سیگنال‌ها و سیستم‌ها

درس ۷

سیستم‌های خطی تغییرناپذیر با زمان (۳)

Linear Time-Invariant (LTI) Systems (3)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

بازنمایی سیگنال‌های پیوسته–زمان برحسب ضربه‌های واحد شیفت‌یافته
Representation of CT Signals in terms of shifted unit impulses
بازنمایی انتگرال کانوولوشن سیستم‌های خطی تغییرناپذیر با زمان پیوسته–زمان
Convolution integral representation of CT LTI systems
خصوصیات و مثال‌ها
Properties and Examples

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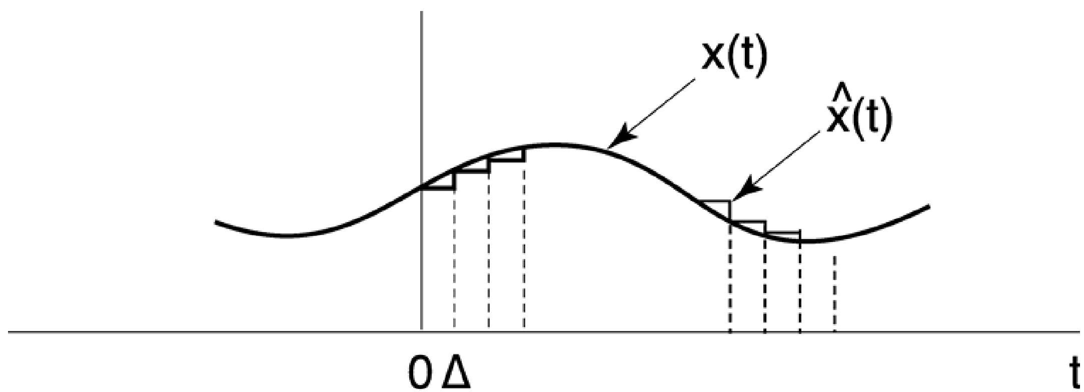
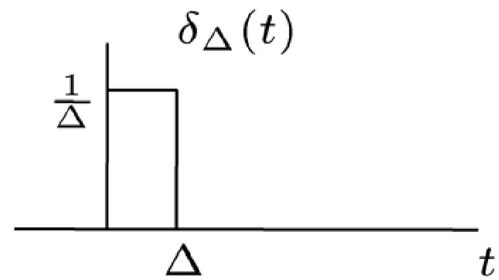
بازنمایی سیگنال‌های پیوسته-زمان برحسب ضربه‌های واحد شیفت‌یافته

Representation of CT Signals

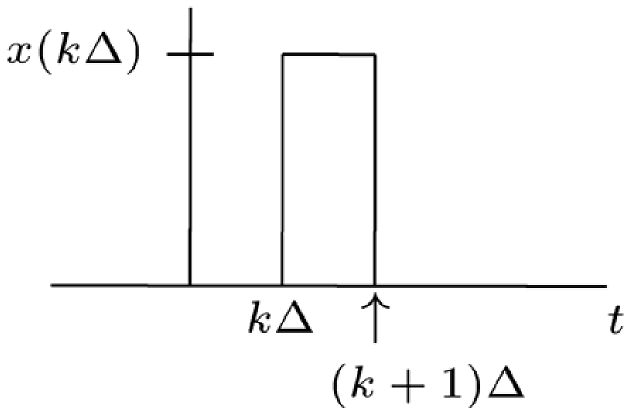
- Approximate any input $x(t)$ as a sum of shifted, scaled pulses



$$\hat{x}(t) = x(k\Delta)\ ,\ k\Delta < t < (k+1)\Delta$$



$\delta_\Delta(t)$ has unit area



$$= \quad x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

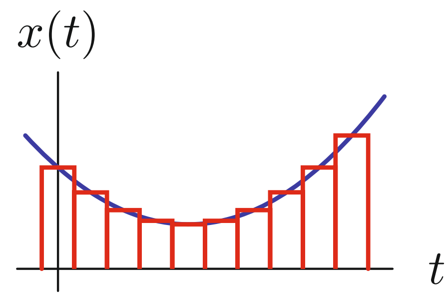
$\downarrow$ limit as $\Delta \to 0$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The Sifting Property of the Unit Impulse

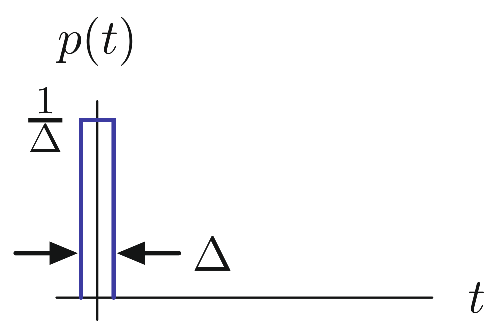
بازنمایی سیگنال‌های پیوسته-زمان برحسب ضربه‌های واحد شیفت‌یافته

The same sort of reasoning applies to CT signals.



$$x(t) = \lim_{\Delta \to 0} \sum_{k} x(k\Delta) p(t - k\Delta) \Delta$$

where



As $\Delta \to 0$, $k\Delta \to \tau$, $\Delta \to d\tau$, and $p(t) \to \delta(t)$:

$$x(t) \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

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بازنمایی انتگرال کانولوشن سیستم‌های خطی تغییرناپذیر با زمان پیوسته-زمان

Response of a CT LTI System

$$\delta_\Delta(t) \longrightarrow h_\Delta(t)$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$

Impulse response: $\boxed{\delta(t) \longrightarrow h(t)}$

Taking limits $\Delta \to 0$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \underbrace{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{\text{Convolution Integral}}$$

ساختار برهم‌نهی

If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

کانوولوشن پیوسته

DT: $$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

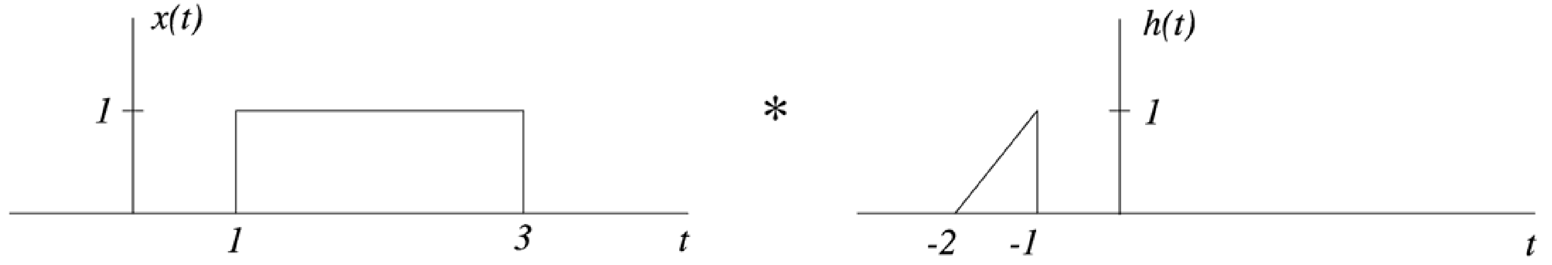
Operation of CT Convolution

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau) \xrightarrow{\text{Slide}} h(t-\tau)$$

$$\xrightarrow{\text{Multiply}} x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Example: CT convolution



Operation of CT Convolution

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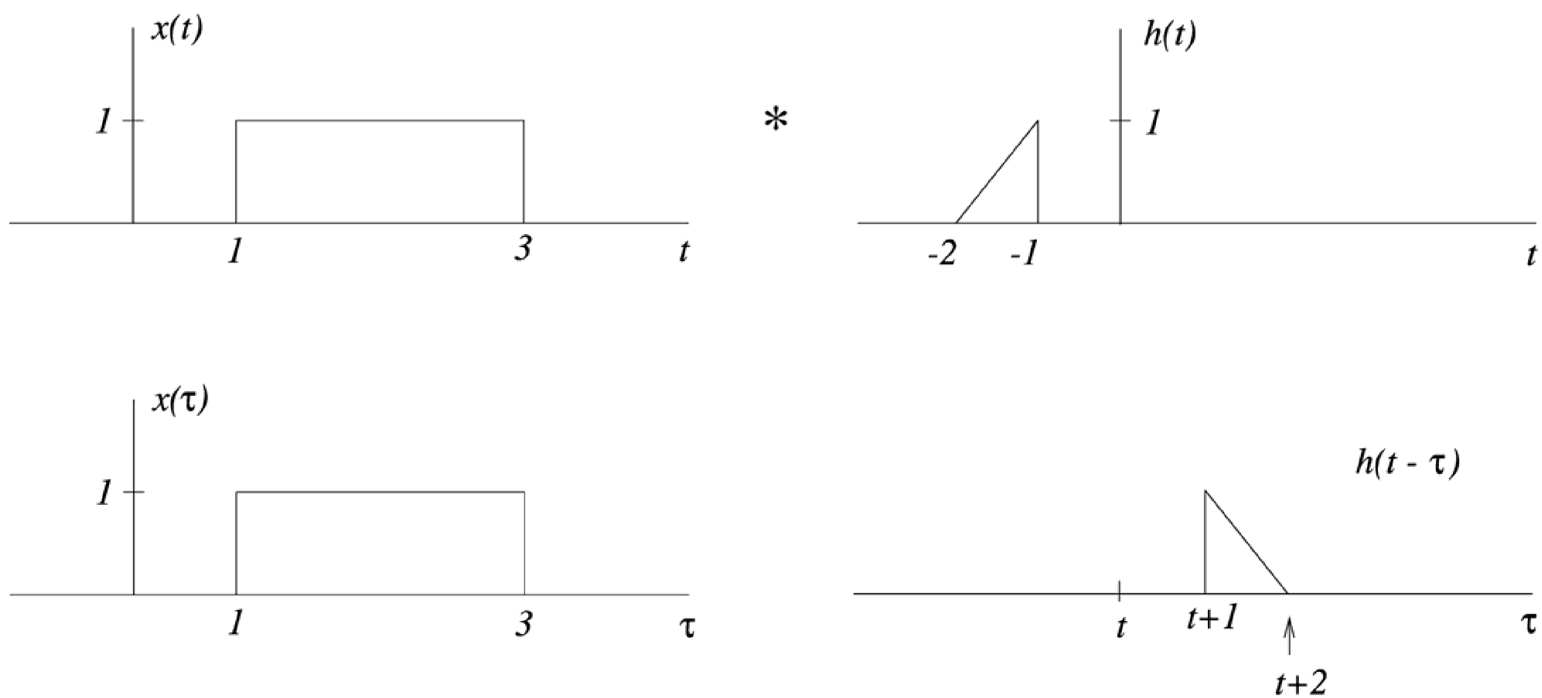
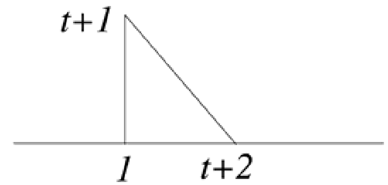
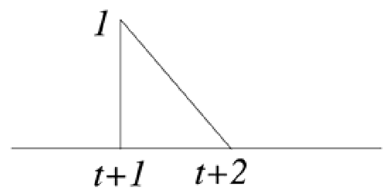
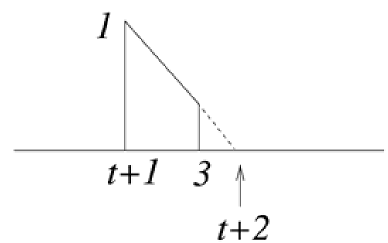
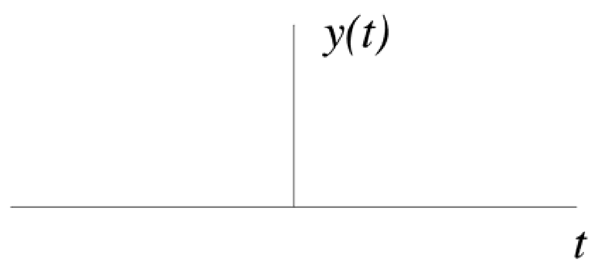
Example: **CT convolution**



Time Interval	$x(\tau) \cdot h(t-\tau)$		Output
$t <$ 1-	*0*	$\Longrightarrow$	$y(t) = 0$
1- $< t <$ 0		$\Longrightarrow$	$y(t) = \frac{1}{2}(t+2)(t+2-1) = \frac{1}{2}(t+1)^2$
0 $< t <$ 1		$\Longrightarrow$	$y(t) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$
1 $< t <$ 2		$\Longrightarrow$	$y(t) = \frac{1}{2} - \frac{1}{2}(t+2-3)(t-1) = \frac{1}{2} - \frac{1}{2}(t-1)^2$
$t >$ 2	*0*	$\Longrightarrow$	$y(t) = 0$



سیستم‌های خطی تغییرناپذیر با زمان (۳)

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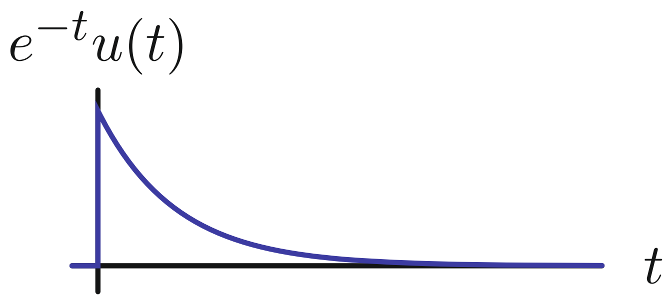
خصوصیات و مثال‌ها

انتگرال کانوولوشن

مثال



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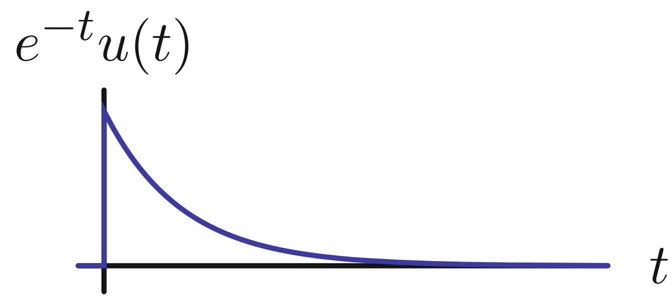
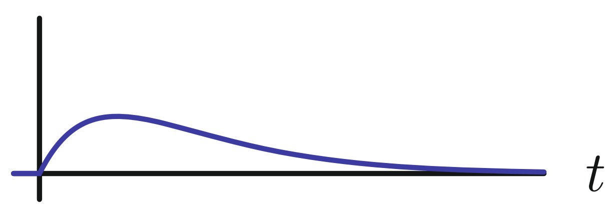
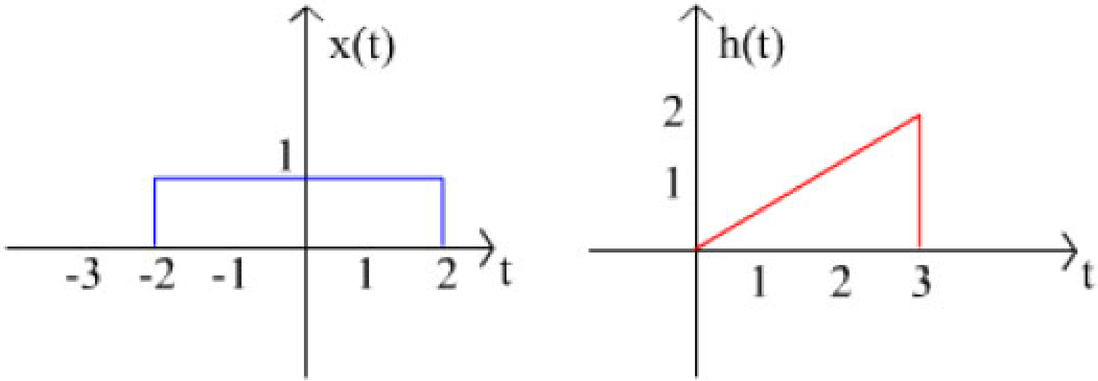


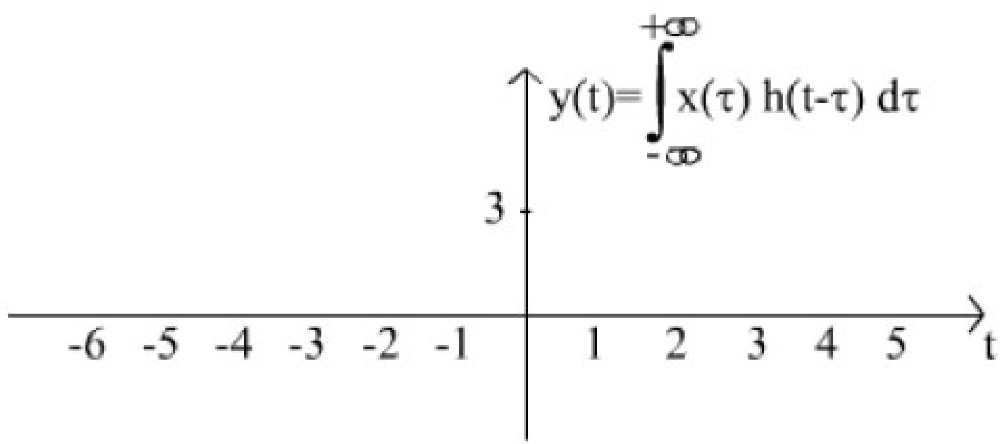
$$\left(e^{-t}u(t)\right) * \left(e^{-t}u(t)\right) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

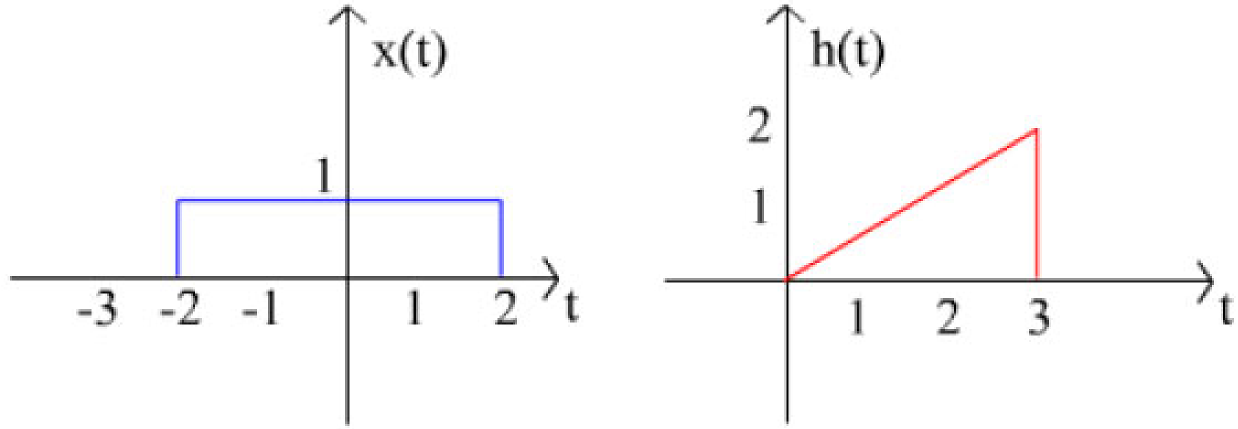
$$= \int_{0}^{t} e^{-\tau}e^{-(t-\tau)}d\tau = e^{-t}\int_{0}^{t} d\tau = te^{-t}u(t)$$

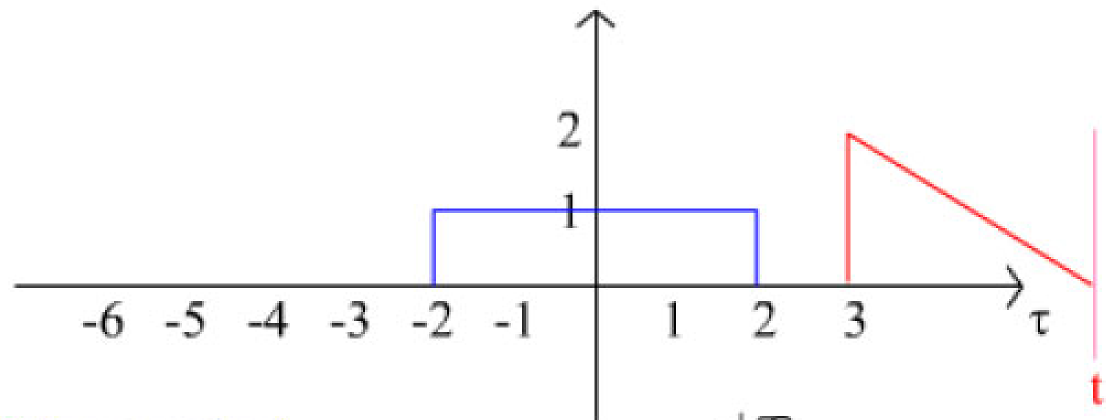


1) Draw h(-τ)


x(t)
1
-3
-2
-1
1
2
t
h(t)
2
1
1
2
3
t


$y(t)=\int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$
3
-6
-5
-4
-3
-2
-1
1
2
3
4
5
t





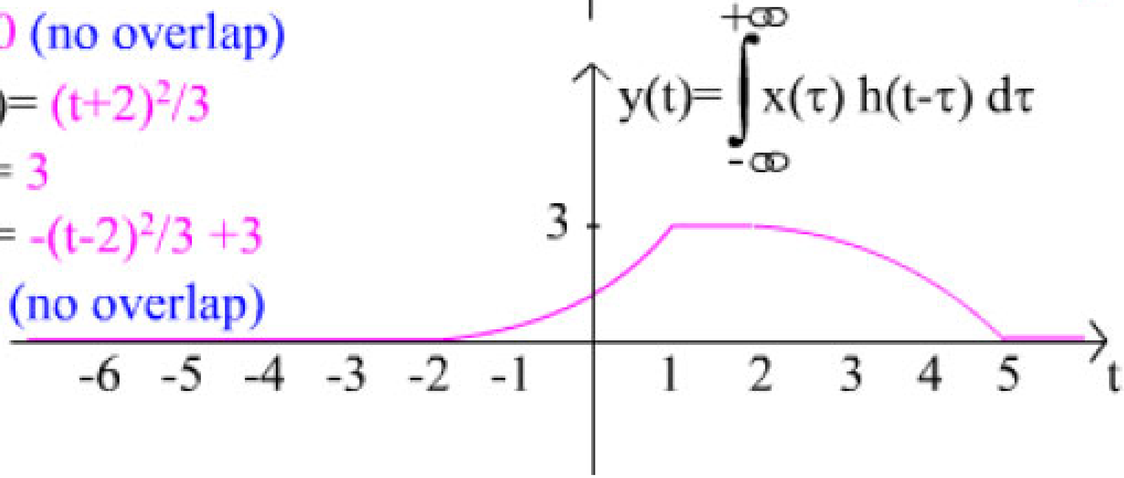
t<-2: y(t)= 0 (no overlap)

-2<t<1: y(t)= $(t+2)^2/3$

1<t<2: y(t)= 3

2<t<5: y(t)= $-(t-2)^2/3 +3$

t>5: y(t)= 0 (no overlap)



انتگرال کانوولوشن

مثال

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$h(t) = u(t)$$

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau} & 0 < \tau < t \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\tfrac{1}{a} e^{-a\tau}\Big|_0^t$$

$$= \tfrac{1}{a}\left(1 - e^{-at}\right)$$

$$y(t) = \tfrac{1}{a}\left(1 - e^{-at}\right)u(t)$$

انتگرال کانوولوشن

مثال

$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t-3)$$

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \tfrac{1}{2} e^{2(t-3)}, \quad t < 3$$

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \tfrac{1}{2}, \quad t \geq 3$$

PROPERTIES AND EXAMPLES

1) Commutativity: $x(t) * h(t) = h(t) * x(t)$

2) $x(t) * \delta(t - t_0) = x(t - t_0)$ Sifting property: $x(t) * \delta(t) = x(t)$

3) An integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

$$\Downarrow$$

So if input $x(t) = \delta(t)$
output $y(t) = h(t)$

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

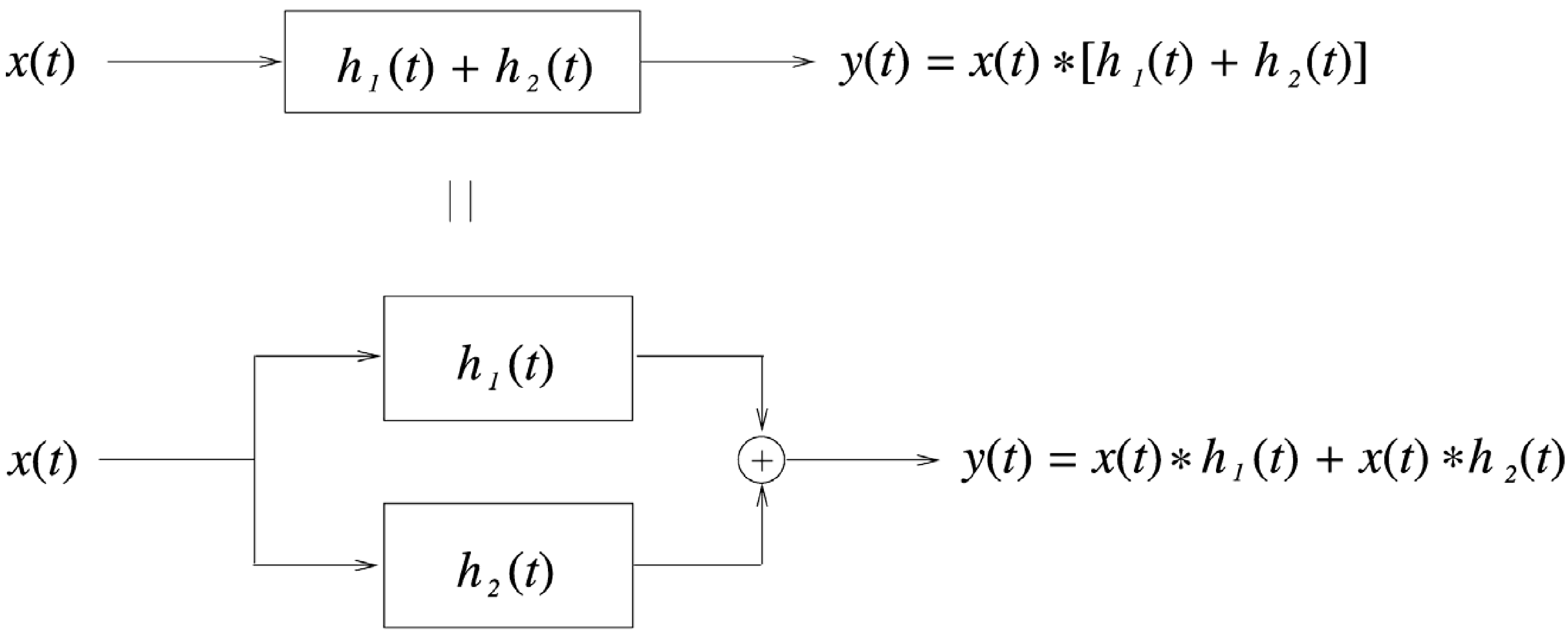
That is

$$y(t) = x(t) * h(t) = \boxed{x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau}$$

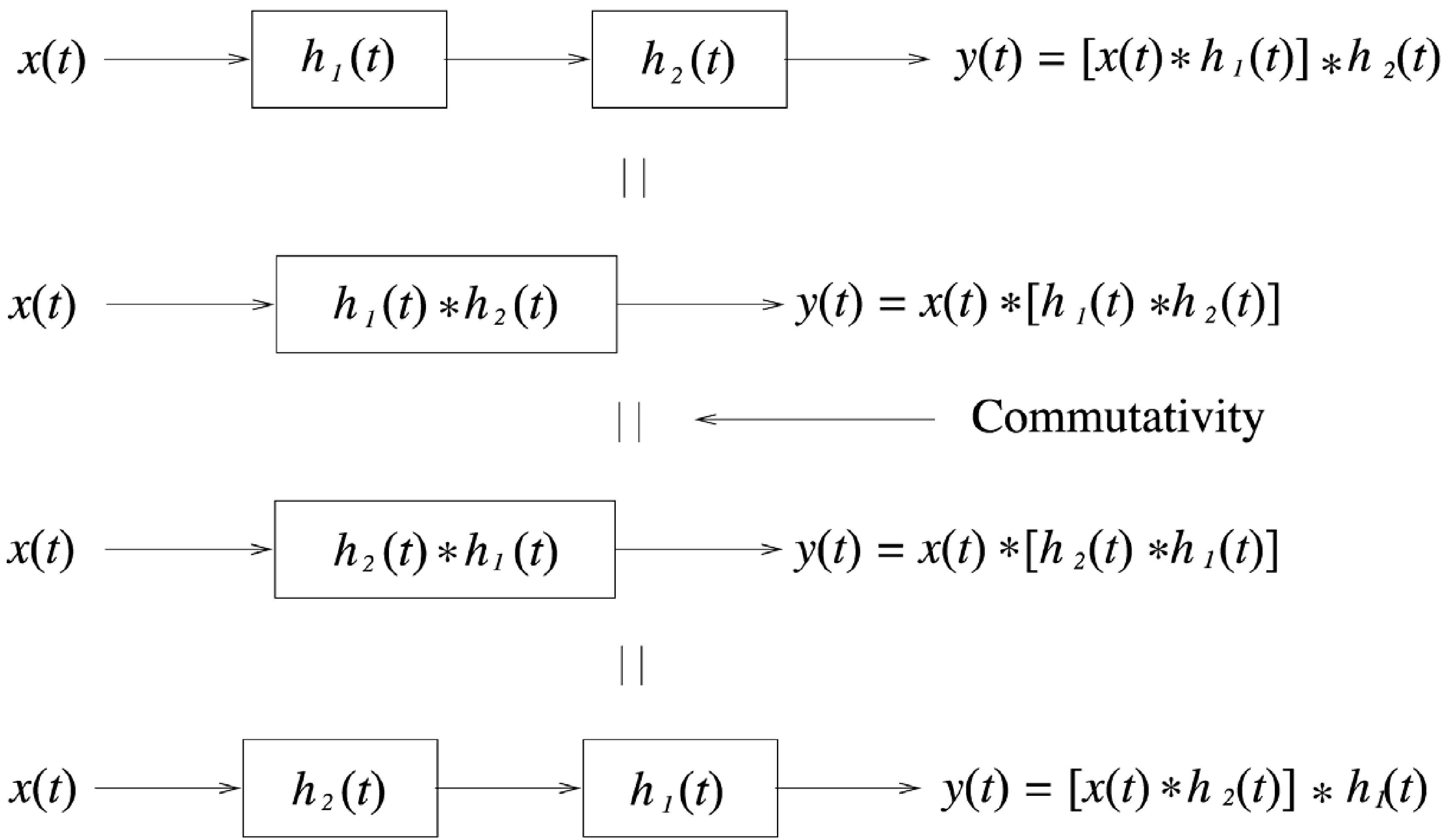
4) Step response:

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

DISTRIBUTIVITY



ASSOCIATIVITY



Causality: CT LTI system is causal $\Leftrightarrow h(t) = 0, t < 0$

Stability: CT LTI system is stable $\Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

خاصیت پایداری

مثال

$$h[n]=\delta[n-n_0]$$
$$h(t)=\delta(t-t_0)$$

$$\sum_{k=-\infty}^{\infty}|h[k]|=\sum_{k=-\infty}^{\infty}|\delta[k-n_0]|=1<\infty$$

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau=\int_{-\infty}^{\infty}|\delta(\tau-t_0)|d\tau=1<\infty$$

✓

$$h[n]=u[n-n_0]$$
$$h(t)=u(t-t_0)$$

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau=\int_{-\infty}^{\infty}|u(\tau-t_0)|d\tau=\int_{t_0}^{\infty}|u(\tau)|d\tau=\infty$$

✗

خاصیت بی‌حافظه بودن

مثال

$$h(t) = K\delta(t)$$

$$\Downarrow$$

$$y(t) = Kx(t)$$

خاصیت وارون‌پذیر بودن

مثال

وجود داشته باشد h_i که:

$$h(t) * h_i(t) = \delta(t)$$

پاسخ پله در سیستم‌های LTI

$$s[n]=\sum_{k=-\infty}^{n}h[k] \qquad h[n]=s[n]-s[n-1]$$

$$s(t)=\int_{-\infty}^{t}h(\tau)\,d\tau \qquad h(t)=\frac{ds(t)}{dt}=s'(t)$$

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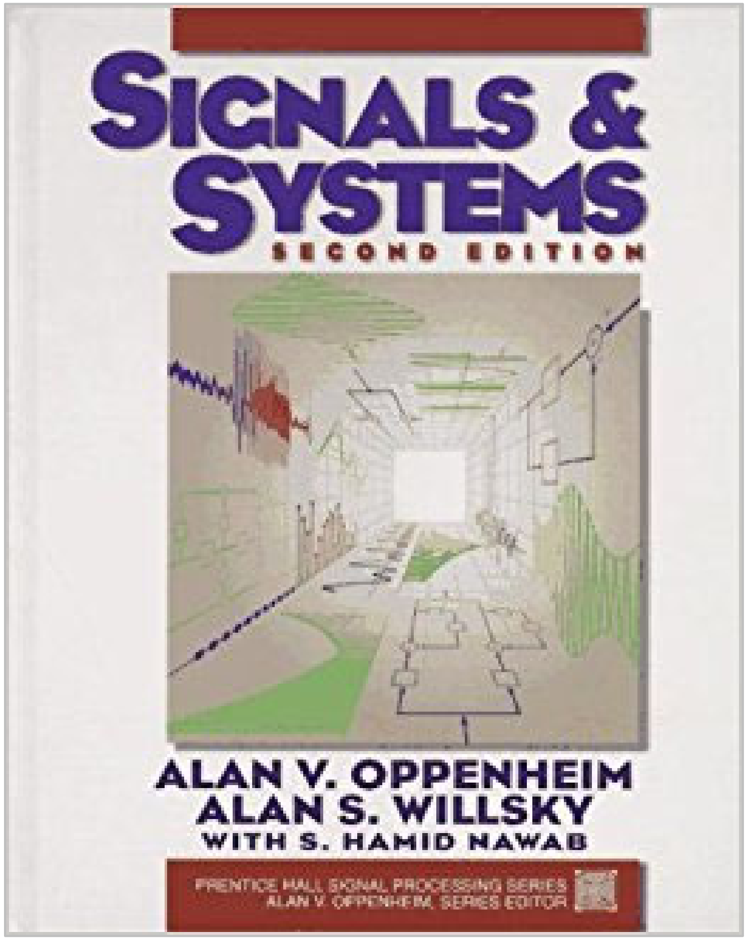
منابع

منبع اصلی



A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
Signals and Systems,
Second Edition, Prentice Hall, 1997.

Chapter 2