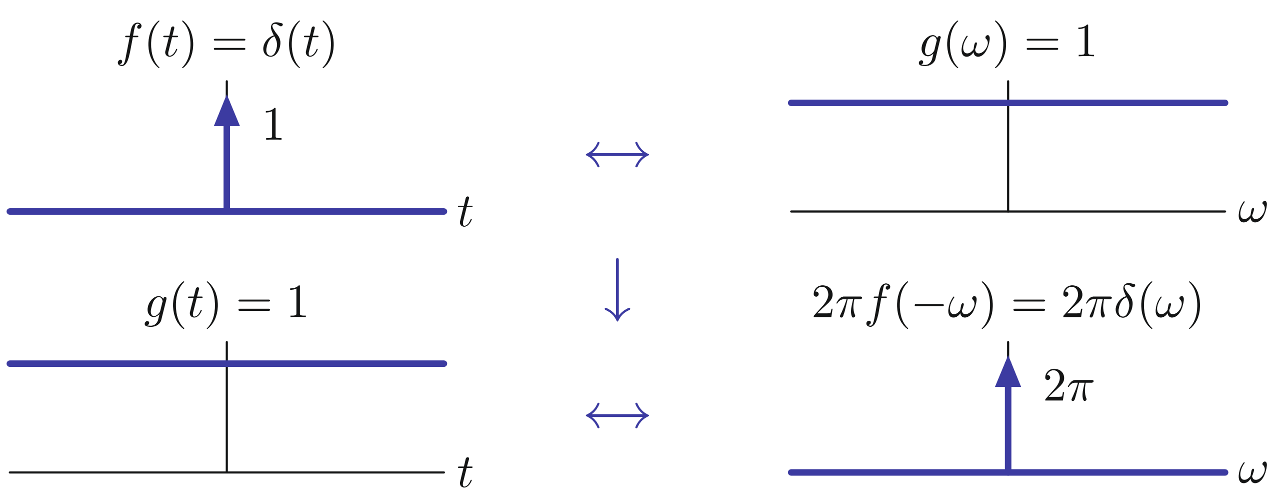
## DUALITY

Using duality to find new transform pairs.

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega)$$

$\omega \rightarrow t$ $\quad$ $t \rightarrow \omega$ ; flip ; $\times 2\pi$

$$x_2(t) = g(t) \leftrightarrow X_2(j\omega) = 2\pi f(-\omega)$$




The function $g(t) = 1$ does not have a Laplace transform!

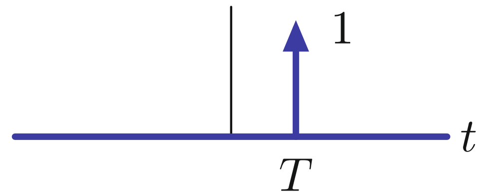
# خاصیت دوگانی

تبدیل مستقیم و تبدیل وارون فوریه

## DUALITY

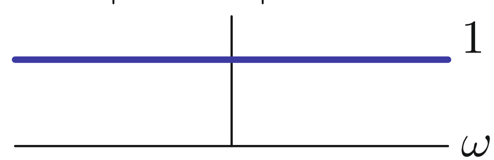
Fourier transform of delayed impulse: $\delta(t-T) \leftrightarrow e^{-j\omega T}$.

$$x(t) = \delta(t-T)$$

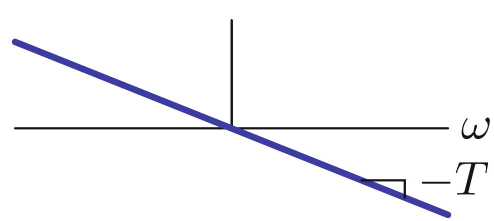



$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t-T) e^{-j\omega t} dt = e^{-j\omega T}$$

$$\left|X(j\omega)\right| = 1$$




$$\angle X(j\omega) = -\omega T$$



# ۳

# تبدیل‌های فوریه‌ی سیگنال‌های متناوب

# تبدیل فوریه‌ی سیگنال‌های متناوب

## تبدیل فوریه‌ی سینوس

Using duality to find the Fourier transform of an eternal sinusoid.

$$\delta(t-T) \quad \leftrightarrow \quad e^{-j\omega T}$$

$\omega \to t$ $\qquad$ $t \to \omega$ ; flip ; $\times 2\pi$

$$e^{-jtT} \quad \leftrightarrow \quad 2\pi\delta(\omega+T)$$

$T \to \omega_0$ :

$$e^{-j\omega_0 t} \quad \leftrightarrow \quad 2\pi\delta(\omega+\omega_0)$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad \stackrel{\text{CTFS}}{\longleftrightarrow} \quad \{a_k\}$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi}{T}k\right)$$

Prepared by Kazim Fouladi | Spring 2024 | 6th Edition

# CT Fourier Transforms of Periodic Signals

Suppose

$$X(j\omega) = \delta(\omega - \omega_0)$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

— periodic in $t$ with frequency $\omega_o$

That is

$$\boxed{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

— All the energy is concentrated in one frequency — $\omega_o$

More generally

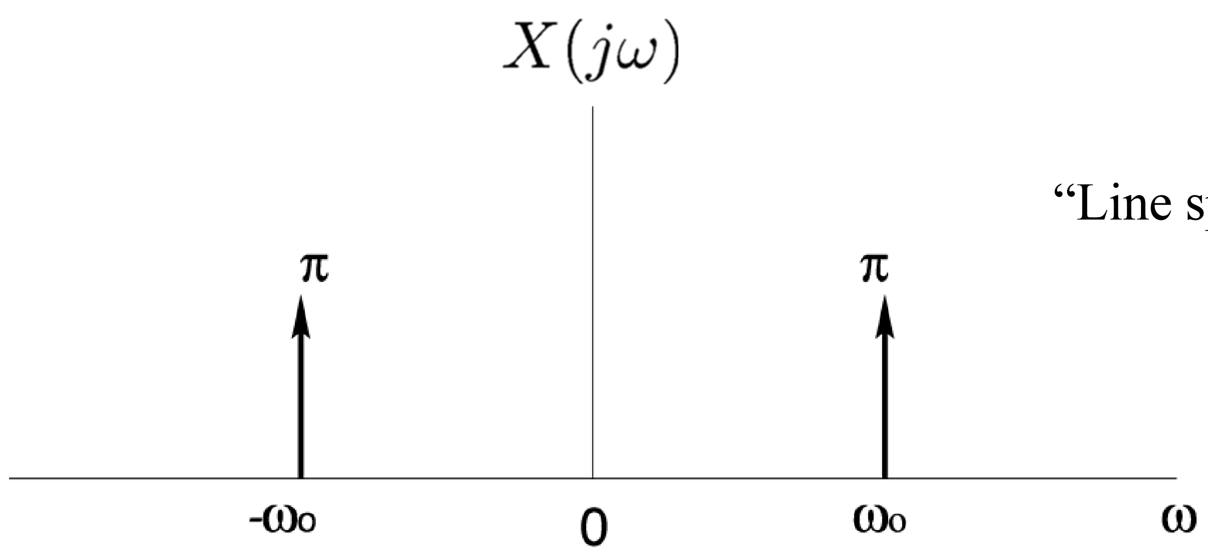
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

**Example #4:**

$$x(t) = \cos \omega_0 t = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$\updownarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



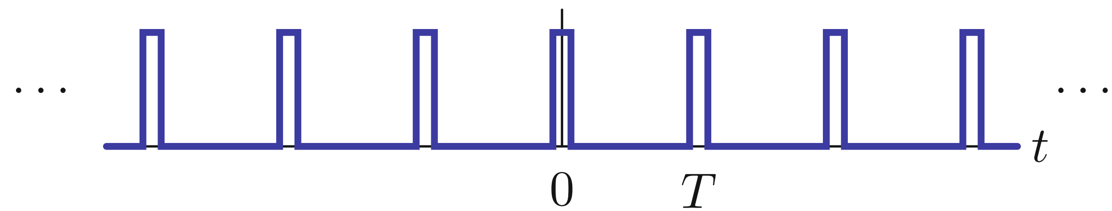

"Line spectrum"

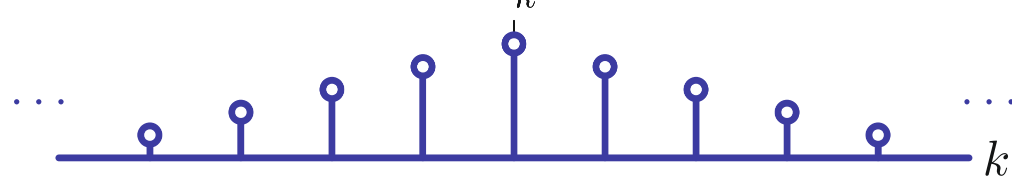
# تبدیل فوریه‌ی سیگنال‌های متناوب

## رابطه‌ی کلی

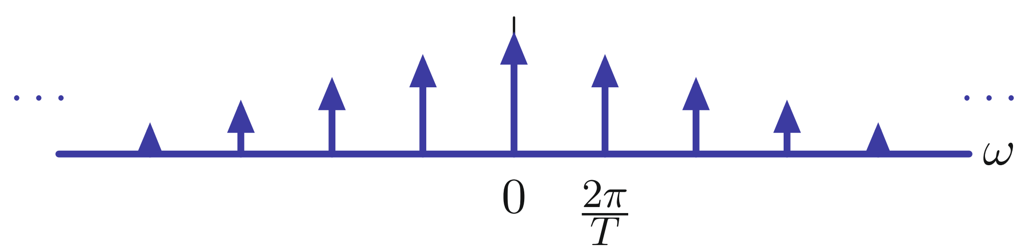
Each term in the Fourier series is replaced by an impulse.

$$x(t) = \sum_{k=-\infty}^{\infty} x_p(t - kT)$$

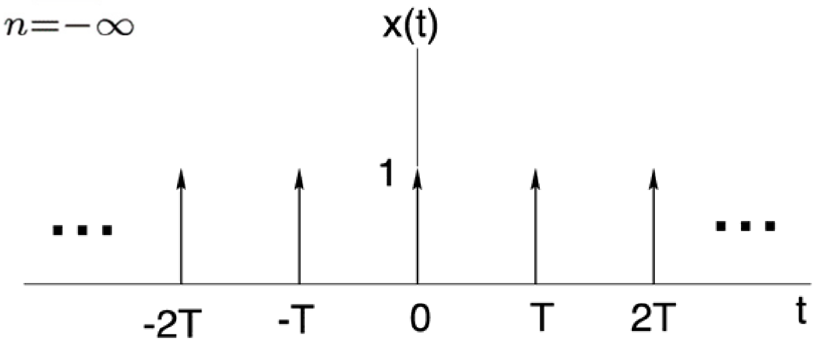



$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\frac{2\pi}{T})$$



**Example #5:** $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ — Sampling function

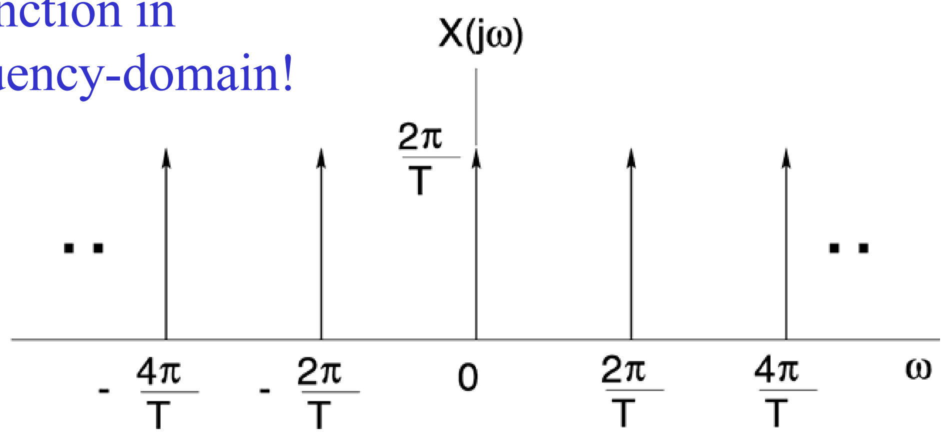



$$x(t) \leftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$$

$$\Downarrow$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{2\pi}{T}}_{2\pi a_k} \delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_0})$$

Same function in
the frequency-domain!



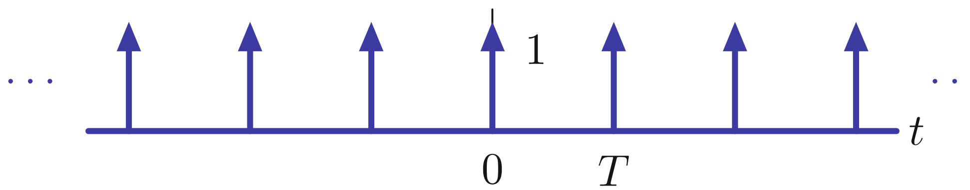

Note: (period in $t$) $T$
⇔ (period in ω) $2\pi/T$
Inverse relationship again!

# تبدیل فوریه‌ی سیگنال‌های متناوب

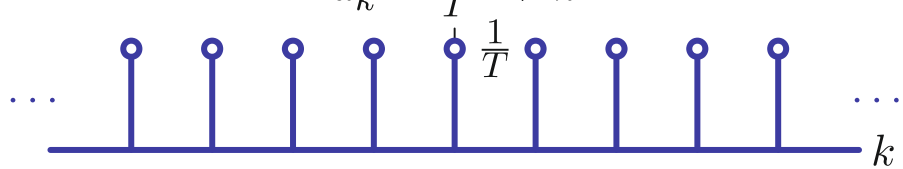
## تبدیل فوریه‌ی قطار ضربه

The Fourier transform of an impulse train is an impulse train.

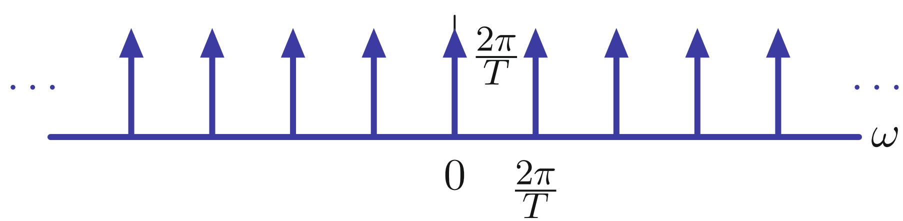
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$




$$a_k = \frac{1}{T} \quad \forall \ k$$




$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\frac{2\pi}{T})$$



# ۴

# خصوصیات تبدیل فوریه‌ی پیوسته-زمان

# Properties of the CT Fourier Transform

1) Linearity $$ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$$

2) Time Shifting $$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$$

Proof: $$\int_{-\infty}^{\infty} x(\underbrace{t - t_0}_{t'}) e^{-j\omega t} dt = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'}_{X(j\omega)}$$

*FT* magnitude unchanged

$$|e^{-j\omega t_0} X(j\omega)| = |X(j\omega)|$$

Linear change in *FT* phase

$$\angle(e^{-j\omega t_0} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

# Properties (continued)

3) Conjugate Symmetry

$$x(t) \text{ real} \leftrightarrow X(-j\omega) = X^*(j\omega)$$

$$\Downarrow$$

$$|X(-j\omega)| = |X(j\omega)|$$ *Even*

$$\angle X(-j\omega) = -\angle X(j\omega)$$ *Odd*

$$Re\{X(-j\omega)\} = Re\{X(j\omega)\}$$ *Even*

$$Im\{X(-j\omega)\} = -Im\{X(j\omega)\}$$ *Odd*

# The Properties Keep on Coming ...

4) Time-Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

E.g. $a > 1 \rightarrow at > t$
compressed in time $\Leftrightarrow$
stretched in frequency

$$\Downarrow a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$$\Downarrow$$

*a)* $x(t)$ real and even

$$x(t) = x(-t)$$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) - \text{ Real \& even}$$

*b)* $x(t)$ real and odd

$$x(t) = -x(-t)$$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) - \text{ Purely imaginary \& odd}$$

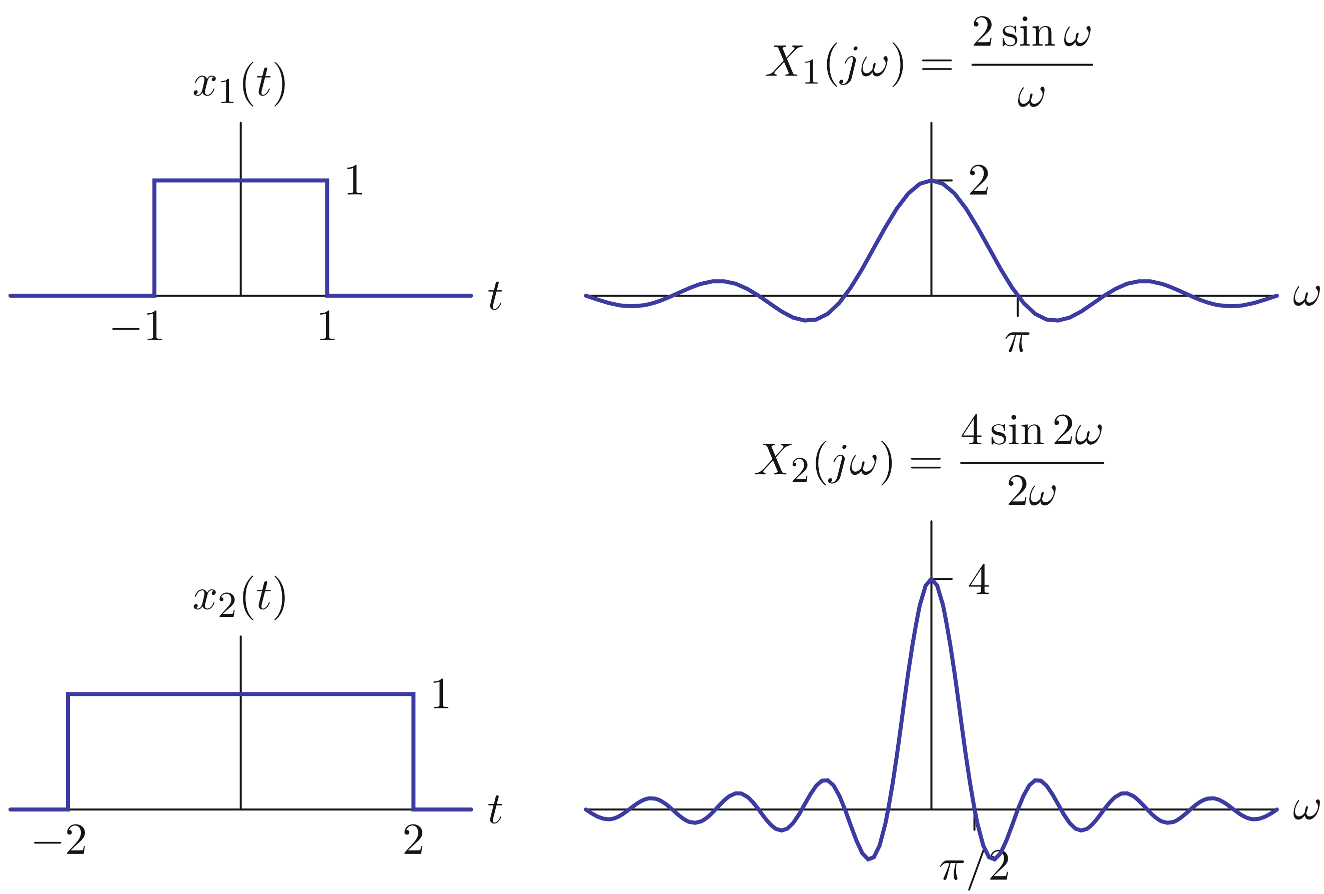
c)

$$\begin{array}{ccccc} X(j\omega) & = & Re\{X(j\omega)\} & + & jIm\{X(j\omega)\} \\ & & \uparrow & & \uparrow \\ \textbf{For real } x(t) & = & Ev\{x(t)\} & + & Od\{x(t)\} \end{array}$$

# خصوصیات تبدیل فوریه‌ی پیوسته-زمان

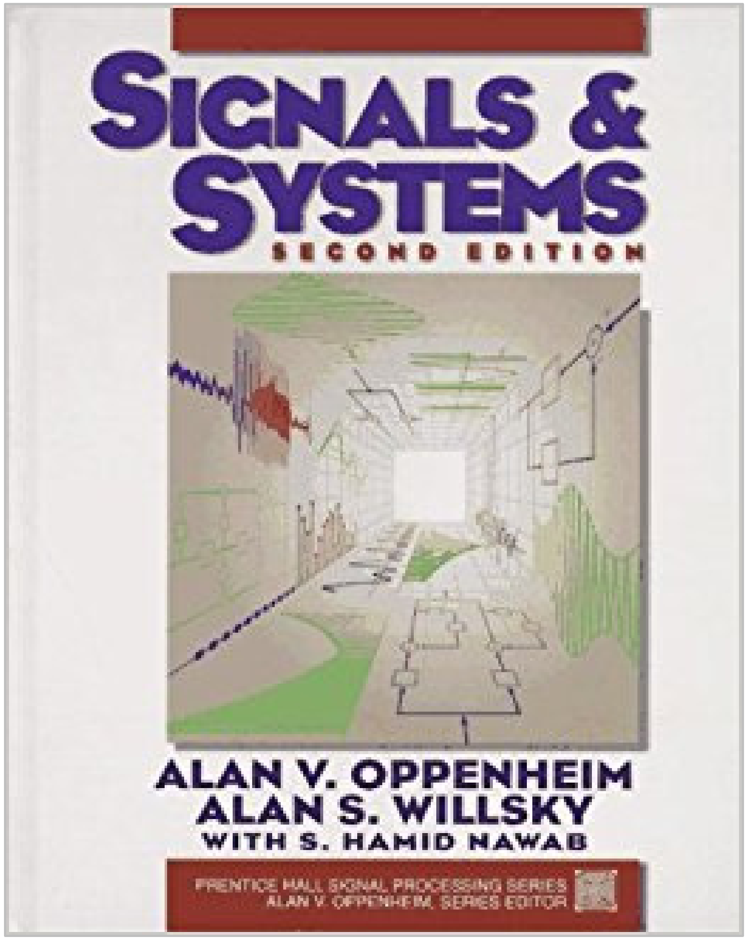
## کشش زمانی موجب فشردگی فرکانسی می‌شود

Stretching time compresses frequency.



# منابع

# منبع اصلی




A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
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**Chapter 4**