

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ



سیگنال‌ها و سیستم‌ها

درس ۱۱

بازنمایی سری فوریه سیگنال‌های متناوب (۳)

Fourier Series Representation of Periodic Signals (3)

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<http://courses.fouladi.ir/sigsys>

طرح درس

COURSE OUTLINE

سری فوریه و سیستم‌های خطی تغییرناپذیر با زمان

Fourier Series and LTI Systems

پاسخ فرکانسی و فیلترینگ

Frequency Response and Filtering

مثال‌ها و نمونه‌های نمایشی

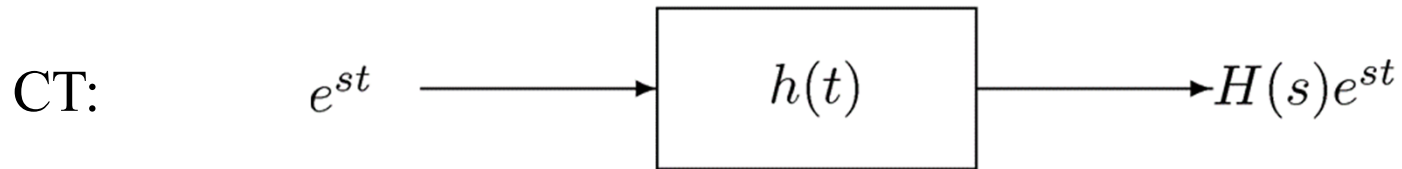
Examples and Demos

بازنمایی سری فوریه سیگنال‌های متناوب (۳)

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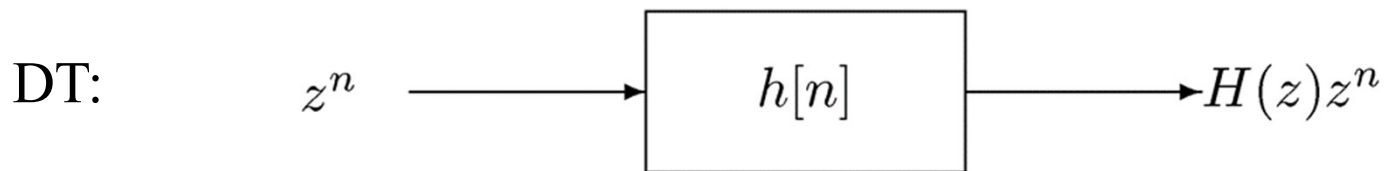
سری فوریه و سیستم‌های خطی تغییرناپذیر با زمان

The Eigenfunction Property of Complex Exponentials



CT
"System Function"

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt$$



DT
"System Function"

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Fourier Series: Periodic Signals and LTI Systems

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}} a_k$$

So $|a_k| \longrightarrow |H(jk\omega_0)| |a_k|$
 or powers of signals get
 modified through filter/system

$$H(jk\omega_0) = |H(jk\omega_0)| e^{j\angle H(jk\omega_0)}$$

includes both amplitude & phase

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=\langle N \rangle} H(e^{jk\omega_0}) a_k e^{jk\omega_0 n}$$

$$a_k \longrightarrow \underbrace{H(e^{jk\omega_0})}_{\text{"gain"}} a_k$$

$$H(e^{jk\omega_0}) = |H(e^{jk\omega_0})| e^{j\angle H(e^{jk\omega_0})}$$

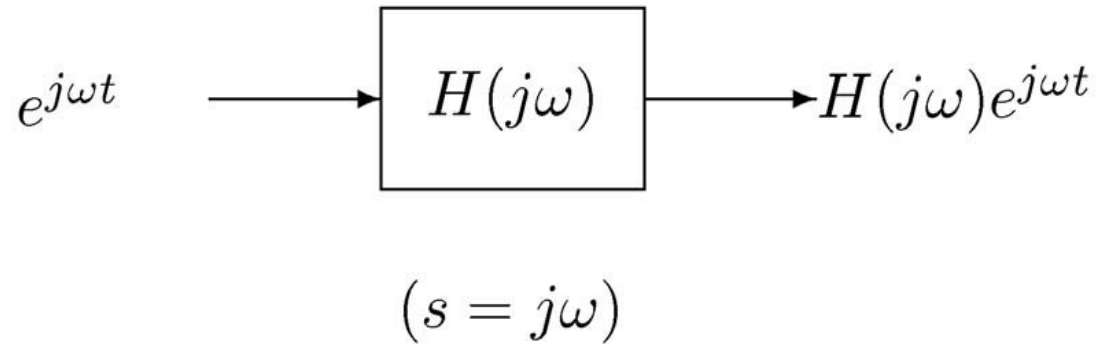
includes both amplitude & phase

بازنمایی سری فوریه سیگنال‌های متناوب (۳)

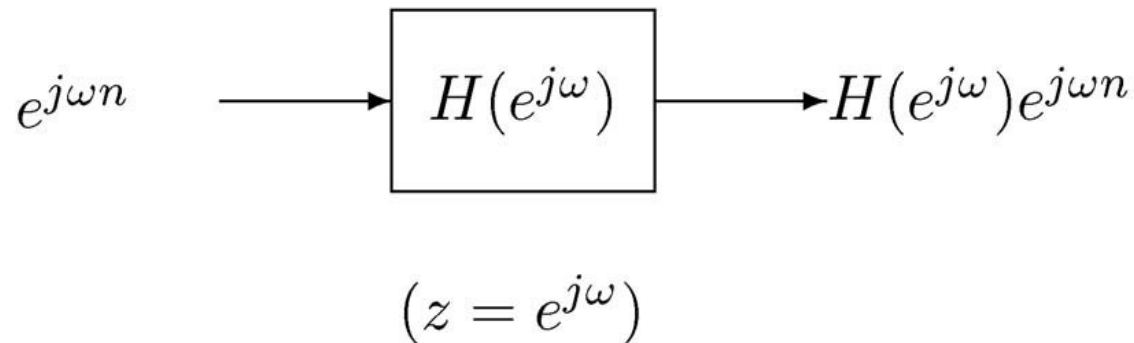
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پاسخ
فرکانسی و
فیلترینگ

The Frequency Response of an LTI System



CT Frequency response: $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$



DT Frequency response: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$

Frequency Shaping and Filtering

- By choice of $H(j\omega)$ (or $H(e^{j\omega})$) as a function of ω , we can *shape* the frequency composition of the output
 - Preferential amplification
 - Selective filtering of some frequencies

بازنمایی سری فوریه سیگنال‌های متناوب (۳)

۳

مثال‌ها و
نمونه‌های
نمایشی

فیلترینگ

FILTERING

سیستم‌های LTI:

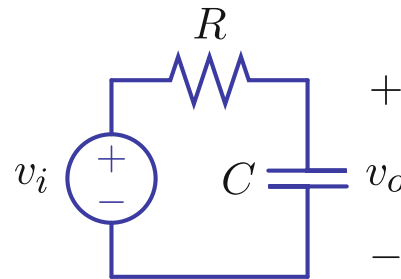
- نمی‌توانند فرکانس جدیدی ایجاد کنند.
- می‌توانند مؤلفه‌های فرکانسی موجود را تغییر اندازه یا شیف‌ت فاز بدهند.

فیلترینگ

مثال

FILTERING

Example: Low-Pass Filtering with an RC circuit

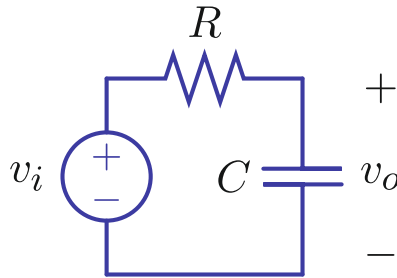


فیلترینگ

مثال

FILTERING

Calculate the frequency response of an RC circuit.



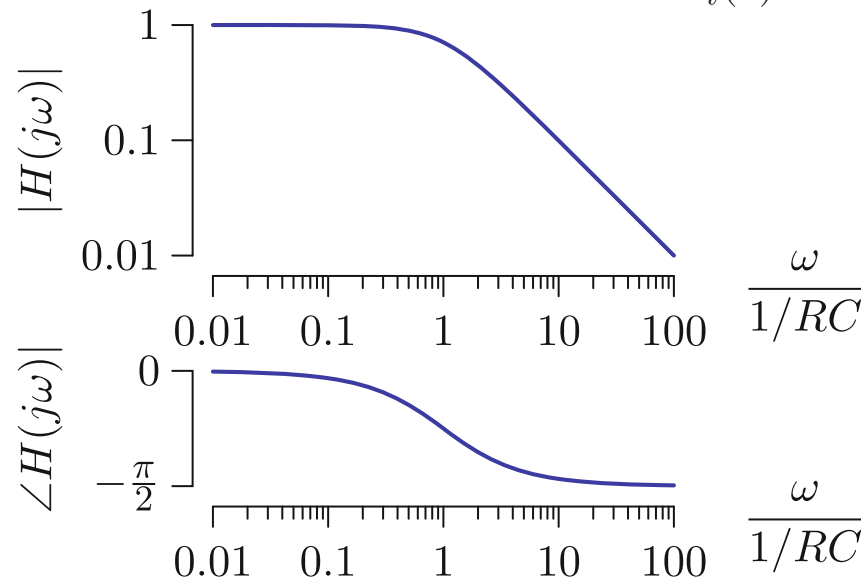
$$\text{KVL: } v_i(t) = Ri(t) + v_o(t)$$

$$\text{C: } i(t) = C\dot{v}_o(t)$$

$$\text{Solving: } v_i(t) = RC\dot{v}_o(t) + v_o(t)$$

$$V_i(s) = (1 + sRC)V_o(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}$$

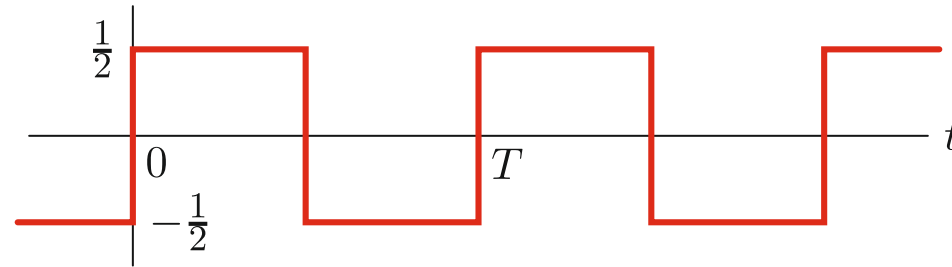


فیلترینگ

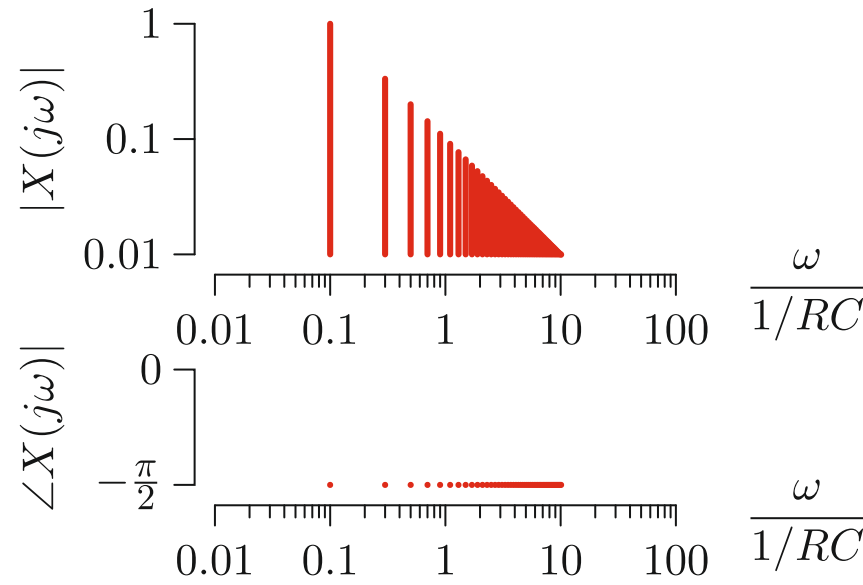
مثال

FILTERING

Let the input be a square wave.



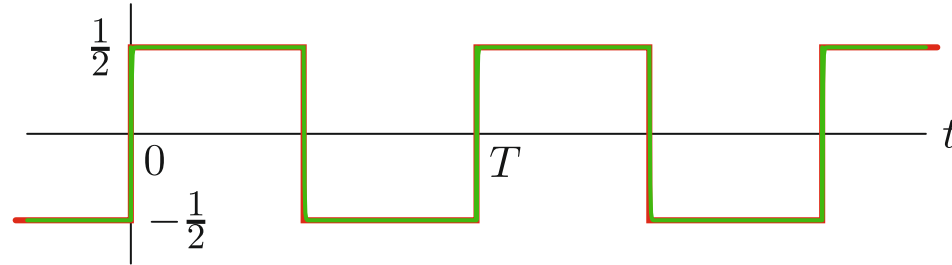
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} ; \quad \omega_0 = \frac{2\pi}{T}$$



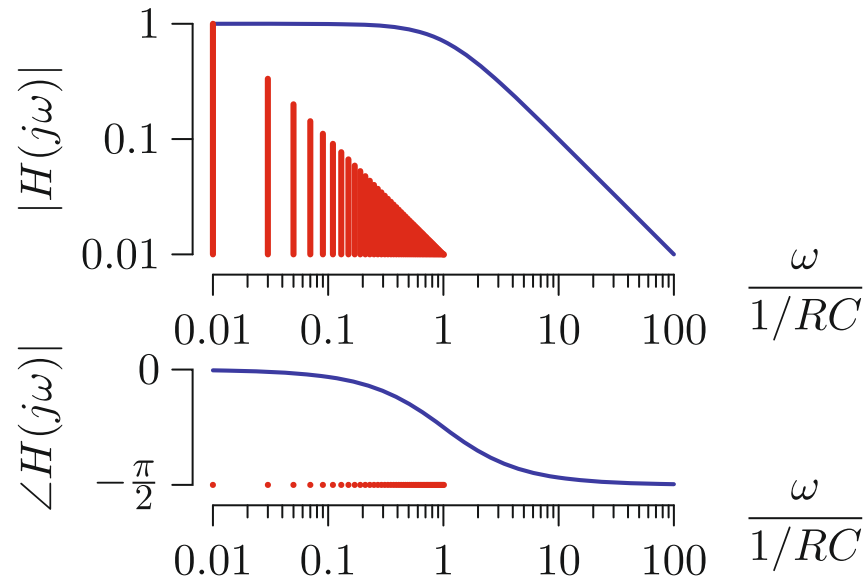
فیلترینگ

مثال

FILTERING

Low frequency square wave: $\omega_0 \ll 1/RC$.

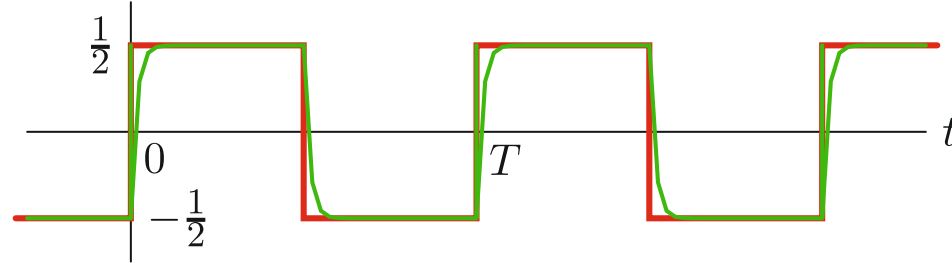
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$



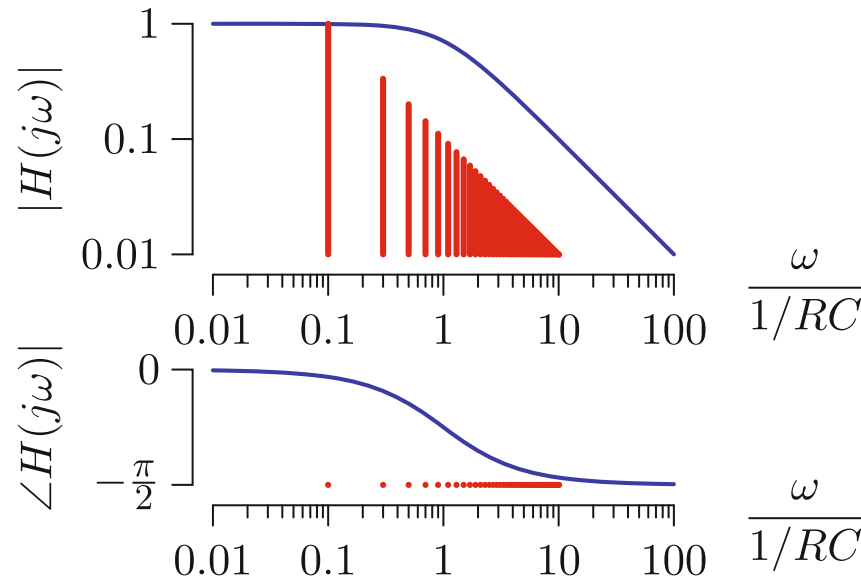
فیلترینگ

مثال

FILTERING

Higher frequency square wave: $\omega_0 < 1/RC$.

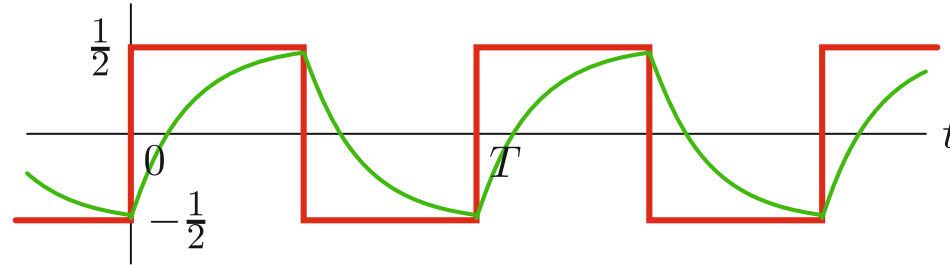
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$



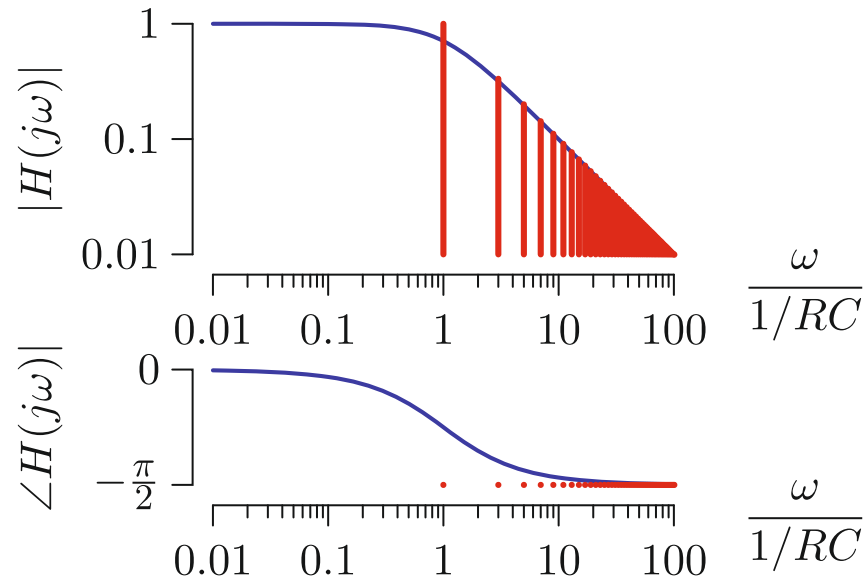
فیلترینگ

مثال

FILTERING

Still higher frequency square wave: $\omega_0 = 1/RC$.

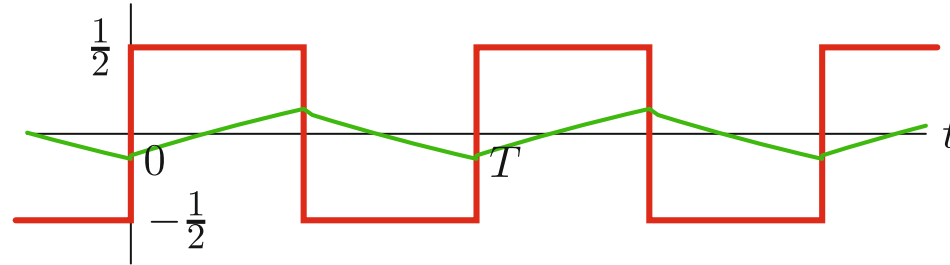
$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$



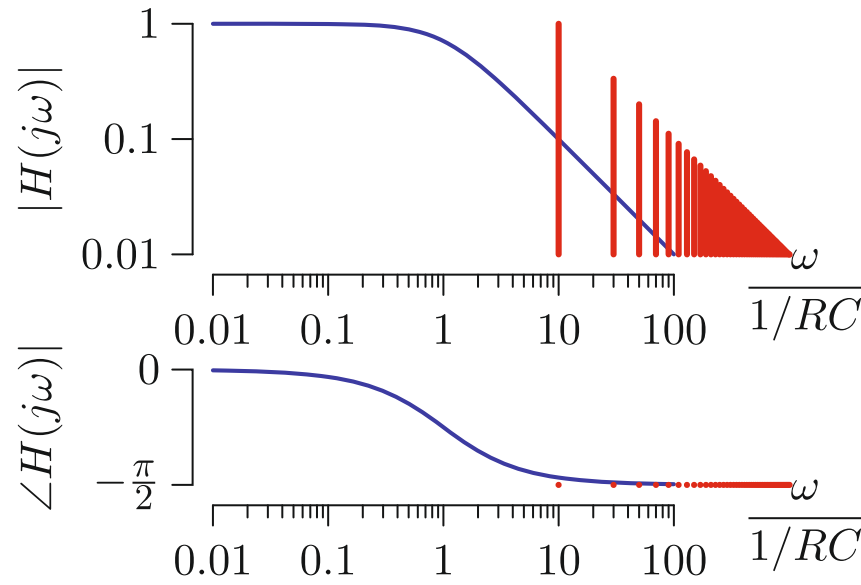
فیلترینگ

مثال

FILTERING

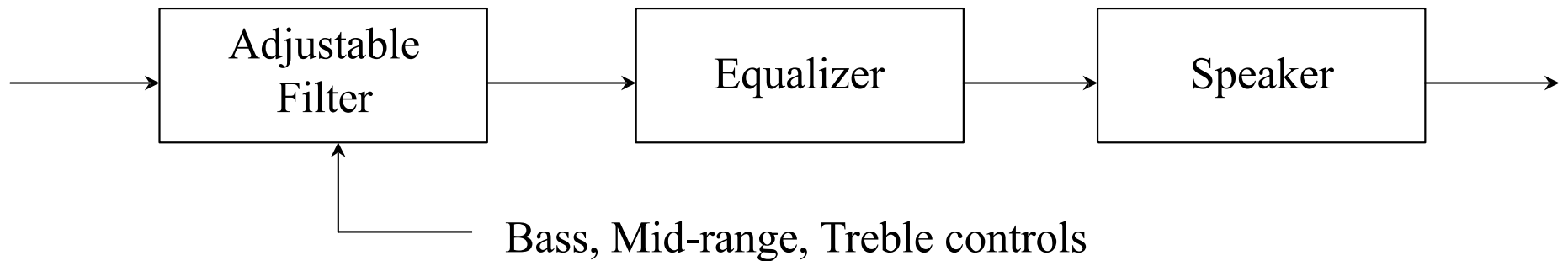
High frequency square wave: $\omega_0 > 1/RC$.

$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}$$



Frequency Shaping and Filtering

Example #1: Audio System

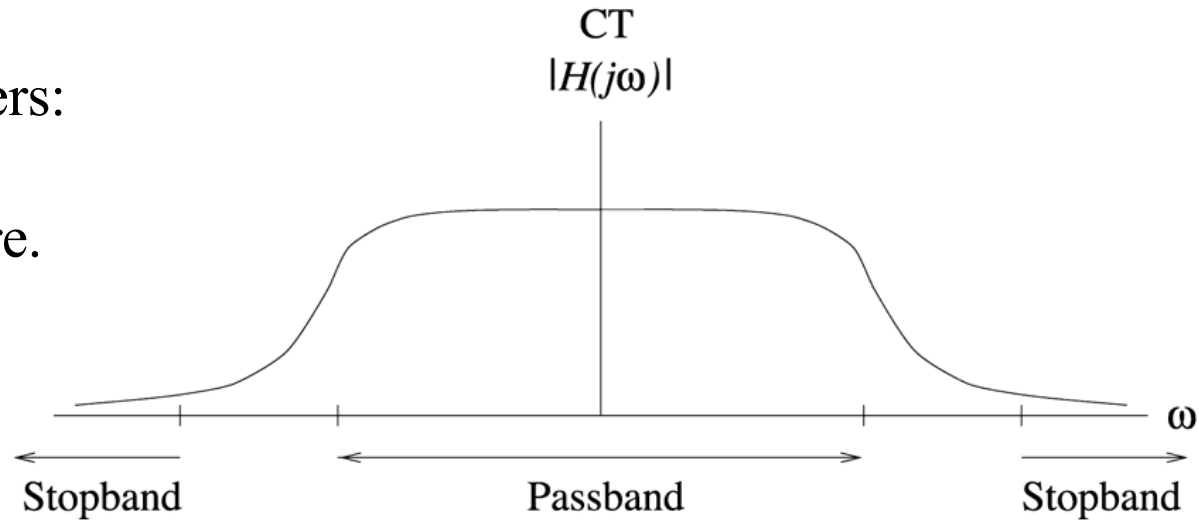


For audio signals, the amplitude is much more important than the phase.

Example #2: Frequency Selective Filters

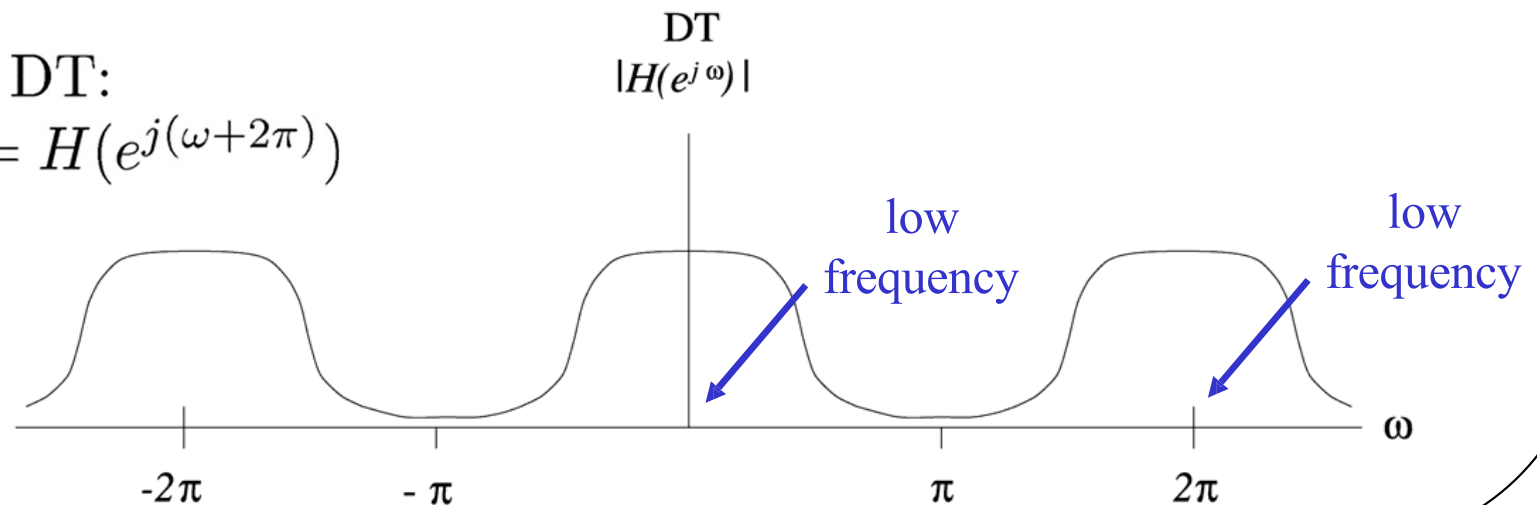
— Filter out signals outside of the frequency range of interest

Lowpass Filters:
Only show
amplitude here.



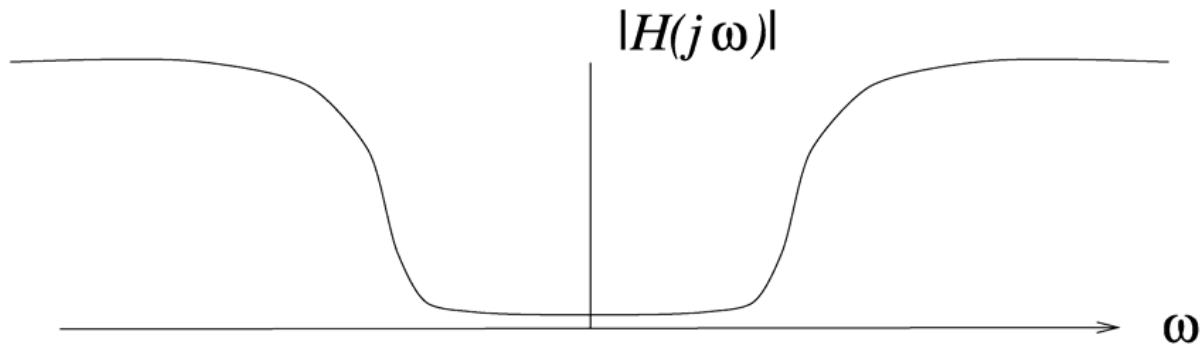
Note for DT:

$$H(e^{j\omega}) = H(e^{j(\omega+2\pi)})$$



Highpass Filters

CT

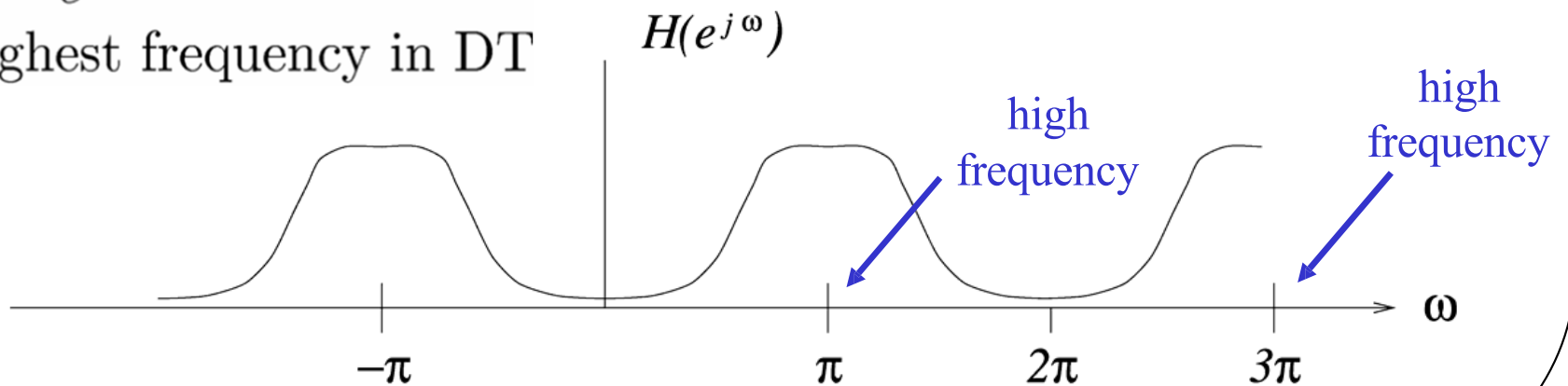


Remember:

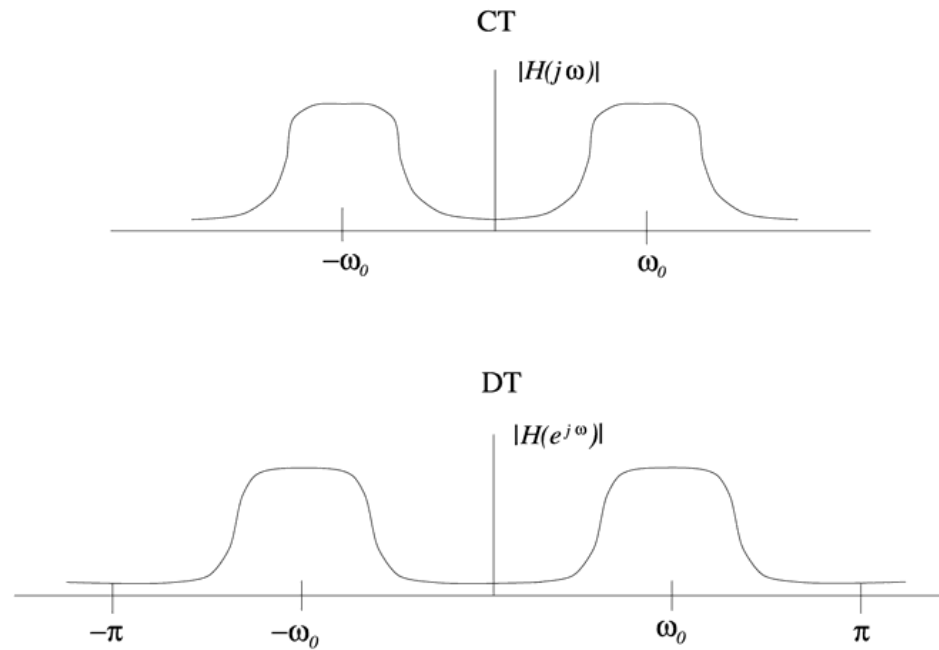
$$(-1)^n = e^{j\pi n}$$

$\pi =$ highest frequency in DT

DT



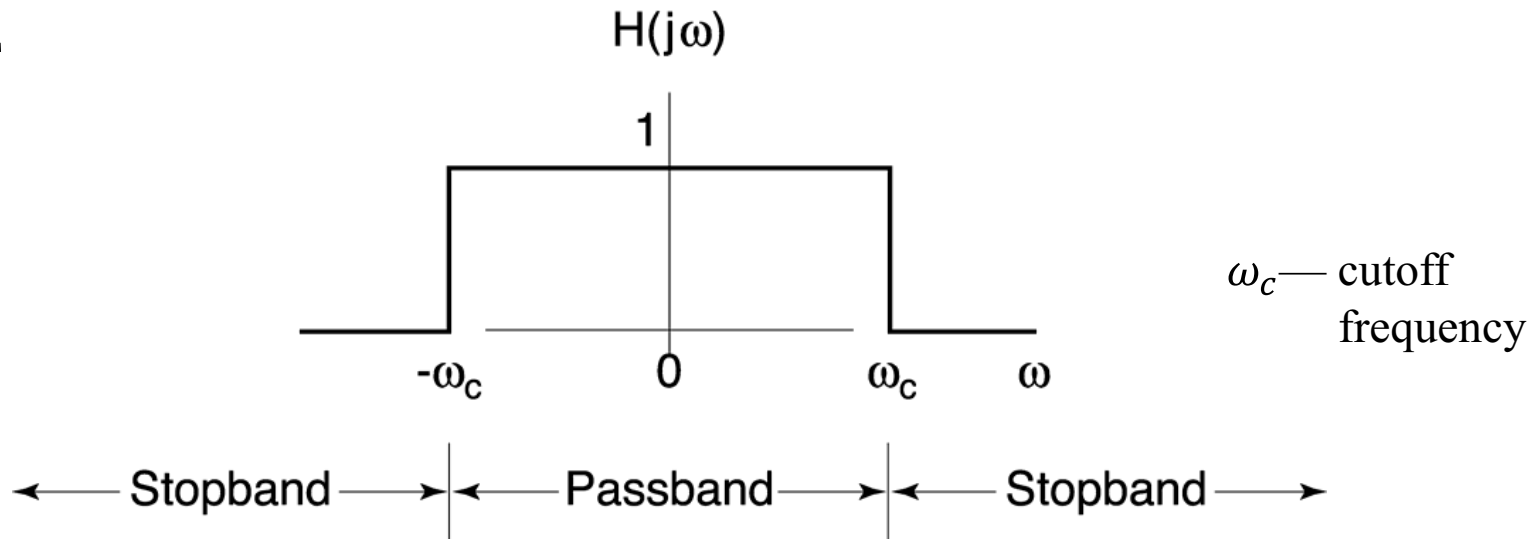
Bandpass Filters



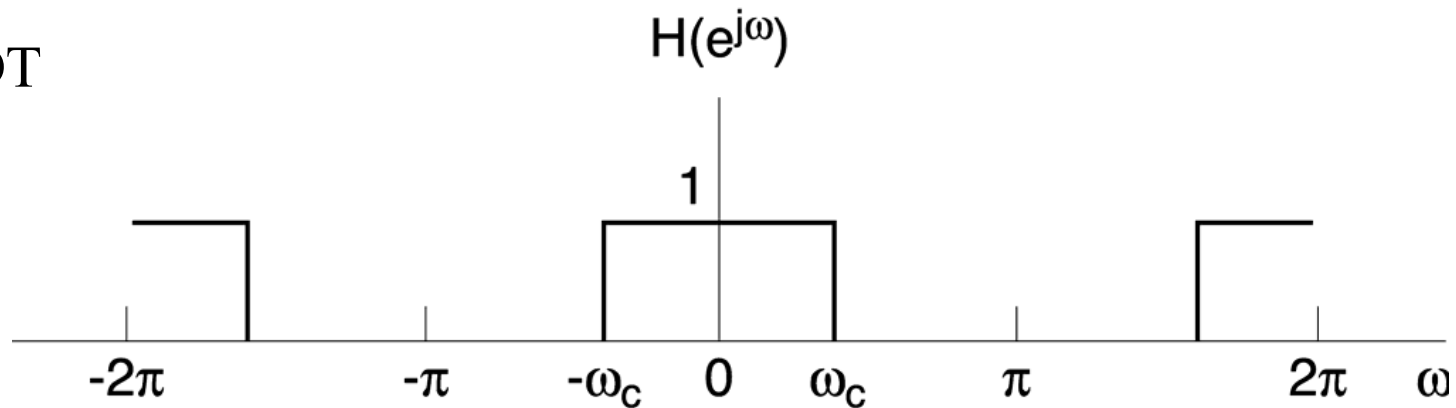
Demo: Filtering effects on audio signals

Idealized Filters

CT



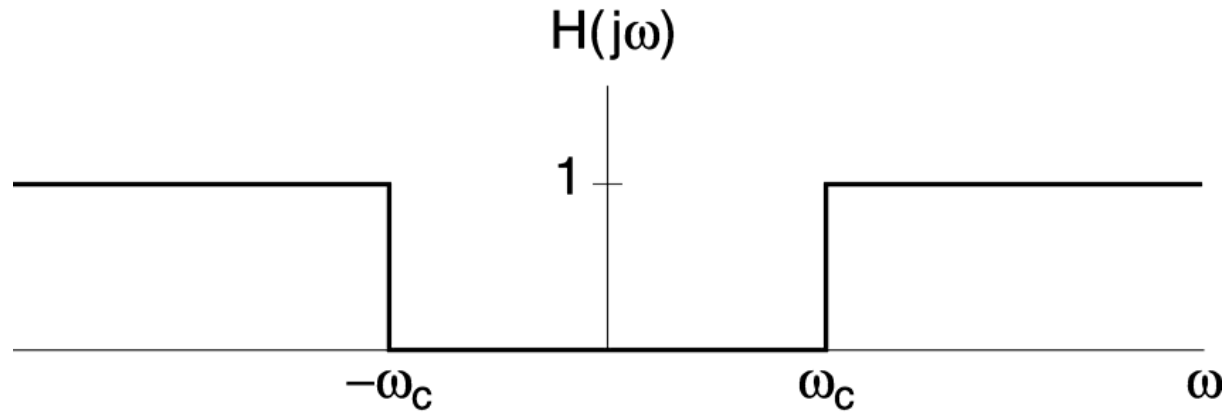
DT



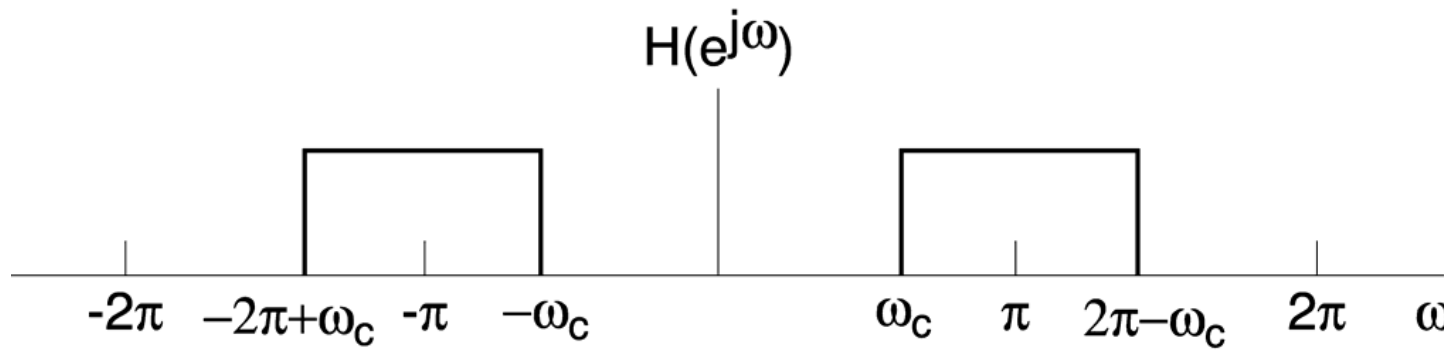
Note: $|H| = 1$ and $\angle H = 0$ for the ideal filters in the passbands, no need for the phase plot.

Highpass

CT

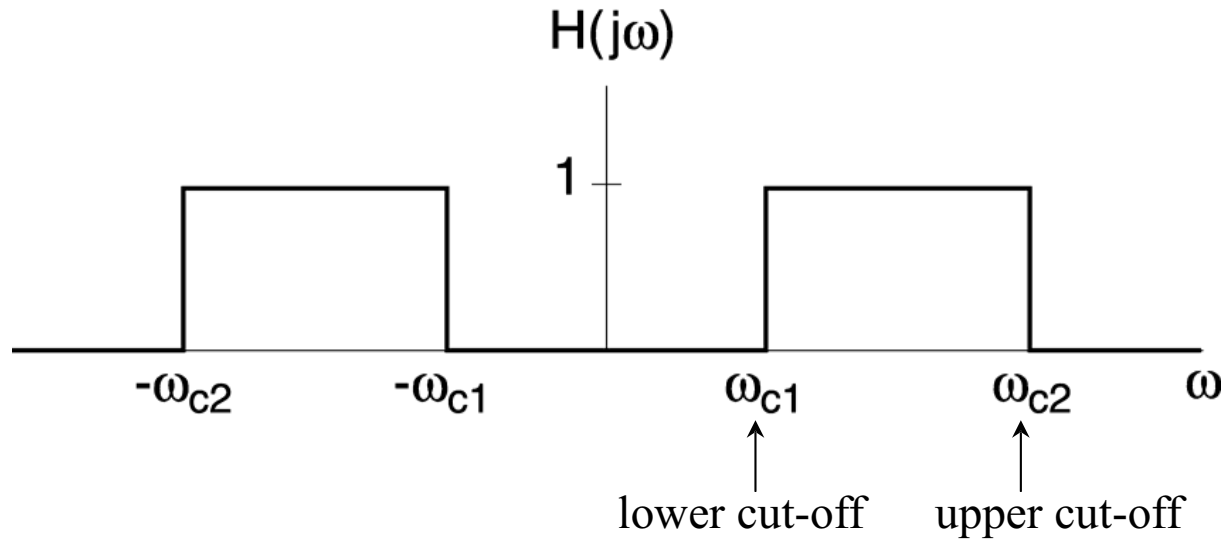


DT

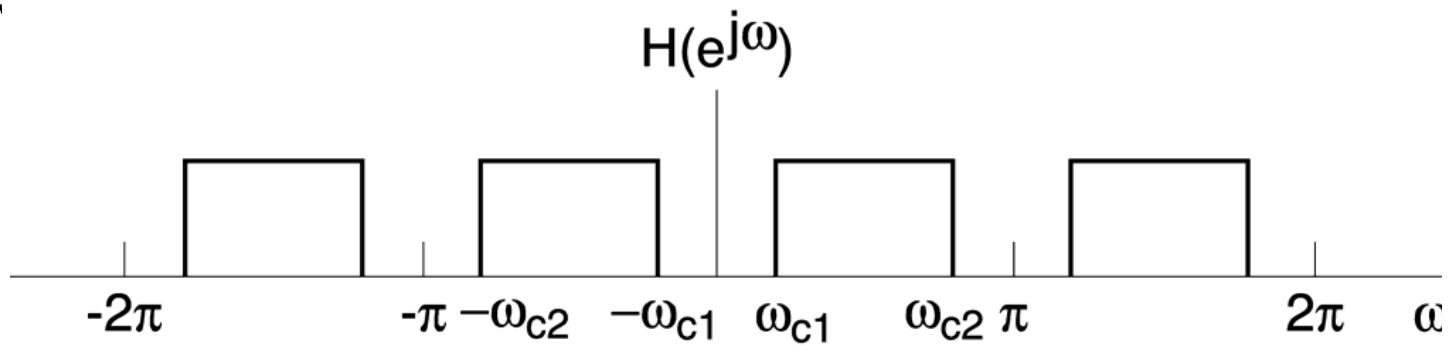


Bandpass

CT



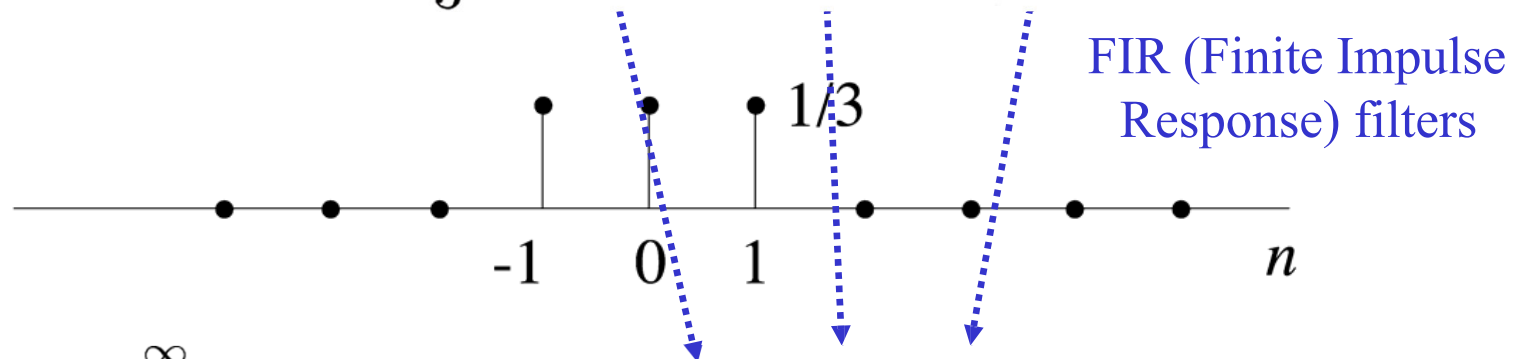
DT



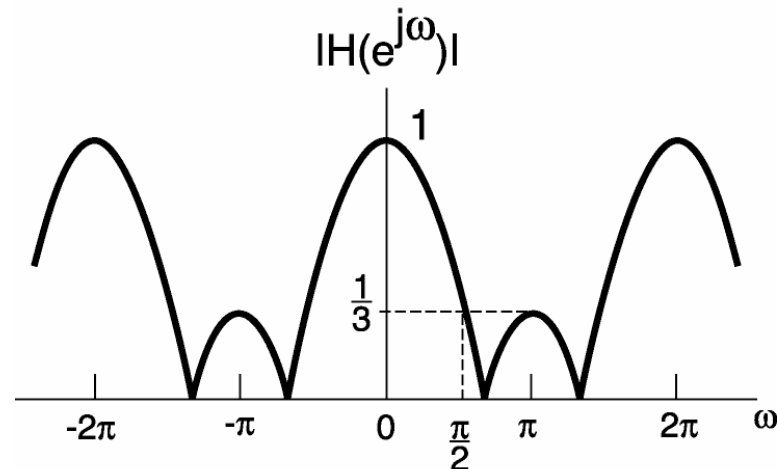
Example #3: DT Averager/Smoothing

$$y[n] = \frac{1}{3} \{x[n-1] + x[n] + x[n+1]\}$$

$$h[n] = \frac{1}{3} \{\delta[n-1] + \delta[n] + \delta[n+1]\}$$



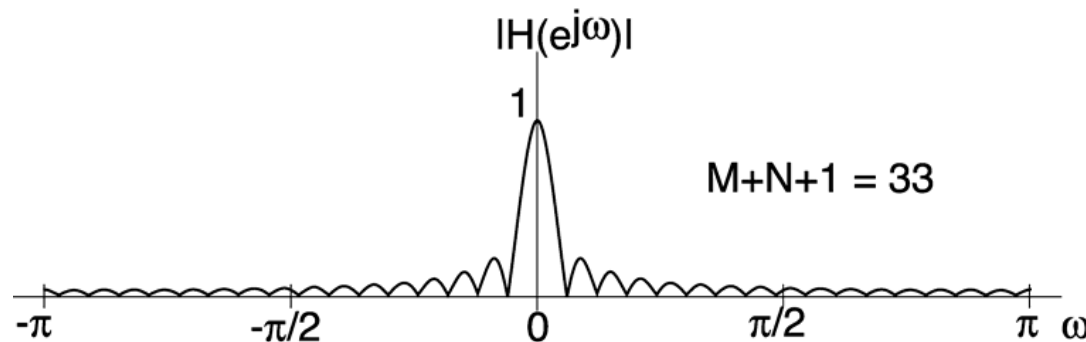
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \frac{1}{3} [e^{-j\omega} + 1 + e^{j\omega}] = \frac{1}{3} + \frac{2}{3} \cos \omega$$



Example #4: Nonrecursive DT (FIR) filters

$$y[n] = \frac{1}{N + M + 1} \sum_{k=-N}^M x[n - k] \longrightarrow h[n] = \frac{1}{N + M + 1} \sum_{k=-N}^M \delta[n - k]$$

$$H(e^{j\omega}) = \frac{1}{N + M + 1} \sum_{k=-N}^M e^{-jk\omega} = \frac{1}{N + M + 1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M + N + 1)/2]}{\sin(\omega/2)}$$

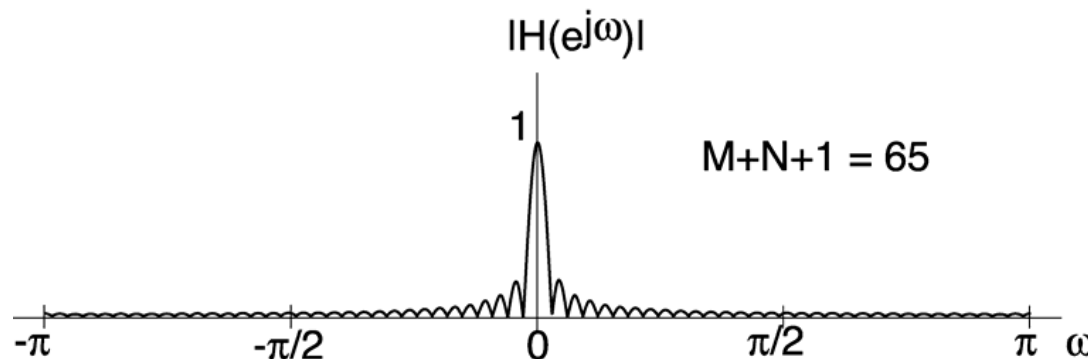
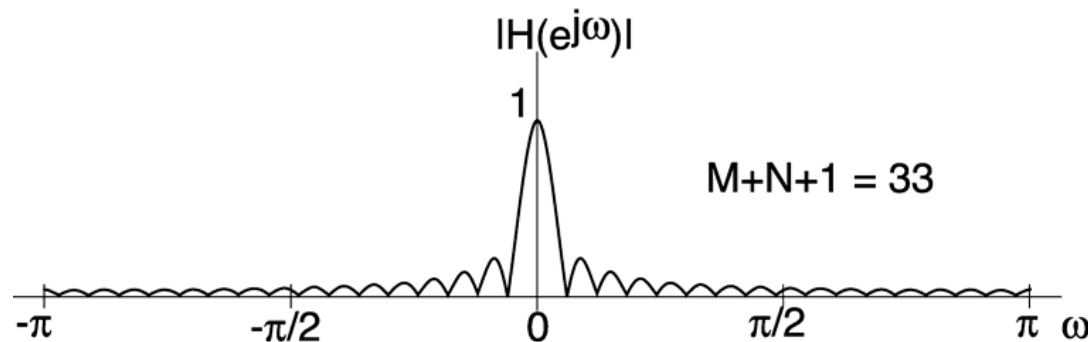


**Rolls off at lower
 ω as $M+N+1$
increases**

Example #4: Nonrecursive DT (*FIR*) filters

$$y[n] = \frac{1}{N + M + 1} \sum_{k=-N}^M x[n - k] \longrightarrow h[n] = \frac{1}{N + M + 1} \sum_{k=-N}^M \delta[n - k]$$

$$H(e^{j\omega}) = \frac{1}{N + M + 1} \sum_{k=-N}^M e^{-jk\omega} = \frac{1}{N + M + 1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M + N + 1)/2]}{\sin(\omega/2)}$$

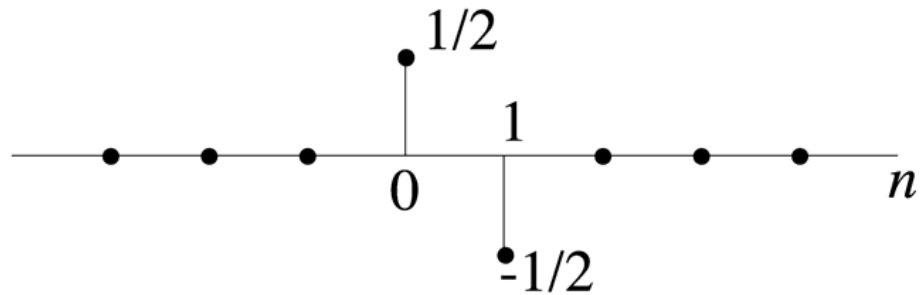


Rolls off at lower ω as $M+N+1$ increases

Example #5: Simple DT “Edge” Detector
 — DT 2-point “differentiator”

$$y[n] = \frac{1}{2} \{x[n] - x[n - 1]\}$$

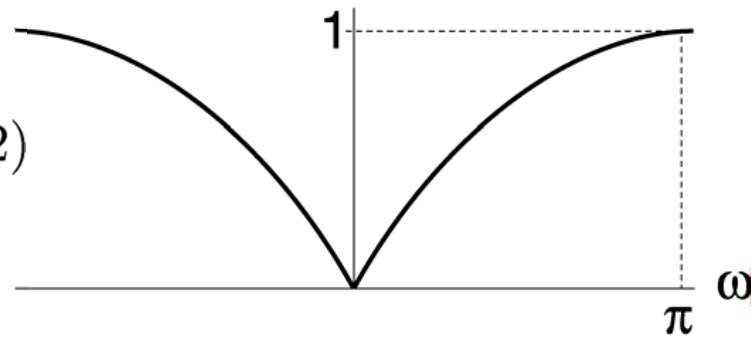
$$h[n] = \frac{1}{2} \{\delta[n] - \delta[n - 1]\}$$



$$\frac{j}{2j} e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2}) = j e^{-j\omega/2} \sin(\omega/2) \quad |H(e^{j\omega})|$$

$$H(e^{j\omega}) = \frac{1}{2} (1 - e^{-j\omega}) = j e^{-j\omega/2} \sin(\omega/2)$$

$$|H(e^{j\omega})| = |\sin(\omega/2)|$$



Passes high-frequency components

Demo: DT filters, LP, HP, and BP applied to DJ Industrial average

Data set

DJIA 1900 - 1991

Lowpass Filters

3-pt avg

Highpass Filters

2-pt diff

Bandpass Filters

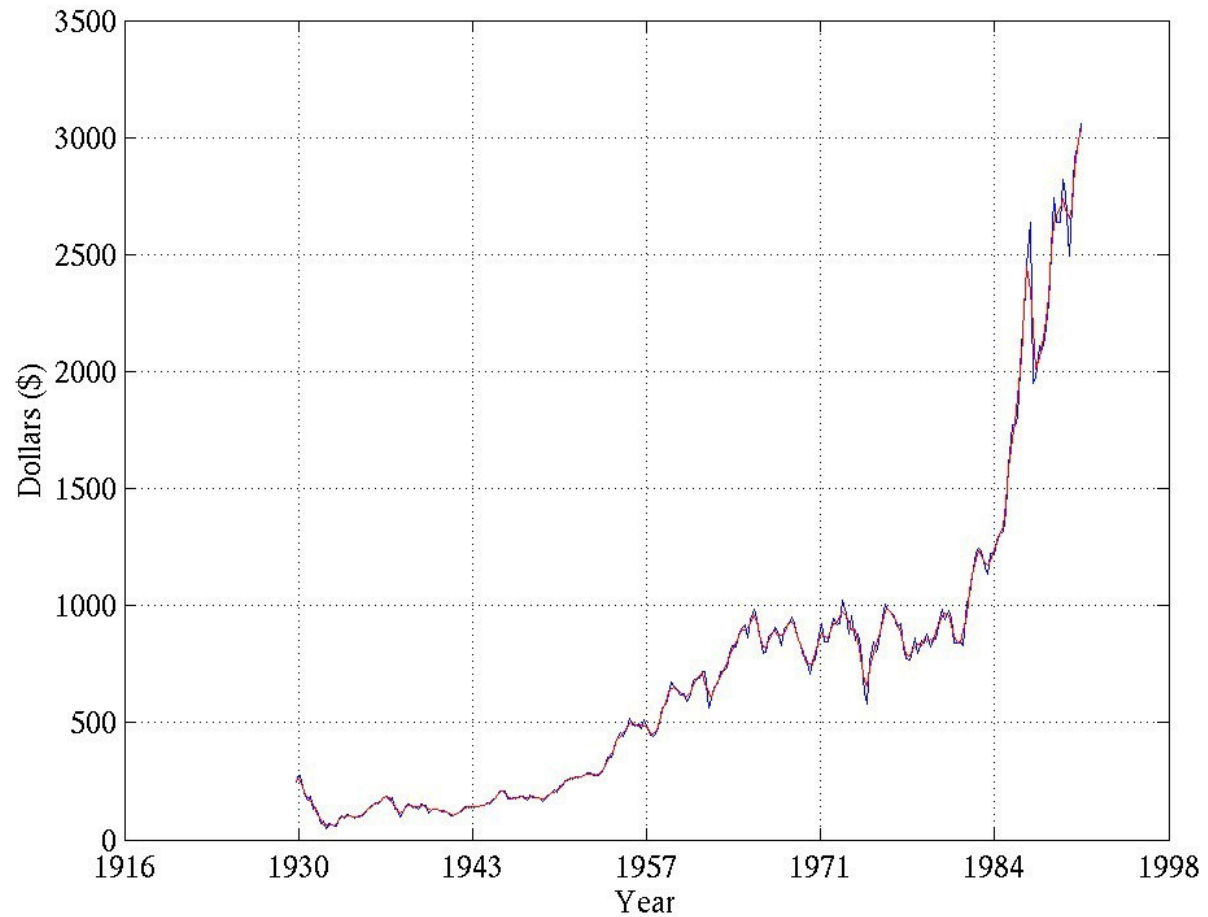
Center Frequency

0.125

Frequency response

Impulse response

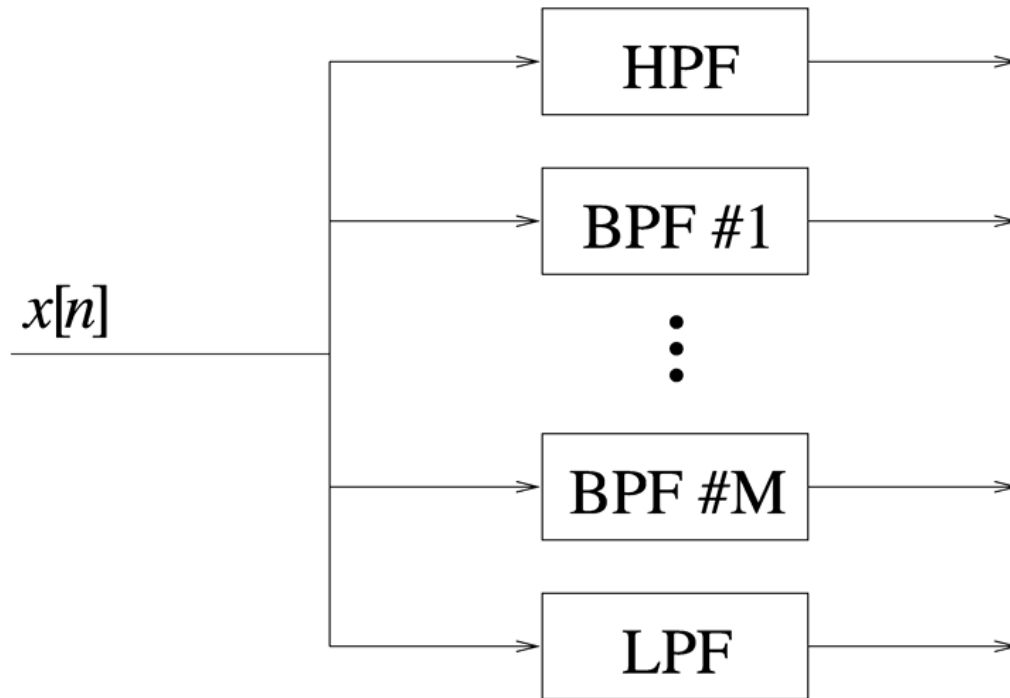
Original sign Filtered sign Zoon



Example #6: Edge enhancement using DT differentiator



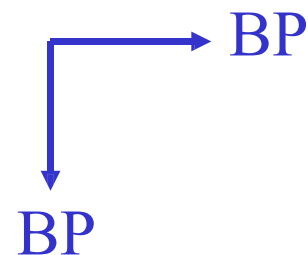
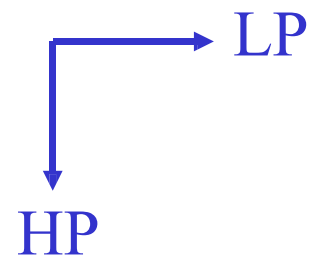
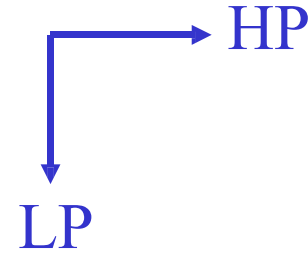
Example #7: A Filter Bank



Demo: Apply different filters to two-dimensional image signals.

Face of a monkey.

Image removed do to
copyright considerations



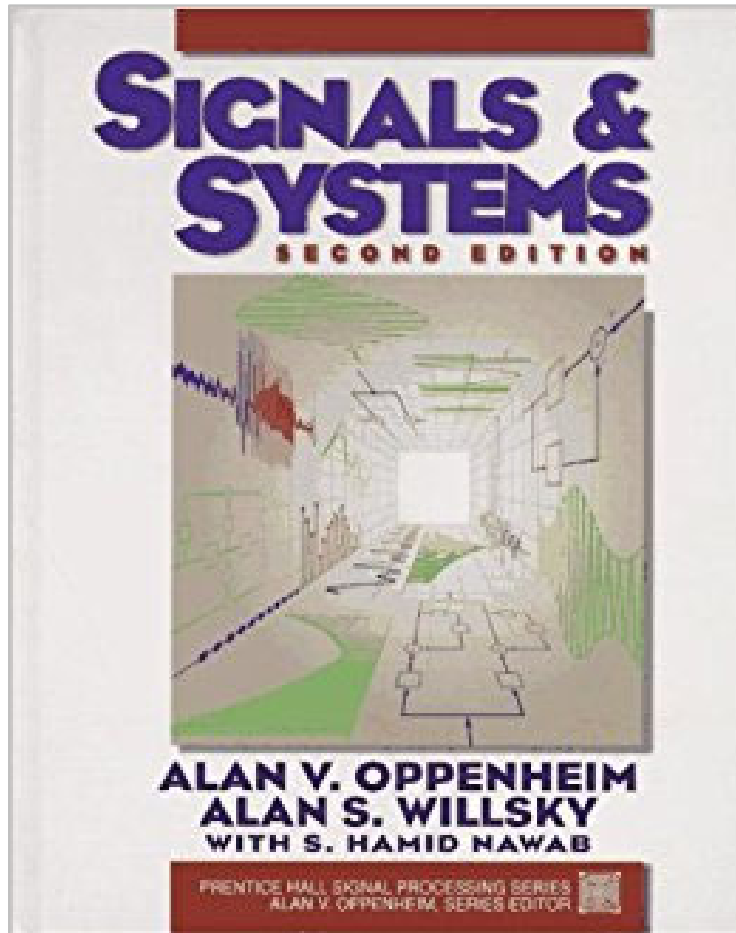
Note: To really understand these examples, we need to understand frequency contents of aperiodic signals \Rightarrow the Fourier Transform

بازنمایی سری فوریه سیگنال‌های متناوب (۳)

۴

منابع

منبع اصلی



A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
Signals and Systems,
Second Edition, Prentice Hall, 1997.

Chapter 3