سیگنال‌ها و سیستم‌ها

درس ۵

سیستم‌های خطی تغییرناپذیر با زمان (۱)

Linear Time-Invariant (LTI) Systems (1)

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http://courses.fouladi.ir/sigsys

طرح درس

COURSE OUTLINE

بازنمایی سیگنال‌های گسسته–زمان برحسب نمونه‌های واحد شیفت‌یافته
Representation of DT signals in terms of shifted unit samples
بازنمایی جمع کانوولوشن سیستم‌های خطی تغییرناپذیر با زمان گسسته–زمان
Convolution sum representation of DT LTI systems
چند مثال
Examples

۱

بازنمایی سیگنال‌های گسسته–زمان برحسب نمونه‌های واحد شیفت‌یافته

Exploiting Superposition and Time-Invariance

$$x[n] = \sum_k a_k x_k[n] \xrightarrow{Linear System} y[n] = \sum_k a_k y_k[n]$$

Question: Are there sets of "basic" signals so that:

a) We can represent rich classes of signals as linear combinations of these building block signals.

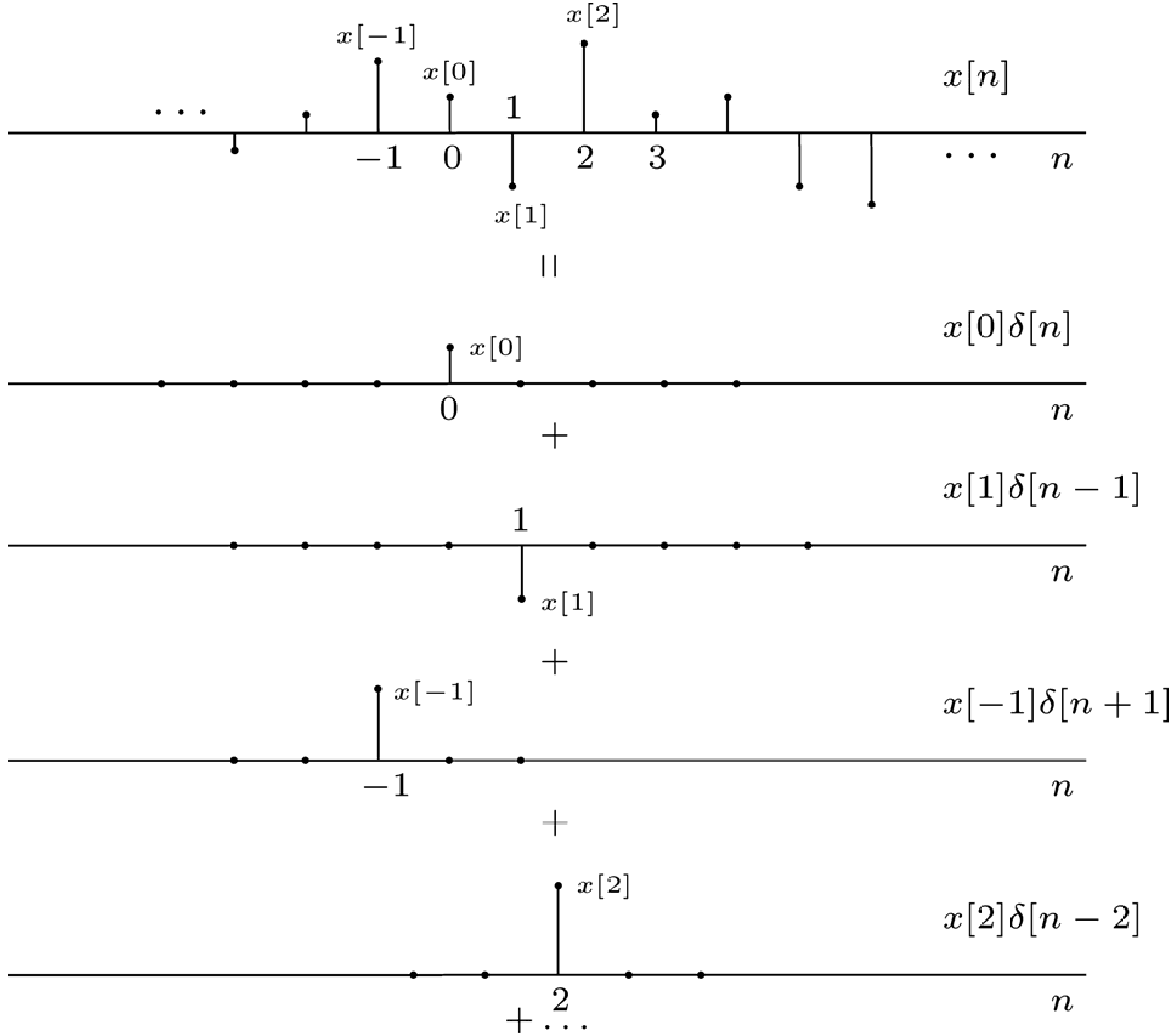
b) The response of LTI Systems to these basic signals are both *simple* and *insightful*.

Fact: For LTI Systems (CT or DT) there are two natural choices for these building blocks

Focus for now:

DT	Shifted unit samples
CT	Shifted unit impulses

Representation of DT Signals Using Unit Samples



That is ...

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$$\Downarrow$$

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{Coefficients}}\,\underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

The Sifting Property of the Unit Sample

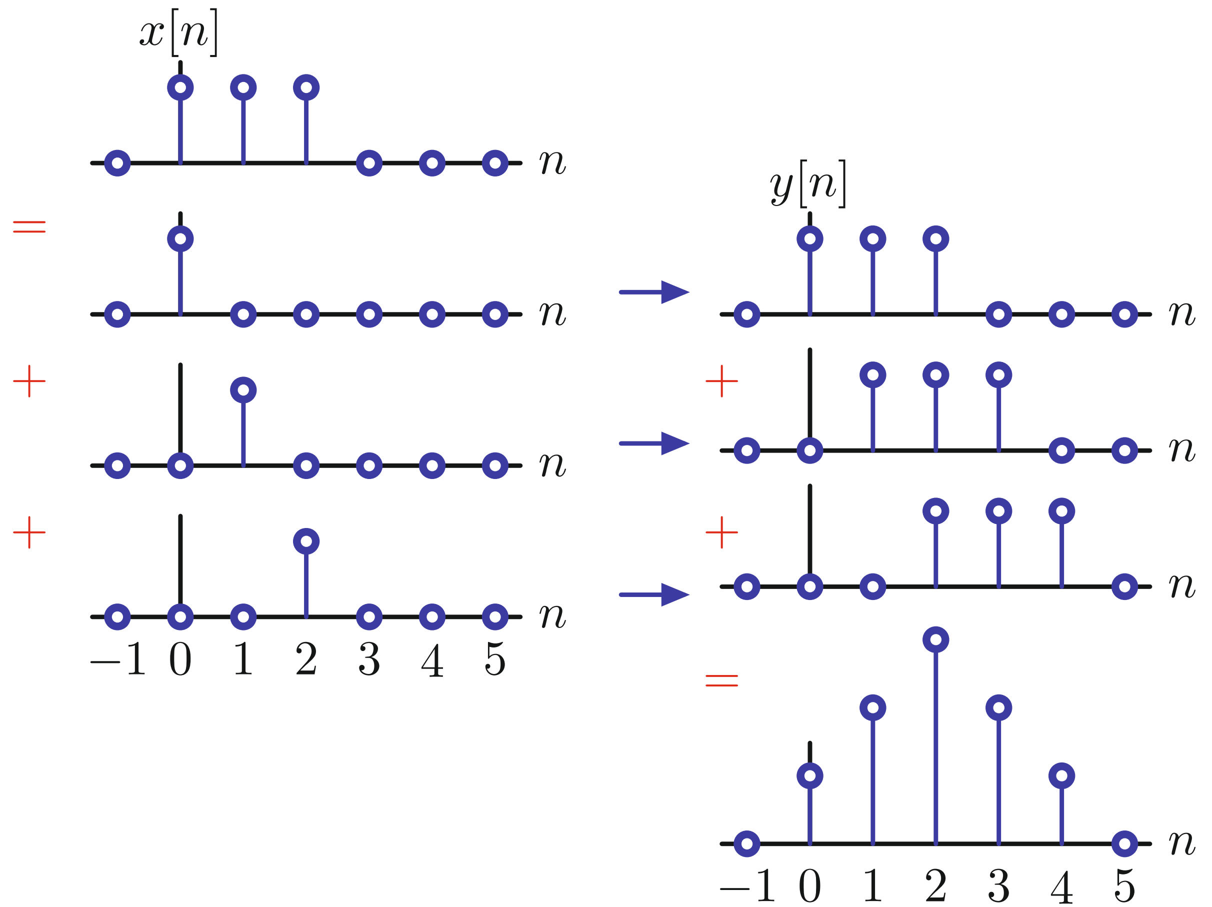
۲

بازنمایی جمع کانوولوشن سیستم‌های خطی تغییرناپذیر با زمان گسسته–زمان

برهم‌نهی

SUPERPOSITION

ورودی را به اجزای جمعی تفکیک می‌کنیم و پاسخ به این اجزا را با هم جمع می‌کنیم:



برهم‌نهی زمانی قابل استفاده است که سیستم **خطی** باشد.

پاسخ به اجزا زمانی به‌سادگی قابل محاسبه است که سیستم **تغییرناپذیر با زمان** باشد.

خطی بودن

LINEARITY

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the system is linear if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β.

تغییرناپذیری با زمان

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

the system is time invariant if

$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

is true for all n_0.

ساختار برهم‌نهی

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit-sample responses.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

جمع کانوولوشن

Convolution Sum

Response of an LTI system to an arbitrary input.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

This operation is called **convolution**.

$$\boxed{(x * h)[n] = x[n] * h[n]}$$

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Suppose the system is **linear** edefin dan ,h_k]n as [ ]δ the response to$n$ $-k$:[

$$\delta[n-k] \rightarrow h_k[n]$$

From superposition: $\Downarrow$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

$$x]n[\longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit sample response h*[*n*]*:*

$$\delta[n] \rightarrow h[n]$$

From TI:

$$\Downarrow$$

$$\delta[n-k] \rightarrow h[n-k]$$

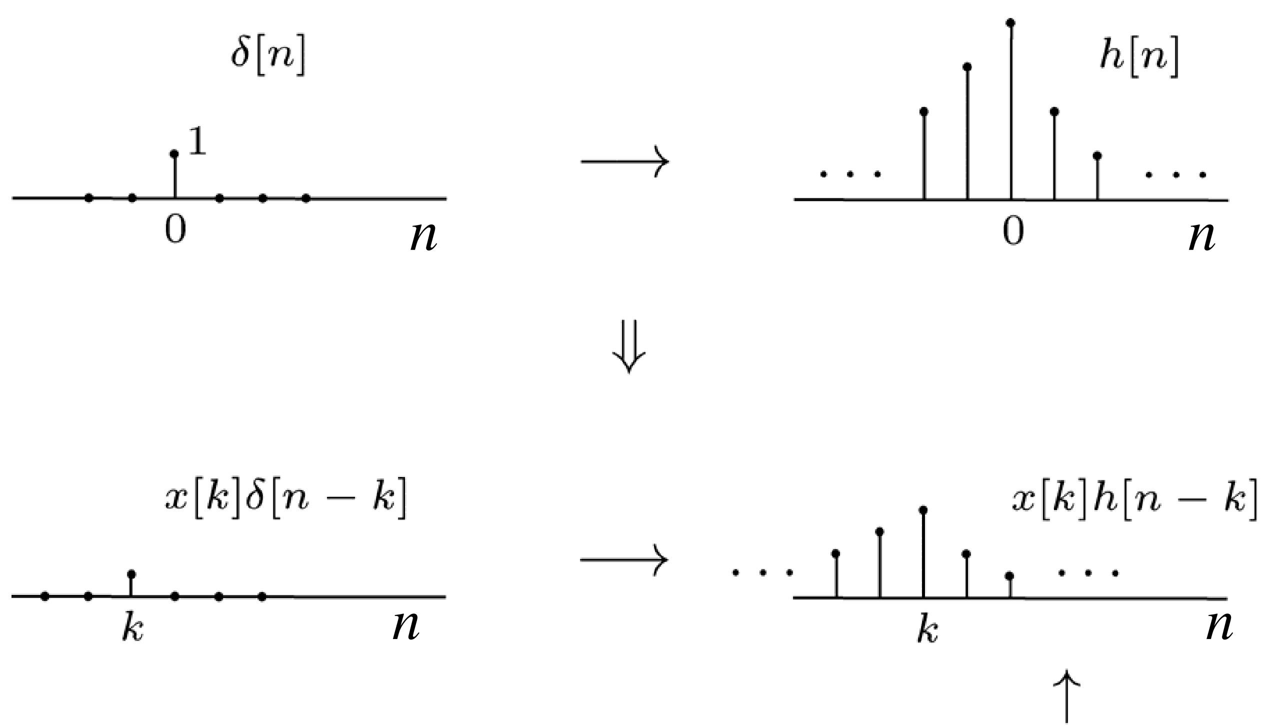
From LTI:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \underbrace{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{\text{Convolution Sum}}$$

Convolution Sum Representation of Response of LTI Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Interpretation



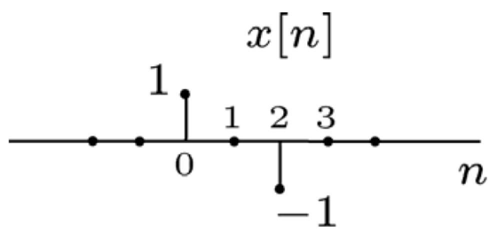
Sum up responses over all k

Visualizing the calculation of $y[n] = x[n] * h[n]$

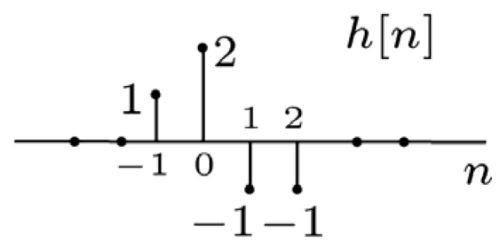
Choose value of n and consider it fixed

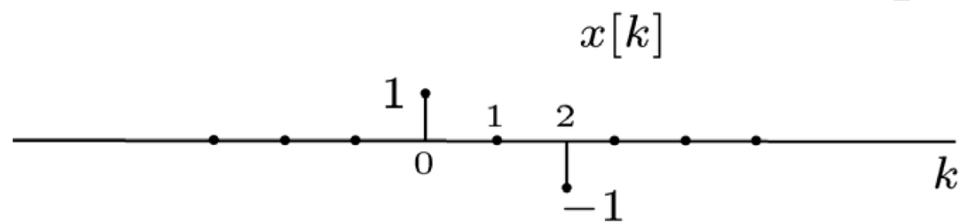
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

View as functions of k with n fixed

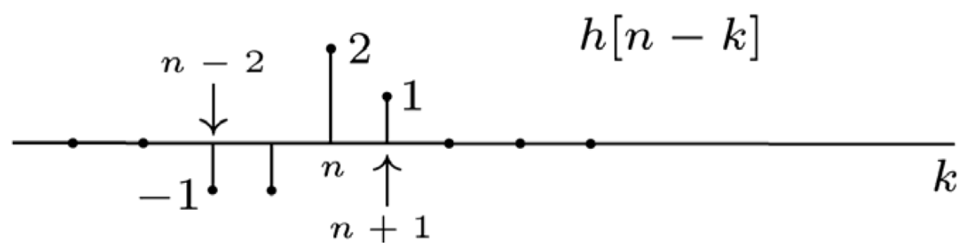


$\Downarrow$





$y[0] = \Sigma$ prod of overlap for $n = 0$



$y[1] = \Sigma$ prod of overlap for $n = 1$

سیستم‌های خطی تغییرناپذیر با زمان (۱)

۳

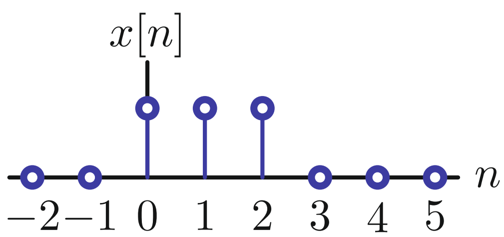
چند مثال

مجموع کانوولوشن

مثال (۱ از ۱۴)

CONVOLUTION SUM

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



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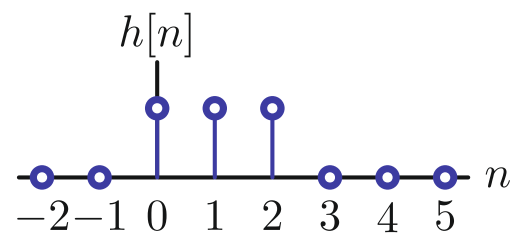
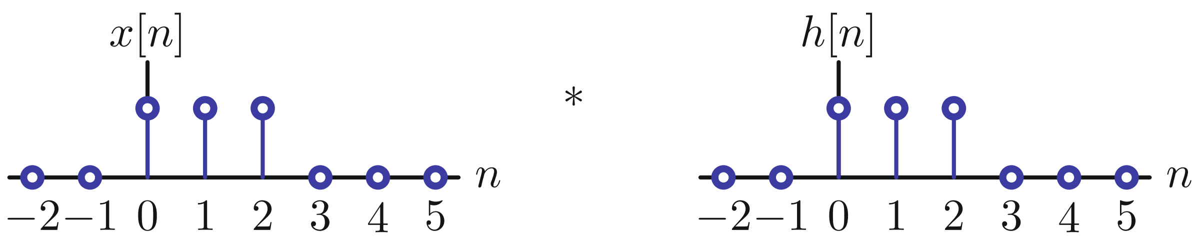


مجموع کانوولوشن

مثال (۲ از ۱۴)

Convolution sum

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

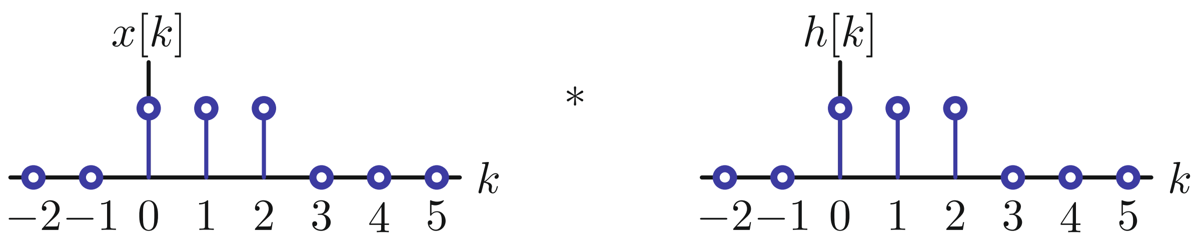


مجموع کانوولوشن

مثال (۳ از ۱۴)

Convolution sum

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

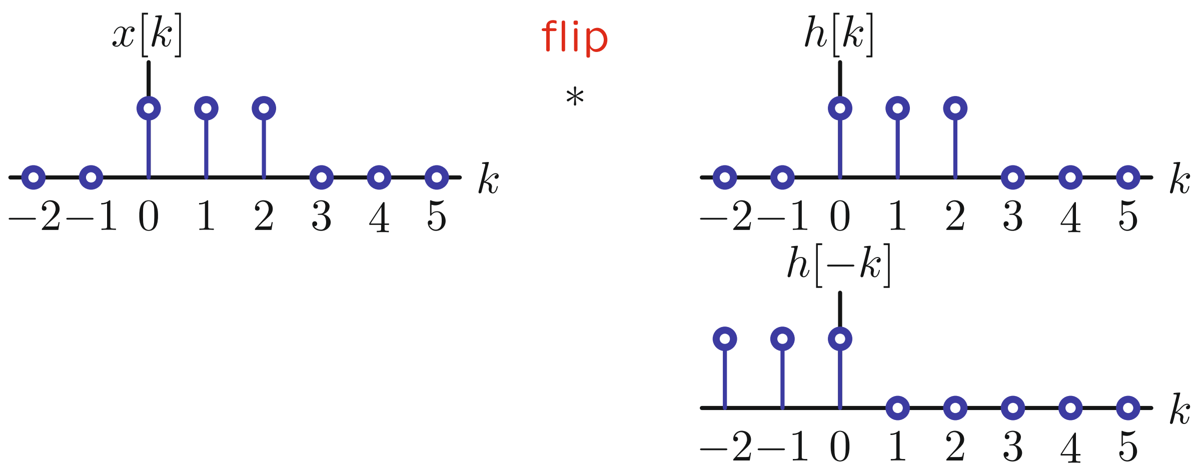


مجموع کانوولوشن

مثال (۴ از ۱۴)

CONVOLUTION SUM

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0 - k]$$

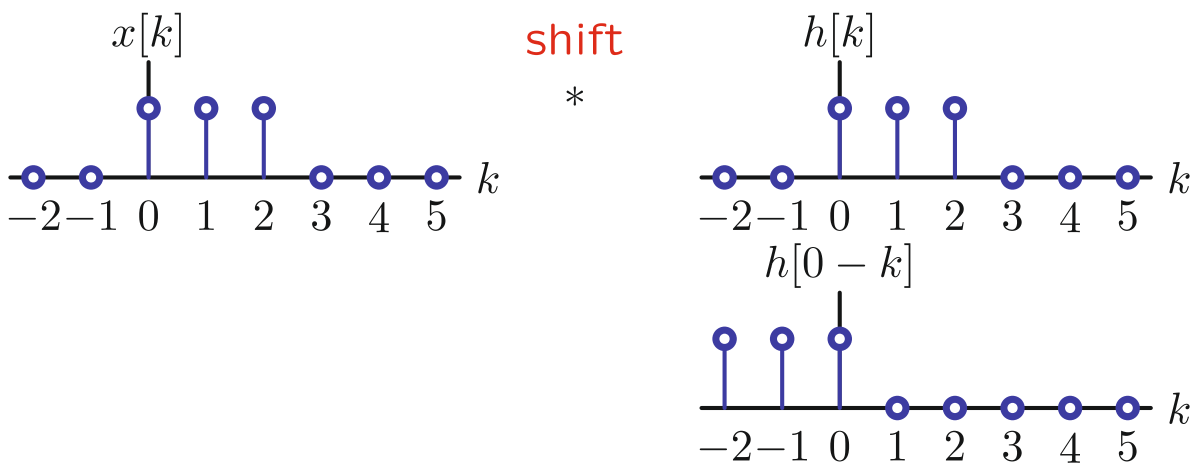


مجموع کانوولوشن

مثال (۵ از ۱۴)

CONVOLUTION SUM

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

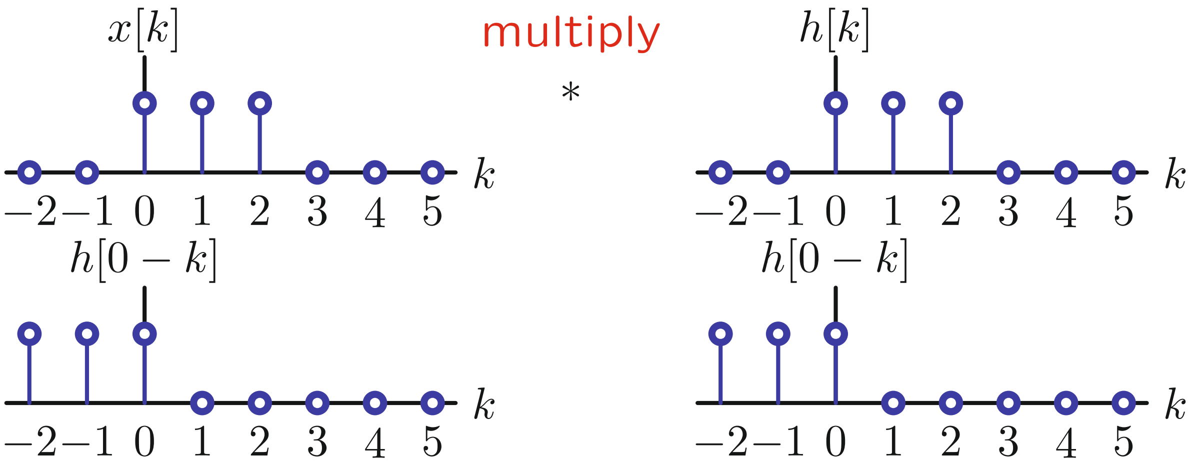


مجموع کانوولوشن

مثال (۶ از ۱۴)

CONVOLUTION SUM

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

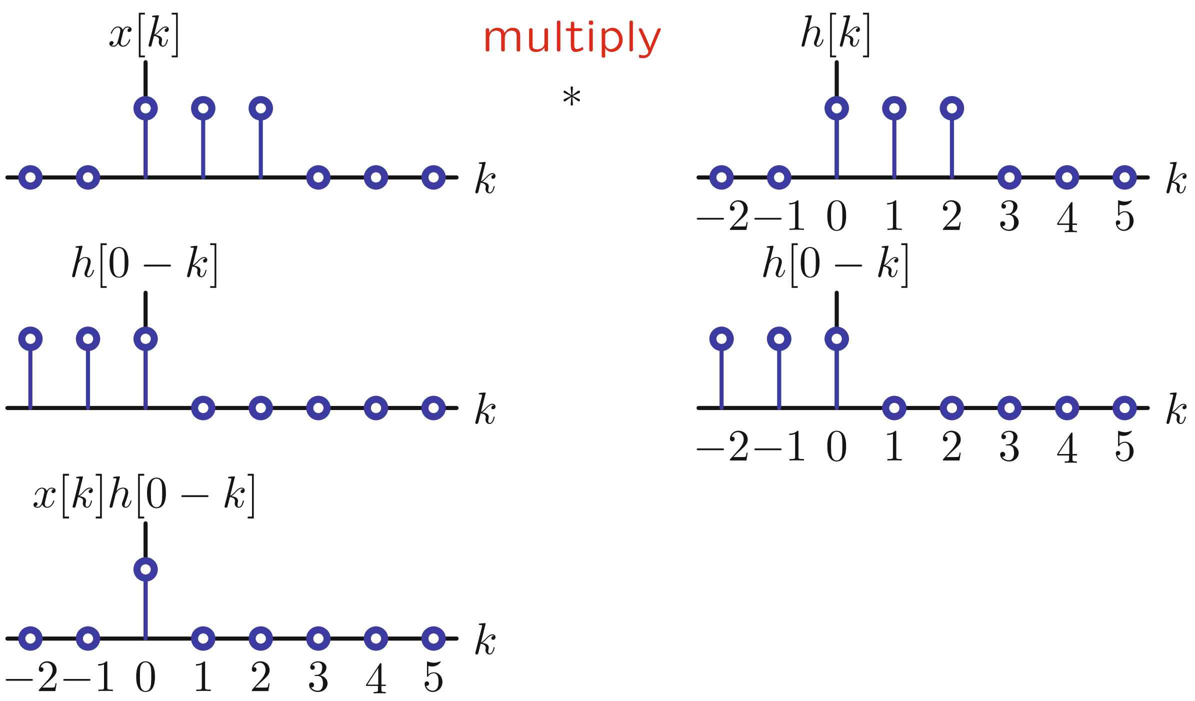


مجموع کانوولوشن

مثال (۷ از ۱۴)

CONVOLUTION SUM

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

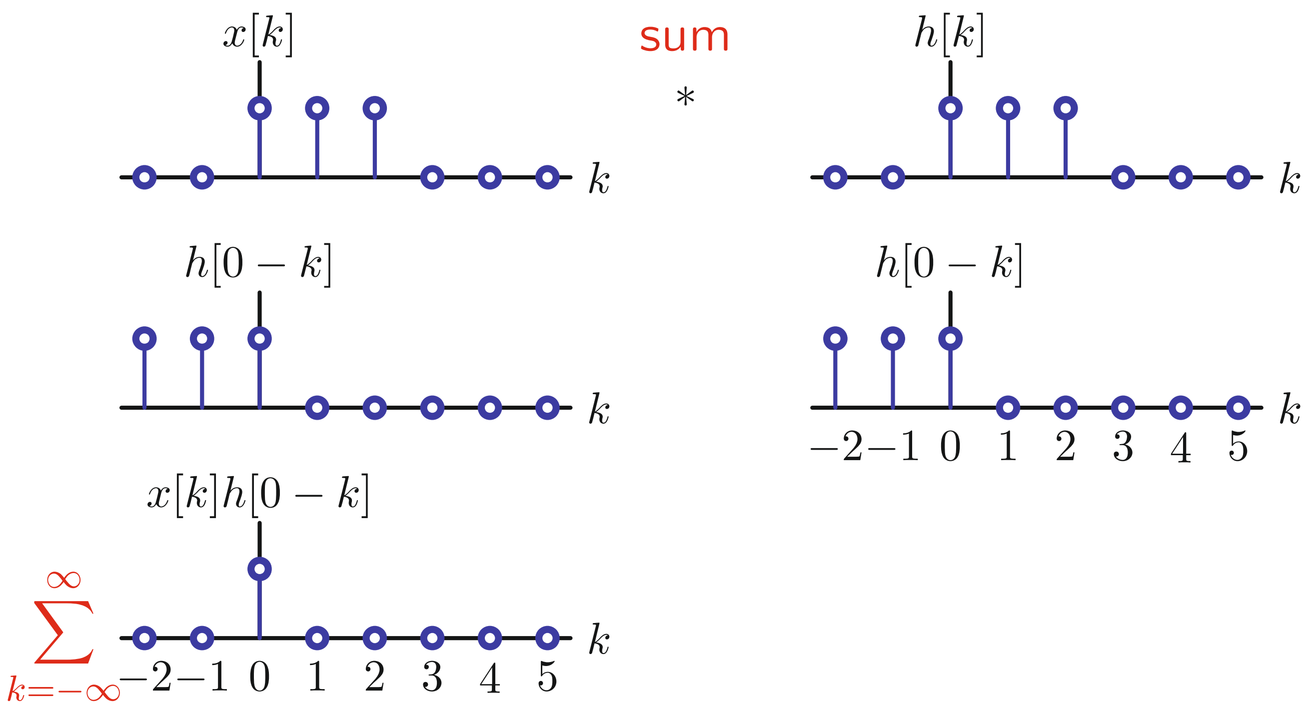


مجموع کانوولوشن

مثال (۸ از ۱۴)

CONVOLUTION SUM

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

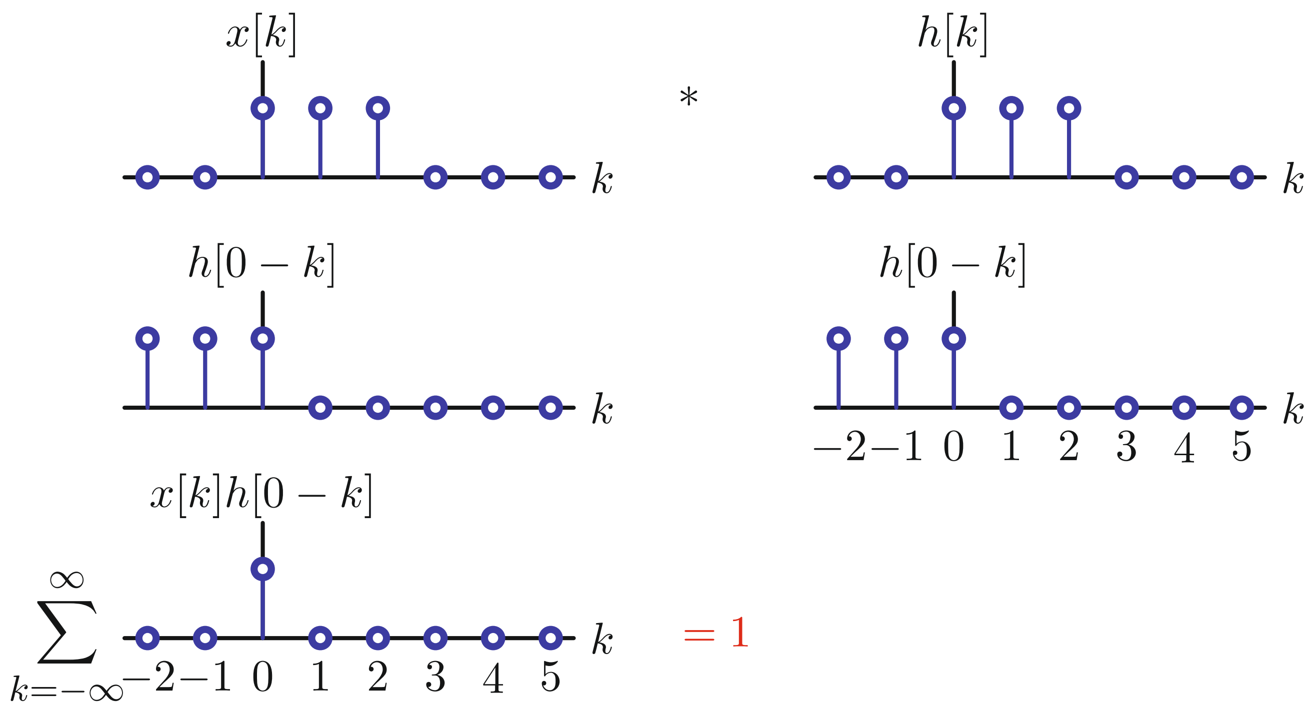


مجموع کانوولوشن

مثال (۹ از ۱۴)

Convolution sum

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

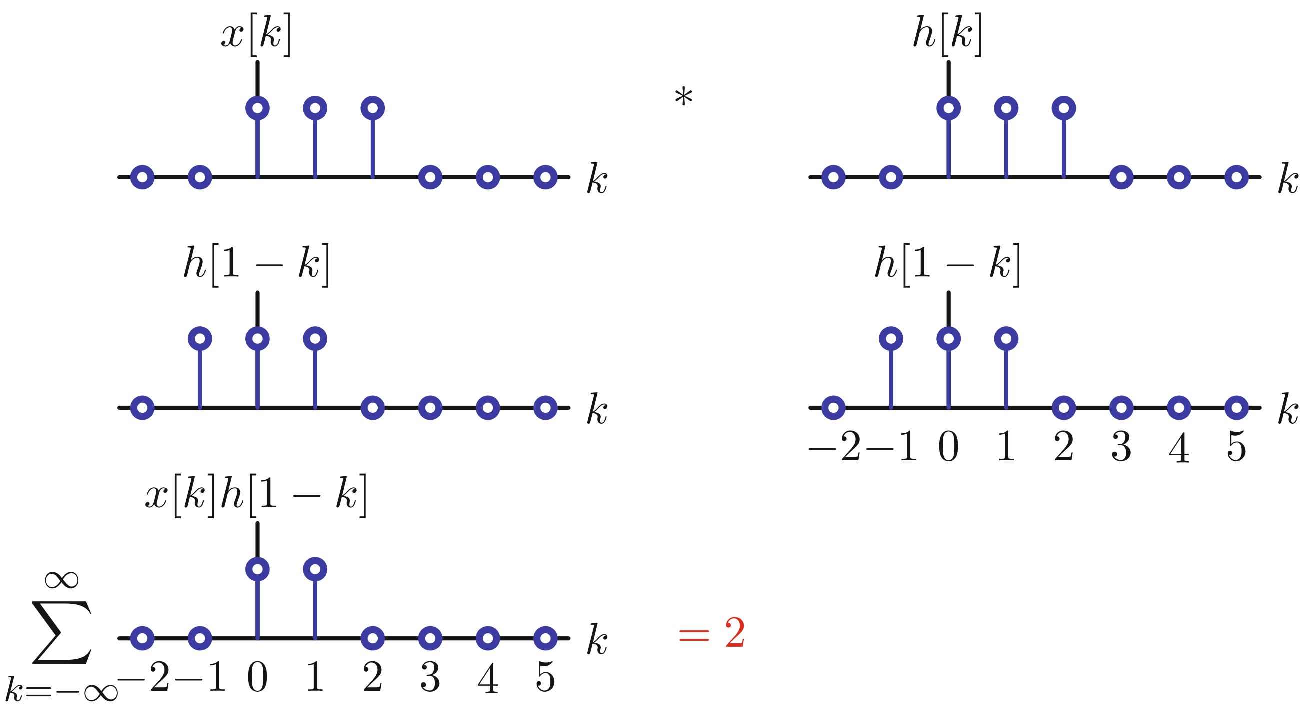


مجموع کانوولوشن

مثال (۱۰ از ۱۴)

Convolution sum

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$

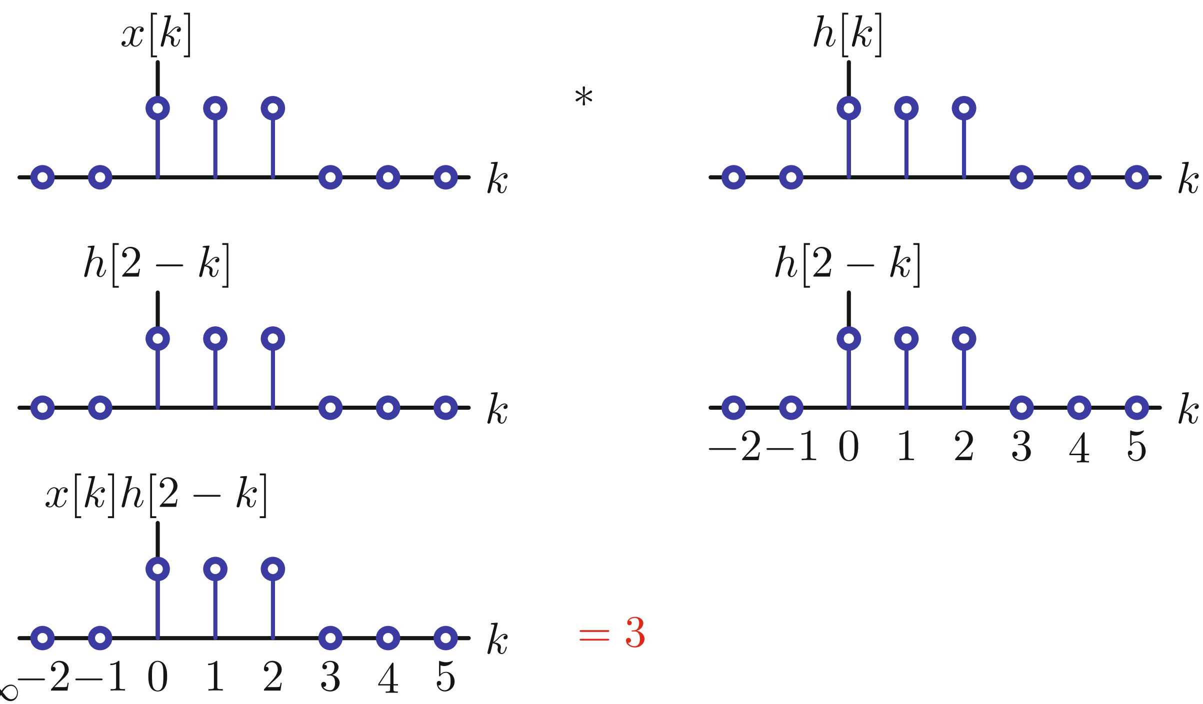


مجموع کانوولوشن

مثال (۱۱ از ۱۴)

Convolution sum

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$



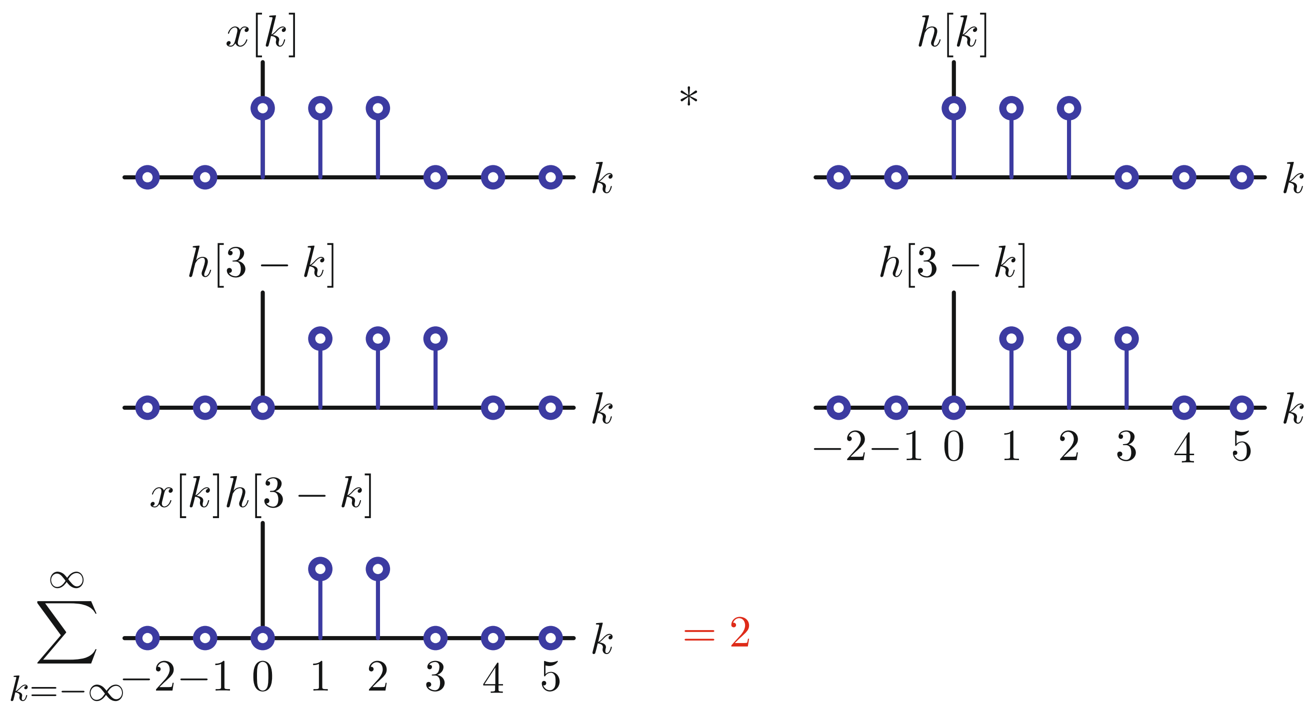
$$\sum_{k=-\infty}^{\infty} x[k]h[2-k] = 3$$

مجموع کانوولوشن

مثال (۱۲ از ۱۴)

CONVOLUTION SUM

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$

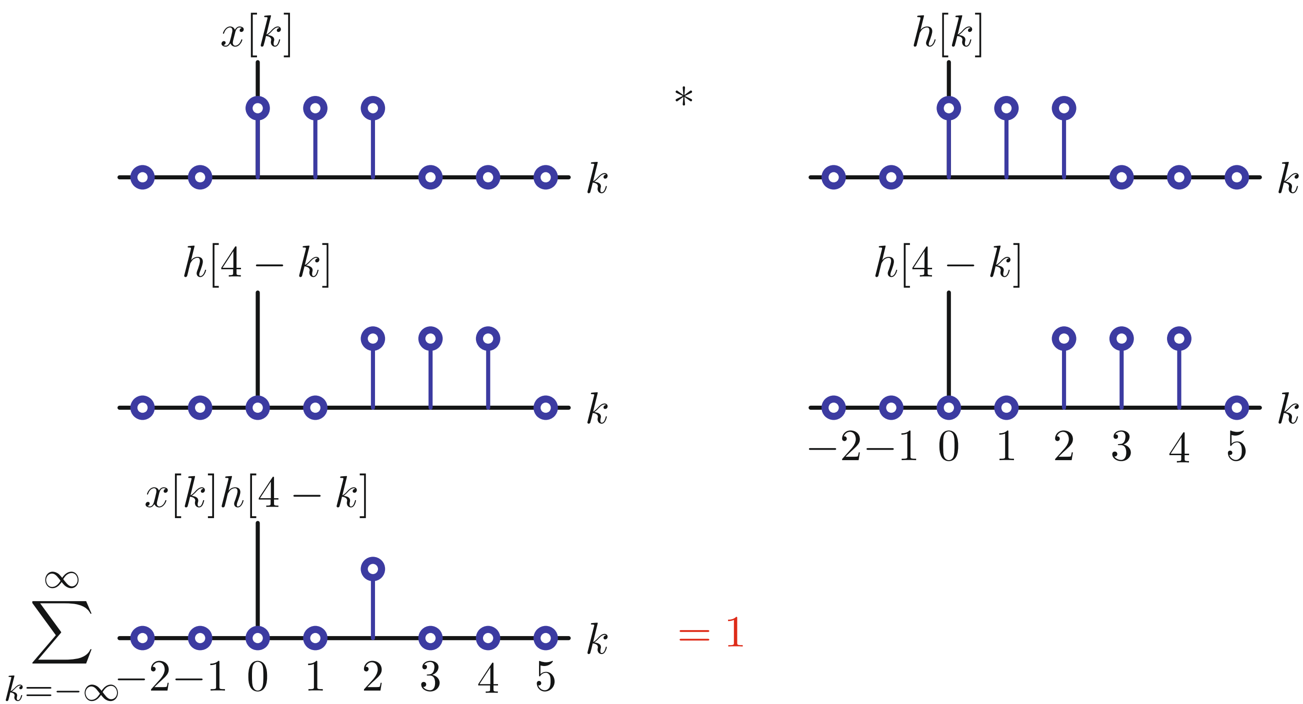


مجموع کانوولوشن

مثال (۱۳ از ۱۴)

Convolution sum

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

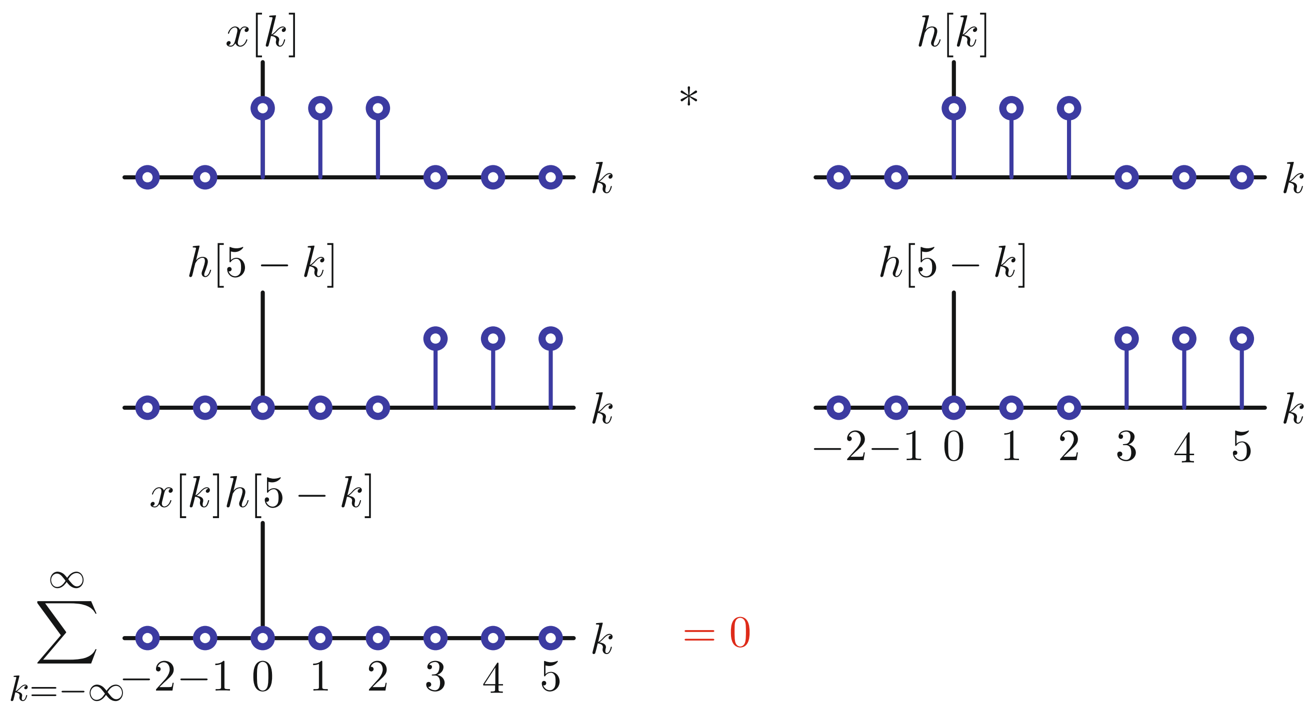


مجموع کانوولوشن

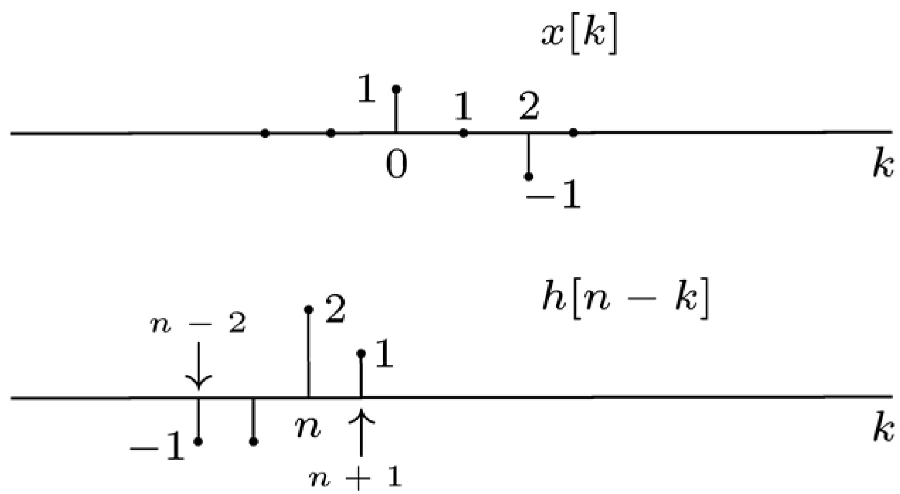
مثال (۱۴ از ۱۴)

Convolution sum

$$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$$



Calculating Successive Values: Shift, Multiply, Sum

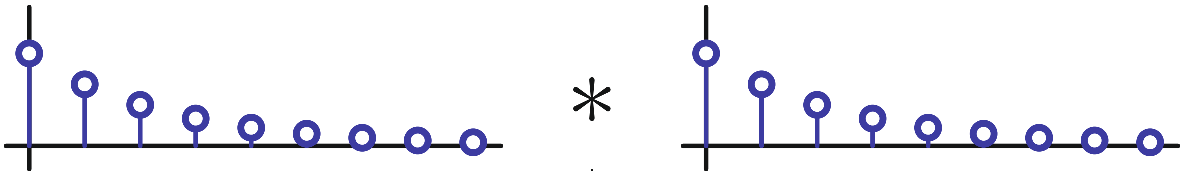


$$
\begin{aligned}
y[n] &= 0 \quad \text{for } n < -1 \\
y[-1] &= 1 \times 1 = 1 \\
y[0] &= 0 \times 1 + 1 \times 2 = 2 \\
y[1] &= (-1) \times 1 + 0 \times 2 + 1 \times (-1) = -2 \\
y[2] &= (-1) \times 2 + 0 \times (-1) + 1 \times (-1) = -3 \\
y[3] &= (-1) \times (-1) + 0 \times (-1) = 1 \\
y[4] &= (-1) \times (-1) = 1 \\
y[n] &= 0 \quad \text{for } n > 4
\end{aligned}
$$

مجموع کانوولوشن

مثال

CONVOLUTION SUM



Express mathematically:

$$\left(\left(\frac{2}{3}\right)^n u[n]\right) * \left(\left(\frac{2}{3}\right)^n u[n]\right) = \sum_{k=-\infty}^{\infty} \left(\left(\frac{2}{3}\right)^k u[k]\right) \times \left(\left(\frac{2}{3}\right)^{n-k} u[n-k]\right)$$

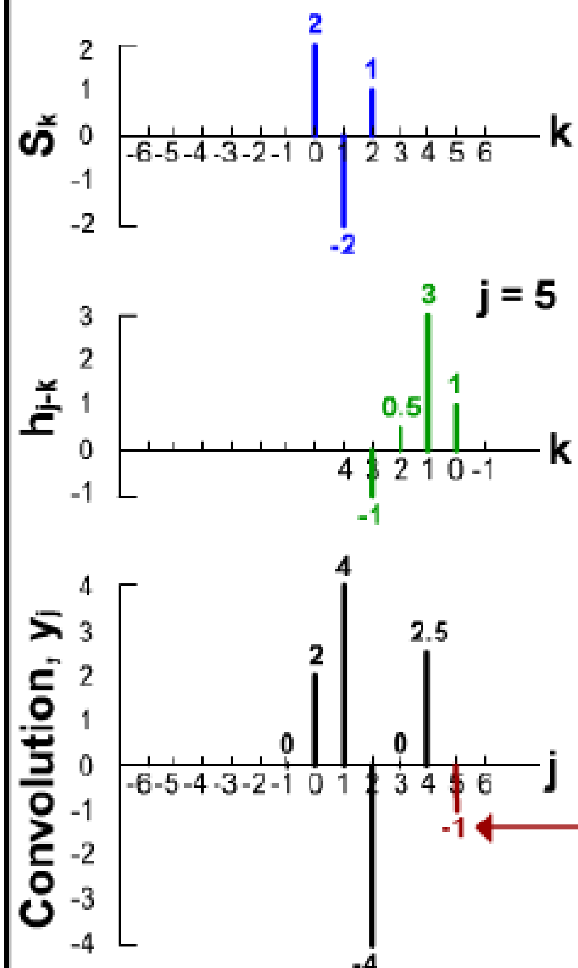
$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \times \left(\frac{2}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n \sum_{k=0}^{n} 1$$

$$= (n+1)\left(\frac{2}{3}\right)^n u[n]$$

$$= 1,\ \frac{4}{3},\ \frac{4}{3},\ \frac{32}{27},\ \frac{80}{81},\ \ldots$$

Convolution Animation (discrete)



$$y_5 = \sum_k s_k h_{5-k}$$

$= 0 \;\; 2 \;\; -2 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0$ *

$-1 \;\; 0.5 \;\; 3 \;\; 1$ =

(0 x -1) + (0 x 0.5) + (0 x 3) + (0 x 1) = 0

(0 x -1) + (0 x 0.5) + (0 x 3) + (2 x 1) = 2

(0 x -1) + (0 x 0.5) + (2 x 3) + (-2 x 1) = 4

(0 x -1) + (2 x 0.5) + (-2 x 3) + (1 x 1) = -4

(2 x -1) + (-2 x 0.5) + (1 x 3) + (0 x 1) = 0

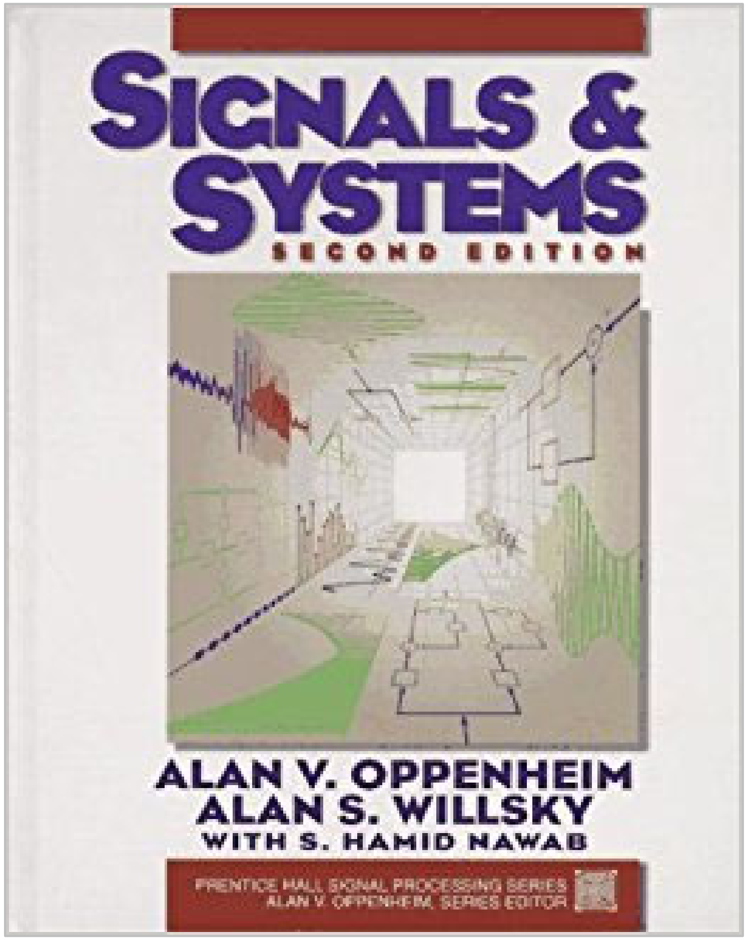
(-2 x -1) + (1 x 0.5) + (0 x 3) + (0 x 1) = 2.5

(1 x -1) + (0 x 0.5) + (0 x 3) + (0 x 1) = -1

۴

منابع

منبع اصلی



A.V. Oppenheim, A.S. Willsky, S.H. Nawab,
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Chapter 2