Marin Marin · Andreas Öchsner

Essentials of Partial Differential Equations

With Applications

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Preface

This book was written based on lectures and tutorials held for mathematics and computer science students. Supplementary notions were added to these notes in order to make the reader refer to the bibliography as little as possible. The present form of the book is due to the essential contributions of our published results. We divided the book into two parts, thus suggesting that the path of the student in mathematics (and other subjects), starting with an introduction into the theory of partial differential equations and then leading into the modern problems of this subject, can be done in two steps.

In the first part, the reader who is already well acquainted with problems from the theory of differential and integral equations learns about the classical notions and problems: differential operators, characteristic surfaces, Levi functions, Green's function, Green's formulas, among others. Moreover, the reader is instructed in the extended potential theory in its three forms, i.e., the volume potential, the surface single-layer potential, and the surface double-layer potential. Furthermore, the book presents the main initial-boundary value problems associated with elliptic, parabolic, and hyperbolic equations.

Compared to other books on the same topic, we added a new chapter that includes operational calculus, with the advantages that it offers for solving initial and boundary value problems.

In the first part, we present the notions and the results in terms of the classical solutions.

The second part of the book, which is addressed first and foremost to those who are already acquainted with the notions and the results from the first part, aims at introducing the reader into the modern aspects of the theory of partial differential equations.

The first tool that is introduced is the theory of distributions and implicitly its advantages with respect to differentiation and integration. In the context of the theory of distributions, the initial-boundary value problems are approached by means of weak solutions. Moreover, the second part presents the basic notions regarding Sobolev spaces.

While in the first part the proofs are, usually, very detailed, in the second part the proofs are more sketch-like, since we assume that the reader already gained some experience while reading the first part.

According to many authors, the approach of the initial-boundary value problems associated with differential operators of order two, in the context of the theory of distributions, has already become classical. Nonetheless, we ensure that the second part deserves to be a modern one.

To this end, a series of recent results is introduced. For instance, we encounter the classical theorem of Lax–Milgram and its recent extension, given by Stampacchia, and the advantages presented by this generalization.

Then, the maximum principle for harmonic functions is presented in the form given by Hopf. The classical embedding theorems of Sobolev were replaced with more recent and easier ones, introduced by renowned mathematicians, such as Gagliardo, Rellich, and Kondrachov.

Obviously, there are many more results, which are more recent and which are related to the theory of distributions and to its problems. If we included these, even in a simplified form, to make them more accessible, then this book would become much longer.

These recent results were not included because we did not want to show a scientific deficit in the rigorous mathematical reasoning of the results presented.

We have to remark our preoccupation with the independence of the book, since the basic notions regarding topology, functional analysis, the geometry of curves and surfaces, and so on are briefly recalled before being used. The two parts of the book are quite independent, and we suggest the reader to go through the book in two parts.

The second part of the book is obviously much more difficult, and therefore, the authors have included more applications in this part.

We are convinced that our efforts will prove useful for students (and not only), who will be attracted by the modern aspects of the theory of partial differential equations.

Brasov, Romania — Marin Marin
Esslingen, Germany — Andreas Öchsner
January, 2018

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Part I
Classical Solutions

Chapter 1
Quasilinear Equations

1.1 Canonical Form in Two-Dimensional Case

In this book, we study mainly second-order partial differential equations. Let $\Omega \subset \mathbb{R}^n$ be bounded. Denote $x = (x_1, x_2, \ldots, x_n) \in R^n$. Let $u: \Omega \to \mathbb{R}$ be a sufficiently smooth function and denote

$$u_{x_i} = \frac{\partial u}{\partial x_i}, i = 1, 2, \ldots, n \quad \text{and} \quad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, i, j = 1, 2, \ldots, n.$$

The general form of second-order partial differential equations is

$$F(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_i x_j}, \ldots, u_{x_{n-1} x_n}) = 0. \tag{1.1.1}$$

In Eq. (1.1.1), the unknown function is $u(x) = u(x_1, x_2, \ldots, x_n)$, $x \in \Omega$, and the scalar function F is a given function with some properties defined below.

Definition 1.1.1 A second-order partial differential equation of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_1, x_2, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x_1, x_2, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) \tag{1.1.2}$$

is called a quasilinear partial differential equation of second order.

In Eq. (1.1.2), the scalar functions $a_{ij}: \Omega \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$ are given as continuous functions such that $a_{ij}(x) = a_{ji}(x)$, $x \in \Omega$. A function $u \in C^2(\Omega, \mathbb{R})$ is a solution of Eq. (1.1.2) if it satisfies Eq. (1.1.2).

In the particular case of $n = 2$, i.e., the case of two independent variables, the quasilinear equation (1.1.2) reduces to

$$a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{1.1.3}$$

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Change the independent variables x, y in Eq. (1.1.3) by the substitution:

$$\begin{aligned} \xi &= \xi(x, y), \\ \eta &= \eta(x, y), \end{aligned} \tag{1.1.4}$$

where the functions $\xi, \eta \in C^2(\Omega, \mathbb{R})$ and

$$\left|\frac{\partial(\xi, \eta)}{\partial(x, y)}\right| = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0, \text{ in } \Omega. \tag{1.1.5}$$

The goal of transformation (1.1.4) is to obtain a new partial differential equation in the form (1.1.3) with variables ξ and η, but with at least one zero coefficient.

Observation 1.1.1 *According to the condition (1.1.5), we can apply the implicit function theorem for any point in* Ω. *Therefore, if* $(x_0, y_0) \in \Omega$ *is an arbitrary point then the system (1.1.4) could be solved with respect to the unknown variables x and y in a neighborhood of the point* (x_0, y_0) *and we obtain*

$$\begin{aligned} x &= x(\xi, \eta) \\ y &= y(\xi, \eta). \end{aligned} \tag{1.1.6}$$

If we denote $\xi_0 = \xi(x_0, y_0)$, $\eta_0 = \eta(x_0, y_0)$, *then we get* $x_0 = x(\xi_0, \eta_0)$ *and* $y_0 = y(\xi_0, \eta_0)$.

Apply equalities

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \\ &\quad + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}, \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) \\ &\quad + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}, \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \\ &\quad + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}. \end{aligned} \tag{1.1.7}$$

to Eq. (1.1.3) and obtain

$$\overline{a_{11}}\frac{\partial^2 u}{\partial \xi^2} + 2\overline{a_{12}}\frac{\partial^2 u}{\partial \xi \partial \eta} + \overline{a_{22}}\frac{\partial^2 u}{\partial \eta^2} = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{1.1.8}$$

In Eq. (1.1.8), the coefficients $\overline{a_{ij}},\ i, j = 1, 2$ are given by

$$\begin{aligned}
\overline{a_{11}} &= a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + a_{22}\left(\frac{\partial \xi}{\partial y}\right)^2, \\
\overline{a_{12}} &= a_{11}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + a_{12}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + a_{22}\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}, \\
\overline{a_{22}} &= a_{11}\left(\frac{\partial \eta}{\partial x}\right)^2 + 2a_{12}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + a_{22}\left(\frac{\partial \eta}{\partial y}\right)^2.
\end{aligned} \tag{1.1.9}$$

Consider the following partial differential equation of first order (compare with the first and the third equations in (1.1.9))

$$a_{11}\left(\frac{\partial z}{\partial x}\right)^2 + 2a_{12}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + a_{22}\left(\frac{\partial z}{\partial y}\right)^2 = 0. \tag{1.1.10}$$

Let $z = \varphi(x, y)$ be a solution of Eq. (1.1.10).

Consider the substitution

$$\begin{aligned}
\xi &= \varphi(x, y) \\
\eta &= \eta(x, y),
\end{aligned}$$

where $\eta(x, y)$ is an arbitrary function such that the condition (1.1.5) is satisfied. Then from Eq. (1.1.9) we get $\overline{a_{11}} = 0$.

Consider the substitution

$$\begin{aligned}
\xi &= \xi(x, y) \\
\eta &= \varphi(x, y),
\end{aligned}$$

where $\varphi(x, y)$ is an arbitrary function such that the condition (1.1.5) is satisfied. Then from Eq. $(1.1.9)_1$ we get $\overline{a_{22}} = 0$.

On the other hand, solving the partial differential equation (1.1.10) is equivalent to solving the following ordinary differential equation:

$$a_{11}\,(dy)^2 - 2a_{12}dy\,dx + a_{22}\,(dx)^2 = 0. \tag{1.1.11}$$

Equation (1.1.11) can be written in the form

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = 0. \tag{1.1.12}$$

Proposition 1.1.1 *(i). Let $\varphi(x, y) = C$ be a prime integral of Eq. (1.1.11), where C is an arbitrary constant.*

Then the function $z = \varphi(x, y)$ is a solution of Eq. (1.1.10).
(ii). Let $z = \varphi(x, y)$ be a solution of Eq. (1.1.10).

Then $\varphi(x, y) = C$, where C is an arbitrary constant, is a prime integral of Eq. (1.1.11).

Proof (i). Let $\varphi(x, y) = C$ be a prime integral of the Eq. (1.1.11). Without loss of generality, we can assume that $\frac{\partial\varphi}{\partial y}(x, y) \neq 0, \forall(x, y) \in \Omega$. Indeed, if there exists a set $\Omega_0 \subset \Omega$ such that $\frac{\partial\varphi}{\partial y}(x, y) = 0, \forall(x, y) \in \Omega_0$, then we consider the domain $\Omega \setminus \Omega_0$ in which the derivative $\frac{\partial\varphi}{\partial y}(x, y)$ is not zero. If $\frac{\partial\varphi}{\partial x}(x, y) = 0, \forall(x, y) \in \Omega$, then we change the variable y by x. If both $\frac{\partial\varphi}{\partial x}(x, y) = 0$ and $\frac{\partial\varphi}{\partial y}(x, y) = 0, \forall(x, y) \in \Omega$, then the function $\varphi(x, y)$ is a constant and Eq. (1.1.10) has a trivial solution.

Therefore, we can assume that $\frac{\partial\varphi}{\partial y}(x, y) \neq 0$ in Ω. Then in a neighborhood of a point (x_0, y_0) so that $\frac{\partial\varphi}{\partial y}(x_0, y_0) \neq 0$, we can have the expression $y = f(x, c_0)$, where $c_0 = \varphi(x_0, y_0)$. Additionally, we get

$$\frac{dy}{dx} = -\frac{\frac{\partial\varphi}{\partial x}(x, y)}{\frac{\partial\varphi}{\partial y}(x, y)}. \tag{1.1.13}$$

We substitute (1.1.13) in (1.1.12), which is equivalent to (1.1.11), and obtain

$$\begin{aligned}
0 &= \left[a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22}\right]_{(x_0, y_0)} \\
&= \left[a_{11}\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2 - 2a_{12}\left(-\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right) + a_{22}\right]_{(x_0, y_0)} \\
&= \left[a_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y} + a_{22}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\frac{1}{\left(\frac{\partial\varphi}{\partial y}\right)^2}(x_0, y_0).
\end{aligned}$$

Therefore, we have

$$\left[a_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial x} + a_{22}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]_{(x_0, y_0)} = 0, \quad \text{for } \forall(x, y) \in \Omega.$$

It proves that $\varphi(x, y)$ is a solution of Eq. (1.1.10).

(ii). Let $z = \varphi(x, y)$ be a solution of Eq. (1.1.10). Therefore,

$$a_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\varphi}{\partial y}\frac{\partial\varphi}{\partial x} + a_{22}\left(\frac{\partial\varphi}{\partial y}\right)^2 = 0.$$

The assumptions of the implicit function theorem are satisfied. Then, we divide the above equation, formally, by $\frac{\partial\varphi}{\partial y}$ and obtain

$$a_{11}\left(\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}}\right)^2 + 2a_{12}\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}} + a_{22} = 0. \tag{1.1.14}$$

Compare Eq. (1.1.14) with (1.1.12) and conclude that

$$\frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}} = -\frac{dy}{dx}$$

or

$$\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = 0.$$

Therefore, $d\varphi(x, y) = 0$ and $\varphi(x, y) = C$, where C is an arbitrary constant. ■

From Proposition 1.1.1, it follows that obtaining the solution of Eq. (1.1.10) is equivalent to obtaining the prime integrals of (1.1.12). At the same time, obtaining the prime integrals of (1.1.12) is equivalent to zero coefficients $\overline{a_{11}}$ and $\overline{a_{22}}$ of (1.1.8).

Equation (1.1.11) is called the *equation of characteristics* and its prime integrals are called *characteristics* or *characteristic curves*. Consider the expression $\Delta = a_{12}^2 - a_{11}a_{22}$ which is important for obtaining prime integrals. There are three possible cases:

1^o. If $\Delta > 0$, then Eq. (1.1.11) admits two distinct real characteristics and the partial differential equation shall be called a *hyperbolic equation*.

2^o. If $\Delta = 0$, then Eq. (1.1.11) admits only one real characteristic and the partial differential equation shall be called a *parabolic equation*.

3^o. If $\Delta < 0$, then Eq. (1.1.11) admits two complex conjugated characteristics and the partial differential equation shall be called an *elliptic equation*.

The above classification was made with respect to the partial differential equation (1.1.3). But the substitution (1.1.4) provided (1.1.5) does not change the type of Eq. (1.1.3). Indeed, $\overline{\Delta}$ for Eq. (1.1.8) will be

$$\overline{\Delta} = \overline{a_{12}}^2 - \overline{a_{11}}\,\overline{a_{22}} = \left(a_{12}^2 - a_{11}a_{22}\right)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2.$$

According to Eq. (1.1.9) for the coefficients $\overline{a_{ij}}$, we get

$$\overline{\Delta} = \Delta \left(\frac{\partial(\xi, \eta)}{\partial(x, y)} \right)^2 . \tag{1.1.15}$$

From (1.1.15), it follows that the sign of $\overline{\Delta}$ is the same as the sign of Δ and therefore, that the substitution (1.1.4) provided (1.1.5) does not change the type of the equation.

Equation (1.1.8) will be called the *canonical form* of the partial differential equation (1.1.3).

It is easy to see that Δ is a continuous function with respect to the variables (x, y).

It is well known that if a continuous function is positive in a point, then it is positive in a whole neighborhood of that point. So, we can divide the whole plane into three disjoint sets. We call the domain of hyperbolicity for Eq. (1.1.3) the set points in plane $\mathbb{R}^2$ for which Eq. (1.1.3) is a hyperbolic equation. Analogously, we can define the domains of parabolicity and ellipticity.

Now consider the following three possible cases:

1^o **The hyperbolic case**: $\Delta = a_{12}^2 - a_{11}a_{22} > 0$. In this case, Eq. (1.1.11) has two real distinct prime integrals $\varphi(x, y) = C_1$, $\psi(x, y) = C_2$, where C_1 and C_2 are arbitrary constants. We substitute

$$\xi = \varphi(x, y), \eta = \psi(x, y) \tag{1.1.16}$$

and according to Proposition 1.1.1 we get $\overline{a_{11}} = 0$ and $\overline{a_{22}} = 0$. Therefore, the canonical form of the hyperbolic equation is

$$\overline{a_{12}} \frac{\partial^2 u}{\partial \xi \partial \eta} = \overline{f} \left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta} \right)$$

or, if we divide by $\overline{a_{12}}$ (which obviously is nonzero):

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = F \left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta} \right). \tag{1.1.17}$$

We must emphasize that the transformation (1.1.16) is non-singular. Indeed,

$$\left| \frac{\partial(\xi, \eta)}{\partial(x, y)} \right| = 0 \Leftrightarrow \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \varphi}{\partial y} = 0 \Leftrightarrow$$
$$-\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} \Leftrightarrow \frac{a_{12} + \sqrt{\Delta}}{a_{11}} = \frac{a_{12} - \sqrt{\Delta}}{a_{11}} \Leftrightarrow$$
$$\Leftrightarrow \sqrt{\Delta} = -\sqrt{\Delta} \Leftrightarrow \Delta = 0,$$

which is a contradiction.

2^o **The parabolic case**: $\Delta = a_{12}^2 - a_{11}a_{22} = 0$. In this case, the characteristic Eq. (1.1.11) only has one real prime integral $\varphi(x, y) = C$, where C is an arbitrary constant. We substitute

$$\xi = \varphi(x, y), \eta = \eta(x, y), \tag{1.1.18}$$

where η is an arbitrary function of class C^2. The transformation (1.1.18) is non-singular

$$\left|\frac{\partial(\xi, \eta)}{\partial(x, y)}\right| = \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \eta}{\partial x}\frac{\partial \varphi}{\partial y} \neq 0. \tag{1.1.19}$$

Since we chose $\xi = \varphi(x, y)$, based on Proposition (1.1.1), we deduce that $\overline{a}_{11} = 0$. Now we will prove that $\overline{a}_{12} = 0$.

Proposition 1.1.2 *If the functions ξ and η have the form (1.1.18) and satisfy the condition (1.1.19), then $\overline{a_{12}} = 0$.*

Proof From $a_{12}^2 = a_{11}a_{22}$, we deduce that a_{11} and a_{22} have simultaneously the same sign and we do not restrict the generality if we assume that $a_{11} > 0$ and $a_{22} > 0$. Then we obtain $a_{12} = \pm\sqrt{a}_{11}\sqrt{a}_{22}$. According to Proposition 1.1.1 we deduce that $\overline{a_{11}} = 0$ and therefore

$$\begin{aligned}
0 &= a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + a_{11}\left(\frac{\partial \xi}{\partial y}\right)^2 \\
&= \left(\sqrt{a}_{11}\frac{\partial \xi}{\partial x}\right)^2 \pm \sqrt{a}_{11}\sqrt{a}_{22}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \left(\sqrt{a}_{22}\frac{\partial \xi}{\partial x}\right)^2 \\
&= \left(\sqrt{a}_{11}\frac{\partial \xi}{\partial x} \pm \sqrt{a}_{22}\frac{\partial \xi}{\partial y}\right)^2.
\end{aligned}$$

From here, we get

$$\sqrt{a_{11}}\frac{\partial \xi}{\partial x} \pm \sqrt{a_{22}}\frac{\partial \xi}{\partial y} = 0. \tag{1.1.20}$$

By using (1.1.9), it follows that

$$\begin{aligned}
\overline{a_{12}} &= a_{11}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + a_{12}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + a_{22}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \\
&= \left(\sqrt{a_{11}}\frac{\partial \xi}{\partial x} \pm \sqrt{a_{22}}\frac{\partial \xi}{\partial y}\right)\left(\sqrt{a_{11}}\frac{\partial \eta}{\partial x} \pm \sqrt{a_{22}}\frac{\partial \eta}{\partial y}\right),
\end{aligned}$$

so that by taking into account (1.1.20) we deduce that $\overline{a_{12}} = 0$. ■

By using that $\overline{a_{11}} = \overline{a_{12}} = 0$, we can conclude that the canonical form of a parabolic equation is

$$\overline{a_{22}}\frac{\partial^2 u}{\partial \eta^2} = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right),$$

or, equivalently

$$\frac{\partial^2 u}{\partial \eta^2} = G\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

Observation 1.1.2 *If instead of transformation (1.1.18), we take the transformation*

$$\xi = \xi(x, y),$$
$$\eta = \varphi(x, y),$$

in which $\varphi(x, y) = C$ is the only prime integral of the characteristic equation (1.1.11), and $\xi(x, y)$ is an arbitrary function of class C^2 and which together with $\varphi(x, y)$ assures the fact that the transformation is non-singular (that is $\xi(x, y)$ and $\varphi(x, y)$ satisfy a condition which is analogous to (1.1.19), then after some calculations similar to those from Proposition 1.1.2, we obtain the following canonical form:

$$\frac{\partial^2 u}{\partial \xi^2} = H\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

3^o **The elliptic case**: $\Delta = a_{12}^2 - a_{11}a_{22} < 0$. In this case, the characteristic Eq. (1.1.11) admits two prime integrals, which are complex conjugate and which can be written in the form

$$\varphi(x, y) = C_1,$$
$$\overline{\varphi}(x, y) = C_2,$$

in which C_1 and C_2 are arbitrary constants and are denoted by $\overline{\varphi}$ the complex conjugate of the function φ. If we proceed by analogy with the hyperbolic case, that is, we take the new variables ξ and η of the form

$$\xi = \varphi(x, y),$$
$$\eta = \overline{\varphi}(x, y),$$

with the condition

$$\left|\frac{\partial(\varphi, \overline{\varphi})}{\partial(x, y)}\right| = \frac{\partial \varphi}{\partial x}\frac{\partial \overline{\varphi}}{\partial y} - \frac{\partial \overline{\varphi}}{\partial x}\frac{\partial \varphi}{\partial y} \neq 0,$$

then we will obtain $\overline{a_{11}} = \overline{a_{22}} = 0$ and therefore the elliptic equation has the canonical form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = F\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right).$$

Unlike the hyperbolic case, the last equation is in the set of complex numbers. Then the natural question of finding other changes in variables arises which should lead us to a canonical form in the set of real numbers. To this aim, we introduce the functions $\alpha(x, y)$ and $\beta(x, y)$ so that

$$\alpha = \text{Re}(\varphi) = \frac{1}{2}(\varphi + \overline{\varphi}),$$
$$\beta = \text{Im}(\varphi) = \frac{1}{2i}(\varphi - \overline{\varphi}),$$

and we take the new variables ξ and η in the form

$$\xi = \alpha + i\beta,$$
$$\eta = \alpha - i\beta. \tag{1.1.21}$$

Proposition 1.1.3 *In the case of elliptic equations, we have*

$$\tilde{a}_{11} = \tilde{a}_{22},\ \tilde{a}_{12} = 0,$$

in which $\tilde{a}_{ij}$ are the coefficients of the canonical equation which were obtained after transformation (1.1.21).

Proof It is obvious that ξ is, in fact, $\xi = \varphi(x, y)$ and then $\overline{a_{11}} = 0$. If we take into account (1.1.21), then we have

$$\begin{aligned}
0 = \overline{a_{11}} &= a_{11}\left(\frac{\partial\xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + a_{22}\left(\frac{\partial\xi}{\partial y}\right)^2 \\
&= a_{11}\left(\frac{\partial\alpha}{\partial x} + i\frac{\partial\beta}{\partial x}\right)^2 + a_{22}\left(\frac{\partial\alpha}{\partial y} + i\frac{\partial\beta}{\partial y}\right)^2 \\
&\quad +2a_{12}\left(\frac{\partial\alpha}{\partial x} + i\frac{\partial\beta}{\partial x}\right)\left(\frac{\partial\alpha}{\partial y} + i\frac{\partial\beta}{\partial y}\right) \\
&= a_{11}\left(\frac{\partial\alpha}{\partial x}\right)^2 + 2a_{12}\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial y} + a_{22}\left(\frac{\partial\alpha}{\partial y}\right) \\
&\quad -\left[a_{11}\left(\frac{\partial\beta}{\partial x}\right)^2 + 2a_{12}\frac{\partial\beta}{\partial x}\frac{\partial\beta}{\partial y} + a_{22}\left(\frac{\partial\beta}{\partial y}\right)^2\right] \\
&\quad +2i\left[a_{11}\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} + a_{12}\left(\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial y} + \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial x}\right) + a_{22}\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y}\right].
\end{aligned}$$

This is an equality in the set of complex numbers, and therefore both the real part and also the imaginary part must be null, so that we obtain the desired result from the statement of Proposition 1.1.3. ■

Using the results of Proposition 1.1.3, we deduce that in the elliptic case the canonical form of the equation is

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = H\left(\alpha, \beta, u, \frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}\right),$$

in which H is a real function.

We conclude that in the domain of ellipticity of Eq. (1.1.3) there is no characteristic direction, while in the domain of hyperbolicity of Eq. (1.1.3) in every point, there are two real distinct characteristic directions, while in every point of the domain of parabolicity there is only one real characteristic direction.

Consequently, if the coefficients a_{11}, a_{12} and a_{22} of Eq. (1.1.3) are smooth enough, the domain of hyperbolicity is covered by a network which consists of two families of characteristic curves, and the domain of parabolicity is covered by only one such family.

For example, let us consider the equation

$$y^m \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where m is a natural odd number. In this case, Eq. (1.1.12) receives the form

$$y^m \left(\frac{dy}{dx}\right)^2 + 1 = 0.$$

It may be immediately noted that there is no characteristic direction in the half-plane $y > 0$. But in every point of the line $y = 0$ and in every point of the half-plane $y < 0$, there are a characteristic direction and two characteristic directions, respectively. We write the equation of the characteristic curves in the form

$$dx \pm (-y)^{\frac{m}{2}} \mathrm{d}y = 0,$$

hence, by integration, we infer that the half-plane $y < 0$ is covered by two families of real characteristic curves described by the equations

$$x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = C_1,$$

and

$$x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = C_2,$$

where C_1 and C_2 are real constants.

1.2 The Canonical Form for $n > 2$

In this paragraph, we present some considerations on the canonical form of a partial differential equation of second order for the case that the unknown function depends on $n > 2$ independent variables.

Let Ω be an open set in n-dimensional space $\mathbb{R}^n$ and consider the quasilinear partial differential equation of second order

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x) = \frac{\partial^2 u}{\partial x_i \partial x_j} = f\left(x_1, x_2, \ldots, x_n, , u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right), \tag{1.2.1}$$

in which $x = (x_1, x_2, \ldots, x_n),\ a_{ij} = a_{ji}(x) \in C(\Omega)$.

Here, u is the unknown function of the equation, $u : \Omega \to \mathbb{R},\ u \in C^2(\Omega)$. The function $f = f(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n)$ is defined and is continuous in any point $(x_1, x_2, \ldots, x_n) \in \Omega$ and we have $-\infty < z, p_1, p_2, \ldots, p_n < \infty$.

We intend to make a change of variables so that in the newly obtained equation (which will be called canonical equation) a part of the new coefficients, that we denote by $\overline{a_{ij}}$, as in the case $n = 2$, must be null.

Consider the transformation

$$\begin{aligned}
&\xi_1 = \xi_1(x_1, x_2, \ldots, x_n),\\
&\xi_2 = \xi_2(x_1, x_2, \ldots, x_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\xi_n = \xi_n(x_1, x_2, \ldots, x_n),
\end{aligned} \tag{1.2.2}$$

with the condition

$$\left|\frac{\partial \xi}{\partial x}\right| = \begin{vmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} & \cdots & \frac{\partial \xi_1}{\partial x_n} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} & \cdots & \frac{\partial \xi_2}{\partial x_n} \\ \frac{\partial \xi_n}{\partial x_1} & \frac{\partial \xi_n}{\partial x_2} & \cdots & \frac{\partial \xi_n}{\partial x_n} \end{vmatrix} \neq 0, \tag{1.2.3}$$

in which $\xi = \xi(x)$ is a vector function, $\xi : \Omega \to \mathbb{R}^n,\ \xi \in C^2(\Omega)$.

Due to condition (1.2.3), based on the theory of implicit functions, we deduce that the system (1.2.2) can be solved with respect to the vector variable x:

$$\begin{aligned}
&x_1 = x_1(\xi_1, \xi_2, \ldots, \xi_n),\\
&x_2 = x_2(\xi_1, \xi_2, \ldots, \xi_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&x_n = x_n(\xi_1, \xi_2, \ldots, \xi_n),
\end{aligned}$$

which will allow the final solution of Eq. (1.2.1) to be expressed as a function of x. From Eq. (1.2.2), we get

$$\frac{\partial u}{\partial x_i}=\sum_{k=1}^{n}\frac{\partial u}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i},\ i=1,2,\ldots,n$$

and then

$$\frac{\partial^2 u}{\partial x_i\partial x_j}=\sum_{k=1}^{n}\sum_{m=1}^{n}\frac{\partial^2 u}{\partial \xi_k\partial \xi_m}\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_m}{\partial x_j}+\sum_{k=1}^{n}\frac{\partial u}{\partial \xi_k}\frac{\partial^2 \xi_k}{\partial x_i\partial x_j}. \tag{1.2.4}$$

We introduce Eq. (1.2.4) into (1.2.1) so we obtain the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n}a_{ij}\frac{\partial^2 u}{\partial \xi_k\partial \xi_m}\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_m}{\partial x_j}=G\left(\xi,u,\frac{\partial u}{\partial \xi_1},\frac{\partial u}{\partial \xi_2},\ldots,\frac{\partial u}{\partial \xi_n}\right). \tag{1.2.5}$$

By using the notation

$$\overline{a_{km}}(\xi)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_m}{\partial x_j} \tag{1.2.6}$$

Equation (1.2.5) becomes

$$\sum_{k=1}^{n}\sum_{m=1}^{n}\overline{a_{km}}(\xi)\frac{\partial^2 u}{\partial \xi_k\partial \xi_m}=G\left(\xi,u,\frac{\partial u}{\partial \xi_1},\frac{\partial u}{\partial \xi_2},\ldots,\frac{\partial u}{\partial \xi_n}\right), \tag{1.2.7}$$

We will fix a point $x^0=(x_1^0,x_2^0,\ldots,x_n^0)\in\Omega$ and by using the notation

$$\lambda_{ik}=\frac{\partial \xi_k}{\partial x_i}(x^0)$$

from (1.2.6) we deduce

$$\overline{a_{km}}(\xi^0)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(x^0)\lambda_{ik}\lambda_{jm}, \tag{1.2.8}$$

where $\xi^0=\xi(x^0)$.

By using the matrix notation

$$\overline{A}=[\overline{a_{km}}],\ A=\left[a_{ij}\right],\ \Lambda=\left[\lambda_{ij}\right],$$

Eq. (1.2.8) becomes

$$\overline{A}=\Lambda^t A\Lambda, \tag{1.2.9}$$

where we denote by Λ^t the transpose of the matrix Λ.

It is known that if in (1.2.9) we make the change of variable

$$\Lambda = TM,$$

where M is the notation for a non-singular matrix and T is an orthogonal matrix (that is, $T^t = T^{-1}$), then matrix $\overline{A}$ is reduced to its diagonal form, therefore a matrix in which the elements that are not on the main diagonal are null. Regarding elements on the main diagonal, we have the Sylvester's law of inertia, which ensures that the number of positive elements on the diagonal is constant. Also, the number of negative elements on the diagonal is constant. We distinguish several possibilities:

1^o If all diagonal elements are strictly positive at a point $\xi^0 \in \Omega$, then the canonical equation becomes

$$\sum_{j=1}^{n} \overline{a_{jj}} \frac{\partial^2 u}{\partial \xi_j^2}(\xi^0) = G\left(\xi_1, \xi_2, \ldots, \xi_n, u, \frac{\partial u}{\partial \xi_1}, \frac{\partial u}{\partial \xi_2}, \ldots, \frac{\partial u}{\partial \xi_n}\right).$$

Then, we say that the quasilinear partial differential equation of second order is *elliptical* in the point $\xi^0 \in \Omega$.

2^o If there are elements on the main diagonal that are not null, but some elements are positive numbers and other negative, then we say that the equation is *hyperbolic*. In particular, if only one element is strictly positive and all others are strictly negative, we say that the equation is *ultra hyperbolic* in that point.

3^o If on the main diagonal there are some null elements, then the equation is called *parabolic* in that point.

4^o If the main diagonal has null elements and the non-null elements have a single sign (therefore, all negative or all positive), then the equation is called *elliptical–parabolic* in that point.

5^o If the main diagonal has null elements and the non-null elements have different signs, then the equation is called *hyperbolic–parabolic* in that point.

It is clear that the interest for the canonical form of quasilinear partial differential equations of second order is given by the fact that this form of the equation facilitates its integration.

Chapter 2
Differential Operators of Second Order

2.1 Green's Formula

Let Ω be a domain (therefore an open and convex set) in the space $\mathbb{R}^n$. The most general form of a differential linear operator of second order is

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \tag{2.1.1}$$

where $a_{ij} = a_{ji}(x) \in C^2(\Omega)$, $b_i = b_i(x) \in C^1(\Omega)$, $c = c(x) \in C^0(\Omega)$ are given functions, and $u = u(x) \in C^2(\Omega)$ is the unknown function.

As usual, we denote by x the vector with n components $x = (x_1, x_2, \ldots, x_n)$.

Definition 2.1.1 (i) Given the differential linear operator of second order, L, we call as its adjoint in Lagrange sense, the operator denoted by M and defined by

$$Mv = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 (a_{ij}(x)v)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (b_i(x)v)}{\partial x_i} + c(x)v. \tag{2.1.2}$$

(ii) The operator L is called self-adjoint if it coincides with its adjoint, that is,

$$Lu = Mu, \ \forall u \in C^2(\Omega).$$

Observation 2.1.1 *1°. The operator L leads a function $u(x) \in C^2(\Omega)$ into another function $(Lu)(x) \in C^0(\Omega)$.*

2°. It is easy to prove that the adjoint operator M can be written in the form

$$Mv = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} - \sum_{i=1}^{n}\left(b_i(x) - 2\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j}\right)\frac{\partial v}{\partial x_i}$$

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$$+\left[c(x)-\sum_{i=1}^{n}\frac{\partial b_i(x)}{\partial x_i}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j}\right]v. \tag{2.1.3}$$

3°. If we use the form (2.1.3) of the adjoint operator, it is easy to prove that the adjoint of the adjoint of an operator L, is even L.

Proposition 2.1.1 *A necessary and sufficient condition that the operator L is self-adjoint is*

$$b_i(x)=\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j},\ i=1,2,\ldots,n. \tag{2.1.4}$$

Proof The operator L is self-adjoint if and only if, according to the above definition, $Lu = Mu,\ \forall u \in C^2(\Omega)$.

If we write Mu, by using formula (2.1.3), in which we replace v with u, we see that the equality $Lu = Mu$ is possible if and only if the coefficients of the derivatives for the two members coincide. We find that these coefficients coincide if and only if we have the equalities

$$\begin{aligned} b_i(x)&=\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j},\ i=1,2,\ldots,n,\\ \sum_{i=1}^{n}\frac{\partial b_i(x)}{\partial x_i}&=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 a_{ij}(x)}{\partial x_i}\partial x_j. \end{aligned} \tag{2.1.5}$$

But the equality $(2.1.5)_2$ is obtained from Eq. $(2.1.5)_1$, by computing the derivative in both members with respect to x_i and then summing the resulting relations after all values of i. ■

In the following theorem, we prove a fundamental formula in the theory of partial differential equations, known as *Green's formula*.

Theorem 2.1.1 *Let Ω be a domain so that its border $\partial\Omega$ is a closed surface which admits a tangent plane, continuously varying, almost everywhere. If L is a differential operator, defined on Ω, and M is the adjoint operator of L, then the following identity holds true:*

$$\begin{aligned} &\int_{\Omega}[v(x)Lu(x)-u(x)Mv(x)]\mathrm{d}x\\ &=\int_{\partial\Omega}\left\{\gamma\left[v(x)\frac{\partial u(x)}{\partial\gamma}-u(x)\frac{\partial v(x)}{\partial\gamma}\right]+b(x)u(x)v(x)\right\}\mathrm{d}\sigma_x, \end{aligned} \tag{2.1.6}$$

where $\frac{\partial u}{\partial\gamma}$ is the derivative of the function u in the direction $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$.

Proof We multiply in (2.1.1), in both members, by v, and in (2.1.2) by u, and the obtained relations are subtracted

$$vLu - uMv = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[va_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - u\frac{\partial^2 (a_{ij}v)}{\partial x_i \partial x_j}\right] + \sum_{i=1}^{n}\left[vb_i\frac{\partial u}{\partial x_i} + u\frac{\partial (b_i v)}{\partial x_i}\right]$$
$$= \sum_{i=1}^{n}\left\{\frac{\partial}{\partial x_i}\left[\sum_{i=1}^{n}\left(va_{ij}\frac{\partial u}{\partial x_i} - u\frac{\partial (a_{ij}v)}{\partial x_j}\right)\right] + \frac{\partial}{\partial x_i}(b_i uv)\right\}$$
$$= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{\sum_{j=1}^{n}\left[va_{ij}\frac{\partial u}{\partial x_j} - u\frac{\partial (a_{ij}v)}{\partial x_j}\right] + b_i uv\right\}.$$

Integrating this equality, member to member, on Ω and we obtain

$$\int_{\Omega}[v(x)Lu(x) - u(x)Mv(x)]\mathrm{d}x = \int_{\Omega}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{\sum_{j=1}^{n}\left[va_{ij}\frac{\partial u}{\partial x_j} - u\frac{\partial (a_{ij}v)}{\partial x_j}\right] + b_i uv\right\}\mathrm{d}x. \tag{2.1.7}$$

Recall Gauss–Ostrogradsky's formula

$$\int_{\Omega}\sum_{i=1}^{n}\frac{\partial Q_i}{\partial x_i}\mathrm{d}x = \int_{\partial\Omega}\sum_{i=1}^{n}Q_i\cos\alpha_i \mathrm{d}\sigma_x,$$

in which $(\cos\alpha_1, \cos\alpha_2, \ldots, \cos\alpha_n)$ are the cosine directors of the outside unit normal to the surface $\partial\Omega$.

Regarding the formula (2.1.7), in the right-hand side we have Q_i given by

$$Q_i = \sum_{j=1}^{n}\left[va_{ij}\frac{\partial u}{\partial x_j} - u\frac{\partial (a_{ij}v)}{\partial x_j}\right] + b_i uv$$

so that if in (2.1.7) we apply Gauss–Ostrogradsky's formula, we get

$$\int_{\Omega}[v(x)Lu(x) - u(x)Mv(x)]\mathrm{d}x = \int_{\partial\Omega}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[va_{ij}\frac{\partial u}{\partial x_i} - ua_{ij}\frac{\partial v}{\partial x_j}\right]\cos\alpha_i\right.$$

$$+\sum_{i=1}^{n}\left(b_i-\sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}\right)\cos\alpha_i uv\right\}\mathrm{d}\sigma_x. \quad (2.1.8)$$

Now recall the definition of the derivative in a direction. If $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is an arbitrary direction (that is, a vector) from $\mathbb{R}^n$, then the derivative of a function u in the direction $\boldsymbol{\lambda}$ is defined by

$$\frac{\partial u}{\partial\lambda}=\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\lambda_j.$$

If we have $(\lambda_1,\lambda_2,\ldots,\lambda_n)=(\cos\alpha_1,\cos\alpha_2,\ldots,\cos\alpha_n)$, that is, the cosine directors of the outside normal to the surface $\partial\Omega$, then we have the derivative in the direction of the normal ν

$$\frac{\partial u}{\partial\nu}=\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\cos\alpha_j.$$

We introduce the notations

$$\gamma=\sqrt{\sum_{j=1}^{n}\left(\sum_{k=1}^{n}a_{kj}\cos\alpha_k\right)^2},$$

$$\gamma_j=\frac{1}{\gamma}\sum_{i=1}^{n}a_{ij}\cos\alpha_i. \quad (2.1.9)$$

It is easy to see that

$$\sum_{j=1}^{n}\gamma_j^2=1,$$

that is, the direction $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ defines a versor (that is, an unit vector).

In the following, we will use also the notation

$$b(x)=\sum_{i=1}^{n}\left[b_i(x)-\sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}(x)\right]\cos\alpha_i. \quad (2.1.10)$$

Taking into account Proposition 2.1.1, we deduce that for a self-adjoint operator we have $b\equiv 0$.

With the help of notations (2.1.9) and (2.1.10), formula (2.1.8) becomes

$$\int_{\Omega} [v(x)Lu(x) - u(x)Mv(x)]\mathrm{d}x$$
$$= \int_{\partial\Omega} \left\{ \gamma \left[v \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\gamma_j - u \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}\gamma_j \right] + buv \right\} \mathrm{d}\sigma_x,$$

so that if we take into account the definition of the derivative in the direction γ, this becomes

$$\int_{\Omega} [v(x)Lu(x) - u(x)Mv(x)]\mathrm{d}x = \int_{\partial\Omega} \left[\gamma \left(v\frac{\partial u}{\partial \gamma} - u\frac{\partial v}{\partial \gamma} \right) + buv \right] \mathrm{d}\sigma_x,$$

which is just Green's formula. ■

Corollary 2.1.1 *In the case of a self-adjoint operator, Green's formula becomes*

$$\int_{\Omega} [v(x)Lu(x) - u(x)Mv(x)]\mathrm{d}x = \int_{\partial\Omega} \gamma \left(v\frac{\partial u}{\partial \gamma} - u\frac{\partial v}{\partial \gamma} \right) \mathrm{d}\sigma_x.$$

Proof The result is obtained immediately, by taking into account that for a self-adjoint operator we have $b \equiv 0$, based on the above observation. ■

Consider now the particular case, when the operator L is the Laplacian Δ, defined by

$$Lu = \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

In this situation, in comparison to the general form (2.1.1) of the linear differential operator of second order, we have

$$a_{ij} = \delta_{ij},\ b_i = 0,\ c = 0,$$

in which δ_{ij} are Kronecker's symbols.

Taking into account formula (2.1.10), we obtain that $b(x) \equiv 0$, and therefore the Laplace operator is self-adjoint. Also, based on notation (2.1.9), we have

$$\gamma = \sqrt{\sum_{j=1}^{n} \left(\sum_{i=1}^{n} \delta_{ij} \cos \alpha_i \right)^2} = \sqrt{\sum_{j=1}^{n} \cos^2 \alpha_j} = 1,$$

and the components γ_j of the direction γ become

$$\gamma_j = \sum_{i=1}^{n} \delta_{ij} \cos \alpha_i = \cos \alpha_j,$$

that is, the direction γ coincides with the direction of the outside normal ν to the surface $\partial\Omega$.

In conclusion, if $L = \Delta \Rightarrow M = \Delta$ and Green's formula becomes

$$\int_{\Omega} (v\Delta u - u\Delta v)\mathrm{d}x = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) \mathrm{d}\sigma_x.$$

2.2 Levi Functions

Definition 2.2.1 The differential linear operator of second order, L, is called elliptic if it satisfies the condition

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\xi)\lambda_i\lambda_j \geq 0, \ \forall \xi \in \Omega, \ \forall (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n, \tag{2.2.1}$$

the equality taking place if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

In algebraic language, the operator L is elliptic if the matrix of its main coefficients, $\{a_{ij}\}_{i,j}$, is positive definite.

Denote by Γ_{ij} the algebraic complement of the element a_{ij} in the matrix of the coefficients $\{a_{ij}\}_{i,j}$. We also use the usual notations

$$|A(\xi)| = \left|\det\{a_{ij}\}_{i,j}\right|, \ A_{ij} = \frac{1}{|A(\xi)|}\Gamma_{ij}.$$

It is known that if the matrix $\{a_{ij}\}_{i,j}$ is positive definite, then the matrix $\{A_{ij}\}_{i,j}$ is positive definite.

Also, we recall that a matrix which is positive definite has a nonzero determinant.

Definition 2.2.2 We call a Levi function of second order, attached to the elliptic operator L, the function $H(x, \xi)$ given by

$$H(x,\xi) = \frac{1}{(n-2)\omega_n\sqrt{|A(\xi)|}} \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(\xi)(x_i - \xi_i)(x_j - \xi_j) \right\}^{(2-n)/2}, \tag{2.2.2}$$

for $n \geq 3$. In the case $n = 2$, $H(x, \xi)$ has the form

$$H(x,\xi) = \frac{1}{2\pi\sqrt{|A(\xi)|}} \ln \frac{1}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)}}, \tag{2.2.3}$$

in which $x = (x_1, x_2, \ldots, x_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \Omega$, and ω_n is the area of the unit sphere from the $n-$dimensional space $\mathbb{R}^n$.

We recall that

$$\omega_n = \frac{2(\sqrt{\pi})^n}{\Gamma(n/2)},$$

where Γ is the Euler's function of second order, defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t.$$

We must outline that the coefficients A_{ij} were built exclusively using the coefficients a_{ij}.

That part of the operator L which corresponds to the coefficients a_{ij} (and, therefore, implicitly, to the coefficients A_{ij}) is called *the main part* of the operator L.

The function H, as we shall see in the following, plays a very important role in the study of elliptic equations.

In the following theorem, we prove a first property of the function H.

Theorem 2.2.1 *The Levi function of second order H satisfies the equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\xi)\frac{\partial^2 H(x,\xi)}{\partial x_i \partial x_j} = 0.$$

Proof We will obtain the result by direct calculation. We consider, specifically the case $n \geq 3$. If we derive (2.2.2), we obtain

$$\frac{\partial H(x,\xi)}{\partial x_k} = \frac{-1}{\omega_n\sqrt{|A(\xi)|}} \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j) \right\}^{-n/2} \times$$
$$\times \left[\sum_{p=1}^{n} A_{kp}(\xi)(x_p-\xi_p) \right],$$

in which we used the symmetry of the coefficients A_{ij} (which is due the symmetry of the coefficients a_{ij}). We derive this equality with respect to x_k

$$\frac{\partial^2 H(x,\xi)}{\partial x_s \partial x_k} = -\frac{A_{sk}(\xi)}{\omega_n\sqrt{|A(\xi)|}}\frac{1}{\left(\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)}\right)^n}$$

$$+\frac{n}{\omega_n\sqrt{|A(\xi)|}}\frac{\sum\limits_{p=1}^{n}\sum\limits_{q=1}^{n} A_{sp}(\xi)A_{kq}(\xi)(x_p-\xi_p)(x_q-\xi_q)}{\left(\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)}\right)^{n+2}}.$$

We multiply this last equality, in both members, by $a_{sk}(\xi)$ and then we sum up with respect to s and k

$$\sum_{s=1}^{n}\sum_{k=1}^{n} a_{sk}(\xi)\frac{\partial^2 H(x,\xi)}{\partial x_s \partial x_k} = \frac{-\sum\limits_{s=1}^{n}\sum\limits_{k=1}^{n} a_{sk}(\xi)A_{sk}(\xi)}{\omega_n\sqrt{|A(\xi)|}\left(\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)}\right)^n}$$

$$+\frac{n}{\omega_n\sqrt{|A(\xi)|}}\frac{\sum\limits_{s=1}^{n}\sum\limits_{k=1}^{n}\sum\limits_{p=1}^{n}\sum\limits_{q=1}^{n} a_{sk}(\xi)A_{sp}(\xi)A_{kq}(\xi)(x_p-\xi_p)(x_q-\xi_q)}{\left(\sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)}\right)^{n+2}}.$$

If we take into account that

$$\sum_{s=1}^{n} a_{ls}A_{sm} = \delta_{lm},$$

then the last relation becomes

$$\sum_{s=1}^{n}\sum_{k=1}^{n} a_{sk}(\xi)\frac{\partial^2 H(x,\xi)}{\partial x_s \partial x_k} = 0.$$

This is the equation from the statement and so the proof of the theorem is completed. ■

Theorem 2.2.2 *The function $H(x,\xi)$ satisfies the following asymptotic evaluations, uniform on compact intervals from Ω:*

$$H(x,\xi) = O\left(r^{2-n}\right),$$
$$\frac{\partial H(x,\xi)}{\partial x_k} = O\left(r^{1-n}\right),\ \forall k = 1,2,\ldots,n,$$

$$\frac{\partial^2 H(x,\xi)}{\partial x_s \partial x_k} = O\left(r^{-n}\right), \ \forall k, s = 1, 2, \ldots, n,$$

where r is given by

$$r = \sqrt{\sum_{i=1}^{n} (x_i - \xi_i)^2}$$

that is, r is the distance between the points x and ξ, $x, \ \xi \in \mathbb{R}^n$.

Proof We recall that we say that a property takes place uniformly on compact subsets from an open set, if the respective property takes place, locally, on the respective open set.

Let O be the notation for the Landau symbol. If we take into account the signification of Landau's symbol, we can give a new formulation for the evaluations from the statement of the theorem.

For any compact set from Ω, there exist the positive constants M_1, M_2, and M_3, which depend only on that compact set, so that

$$\left| \frac{\partial H(x,\xi)}{r^{2-n}} \right| \le M_1, \tag{2.2.4}$$

$$\left| \frac{\partial H(x,\xi)}{\partial x_k} \frac{1}{r^{1-n}} \right| \le M_2, \tag{2.2.5}$$

$$\left| \frac{\partial^2 H(x,\xi)}{\partial x_s \partial x_k} \frac{1}{r^{-n}} \right| \le M_3. \tag{2.2.6}$$

Note that these evaluations are made for $x \to \xi$, and the constants M_1, M_2, and M_3 do not depend on x or ξ.

We made the change of variables

$$x_i - \xi_i = \lambda_i r,$$

so that we have

$$\sum_{i=1}^{n} \lambda_i^2 = \frac{1}{r^2} \sum_{i=1}^{n} (x_i - \xi_i)^2 = 1. \tag{2.2.7}$$

This proves that $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be the cosine directors of a unit direction to the surface $\partial\Omega$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. On the other hand, if we consider $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ as a point in the $n-$dimensional space $\mathbb{R}^n$, then $\boldsymbol{\lambda}$ belongs to the sphere with center at the origin and radius 1, $S(0, 1)$, in $\mathbb{R}^n$.

Anyway, from (2.2.7) we deduce that not all λ_i can be simultaneously null. Using the definition (2.2.2) of the function $H(x, \xi)$, we have

$$\left|\frac{H(x,\xi)}{r^{2-n}}\right| = \frac{1}{(n-2)\omega_n\sqrt{|A(\xi)|}}\frac{1}{\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\xi)\lambda_i\lambda_j}\right)^{n-2}}. \tag{2.2.8}$$

The denominator on the right-hand side of the relation (2.2.8) is positive because the operator L is elliptic. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\xi)\lambda_i\lambda_j \geq 0 \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\xi)\lambda_i\lambda_j \geq 0,$$

and the equality takes place if and only if all λ_i are null, and this, as we saw, is impossible.

If we denote

$$g(\xi,\lambda) = \sqrt{|A(\xi)|}\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\xi)\lambda_i\lambda_j}\right)^{n-2}, \tag{2.2.9}$$

then we deduce that g is a strictly positive and continuous function with respect to the variable ξ and also to the variable λ. If we fix an arbitrary compact set $Q \subset \Omega$, then g is continuous for any $(\xi,\lambda) \in Q \times S(0,1)$. Therefore, we can use the classical result from Weierstrass's theorem and deduce that g has an effective minimum value

$$\inf_{(\xi,\lambda)\in Q\times S(0,1)} g(\xi,\lambda) = \mu > 0,$$

and then the estimate (2.2.4) is proved with $M_1 = 1/\mu$.

To prove the estimate (2.2.5), let us first observe that we have

$$\left|\frac{\partial H(x,\xi)}{\partial x_k}\frac{1}{r^{1-n}}\right| \leq \frac{\sum_{s=1}^{n}|A_{sk}|\,|x_s-\xi_s|\,r^{n-1}}{\omega_n\sqrt{|A(\xi)|}\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)}\right)^{n}}$$

$$= \frac{\sum_{s=1}^{n}|A_{sk}|\,|\lambda_s|\,r^{1-n}}{r^{1-n}\omega_n\sqrt{|A(\xi)|}\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\xi)(x_i-\xi_i)(x_j-\xi_j)}\right)^{n}}.$$

We now introduce the notations

$$h_1(\xi, \lambda) = \sum_{s=1}^{n} |A_{sk}|\,|\lambda_s|\,,$$

$$h_2(\xi, \lambda) = \omega_n \sqrt{|A(\xi)|} \left(\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}(\xi) \lambda_i \lambda_j} \right)^{n},$$

and then, for a compact set Q arbitrarily fixed in Ω, we can make analogous considerations for the functions $h_1(\xi, \lambda)$ and $h_2(\xi, \lambda)$ to those made for function $g(\xi, \lambda)$ from (2.2.9).

So, we deduce that there exist the positive constants m_1 and m_2 depending only on Q so that $h_1(\xi, \lambda) \le m_1$ and $h_2(\xi, \lambda) \ge m_2$ and then the proof of the estimate (2.2.5) is complete if we take $M_2 = m_1/m_2$. In an analogous way, we can prove the estimate (2.2.6). ∎

We obtain a particular form for the Levi function $H(x, \xi)$ if we take the Laplace operator $L = \Delta$ instead of the arbitrary elliptic operator L. As we have already seen, in this case we have

$$a_{ij} = a_{ji} = \delta_{ij} \Rightarrow A_{ij} = \delta_{ij} \Rightarrow |A(\xi)| = 1$$

and then $H(x, \xi)$ becomes

$$H(x, \xi) = \frac{1}{(n-2)\omega_n} \frac{1}{\left(\sqrt{\sum_{i=1}^{n} (x_i = \xi_i)^2} \right)^{n-2}} - \frac{1}{(n-2)\omega_n} \frac{1}{r^{n-2}}.$$

With the help of the Levi function of second order $H(x, \xi)$, we can define the Levi function of first order, denoted by $\Lambda(x, \xi)$.

Definition 2.2.3 We call a Levi function of first order, attached to the elliptic operator L and the domain Ω, the function $\Lambda(x, \xi)$, $\Lambda : \Omega \times \Omega \to R$, with the properties

1° $\Lambda(x, .) \in C^2(\Omega)$, for any x fixed in Ω, $x \neq \xi$;
2° $\Lambda(., \xi) \in C^2(\Omega)$, for any ξ fixed in Ω, $\xi \neq x$;
3° $\exists \alpha \in (0, 1]$, so that

$$\Lambda(x, \xi) - H(x, \xi) = O\left(r^{\alpha+2-n}\right),$$
$$\frac{\partial[\Lambda(x, \xi) - H(x, \xi)]}{\partial x_k} = O\left(r^{\alpha+1-n}\right),$$
$$\frac{\partial^2[\Lambda(x, \xi) - H(x, \xi)]}{\partial x_k \partial x_s} = O\left(r^{\alpha-n}\right),$$

evaluations taking place for $x \to \xi$, uniformly on compact sets from Ω.

Observation 2.2.1 *1° In many specialized books, the authors call the Levi function only the Levi function of first order, that is,* $\Lambda(x,\xi)$ *from the Definition 2.2.3. We adopt this point of view, in the following.*

2° Using the significance of the symbol of Landau, O, we deduce that for any compact set Q, arbitrarily fixed in Ω*, there exist the positive constants* M_1, M_2*, and* M_3*, depending only on Q (not on x or* ξ*), so that*

$$\left|\frac{\Lambda(x,\xi)-H(x,\xi)}{r^{\alpha+2-n}}\right| \le M_1, \tag{2.2.10}$$

$$\left|\frac{\partial[\Lambda(x,\xi)-H(x,\xi)]}{\partial x_k}\frac{1}{r^{\alpha+1-n}}\right| \le M_2,\ \forall k=1,2,\ldots,n, \tag{2.2.11}$$

$$\left|\frac{\partial^2[\Lambda(x,\xi)-H(x,\xi)]}{\partial x_k\partial x_s}\frac{1}{r^{\alpha-n}}\right| \le M_3,\ \forall k,s=1,2,\ldots,n. \tag{2.2.12}$$

The Levi function is useful to define fundamental solutions for elliptic operators.

Definition 2.2.4 A fundamental solution of the elliptic operator L on the domain Ω, is a Levi function $\Lambda(x,\xi)$ which makes zero the adjoint operator of the operator L, that is,

$$M_x\Lambda(x,\xi)=0.$$

We deduce that to find the fundamental solution of an elliptic operator it is required to prove that there exists a Levi function for the respective operator.

Theorem 2.2.3 *For any elliptic operator L, assuming that the standard conditions on the coefficients* a_{ij}, b_i*, and c that define it are satisfied, there is at least a Levi function (of first order).*

Proof In fact, we will prove that the function $H(\xi,x)$, obtained from the Levi function of second order $H(x,\xi)$, by changing the arguments (we recall that a Levi function of second order is not symmetrical in its arguments), is a Levi function (of first order), with the exponent $\alpha=1$. Thus, we take

$$H(\xi,x)=\frac{1}{(n-2)\omega_n\sqrt{|A(x)|}}\frac{1}{\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(x)(\xi_i-x_i)(\xi_j-x_j)}\right)^{n-2}}.$$

Obviously, if we take $\Lambda(x,\xi)=H(\xi,x)$, then $\Lambda(x,\xi)$ satisfies the properties 1° and 2° from Definition 2.2.3 because the operator L is elliptic and then, by definition, $a_{ij}\in C^2$ and the matrix $\{a_{ij}\}$ is positive definite. This involves the fact that the matrix $\{A_{ij}\}$ is also positive definite and then $H(\xi,x)$ is of class C^2 with respect to both variables ξ and x, for $x\ne\xi$. Let us prove now the estimate (2.2.10) in the particular case $\alpha=1$. We have

$$\left|\frac{H(\xi,x)-H(x,\xi)}{r^{3-n}}\right| =$$

$$= \frac{1}{r^{3-n}(n-2)\omega_n}\left|\frac{1}{\sqrt{|A(x)|}\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(x)(\xi_i-x_i)(\xi_j-x_j)}\right)^{n-2}} - \right.$$

$$\left. - \frac{1}{\sqrt{|A(\xi)|}\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\xi)(\xi_i-x_i)(\xi_j-x_j)}\right)^{n-2}}\right|,$$

so that if we make the change of variables $x_i - \xi_i = \lambda_i r$, we get

$$\left|\frac{H(\xi,x)-H(x,\xi)}{r^{3-n}}\right| = \frac{1}{r(n-2)\omega_n}\left|h(\xi,x)-h(x,\xi)\right|,$$

for which we used the notations

$$h(\xi,x) = \frac{1}{\sqrt{|A(x)|}\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(x)\lambda_i\lambda_j}\right)^{n-2}},$$

$$h(x,\xi) = \frac{1}{\sqrt{|A(\xi)|}\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}(\xi)\lambda_i\lambda_j}\right)^{n-2}}.$$

Because for $x \neq \xi$, the functions A and A_{ij}, which appear in $h(\xi,x)$ and $h(x,\xi)$, are of class C^1, we can apply the theorem of finite increases (Theorem of Lagrange) for the functions $h(\xi,x)$ and $h(x,\xi)$. Thus, we deduce the existence of the points $\eta_1^*, \eta_2^* \in (\xi,x)$ as well as of the points $\eta_3^*, \eta_4^* \in (x,\xi)$ so that

$$\left|\frac{H(\xi,x)-H(x,\xi)}{r^{3-n}}\right| =$$

$$= \frac{1}{r(n-2)\omega_n}\sum_{k=1}^{n}\left\{\left|\frac{\partial h(\eta_1^*,\eta_2^*)}{\partial x_k}\right||x_k-\xi_k| + \left|\frac{\partial h(\eta_3^*,\eta_4^*)}{\partial x_k}\right||\xi_k-x_k|\right\}.$$

If we use r as majorant for $|x_k-\xi_k|$ and $|\xi_k-x_k|$, we obtain

$$\left| \frac{H(\xi, x) - H(x, \xi)}{r^{3-n}} \right| \leq$$

$$\leq \frac{r}{r(n-2)\omega_n} \sup_{Q \subset \Omega} \left\{ \sum_{k=1}^{n} \left| \frac{\partial h(\eta_1^*, \eta_2^*)}{\partial x_k} \right| + \sum_{k=1}^{n} \left| \frac{\partial h(\eta_3^*, \eta_4^*)}{\partial x_k} \right| \right\} \leq$$

$$\leq \frac{c_0}{(n-2)\omega_n},$$

that is, the proof of the estimate (2.2.10) is complete. The other two estimates, (2.2.11) and (2.2.12), are proven analogously. ■

We want to outline again that the estimates (2.2.10), (2.2.11), and (2.2.12) are made for values of x very close to ξ. For a point ξ arbitrarily fixed in Ω, we can take a ball centered in ξ as a compact set which contains ξ. Then the evaluations, deduced above, are made on this ball. Outside it the estimates (2.2.10), (2.2.11), and (2.2.12) are trivial.

We have seen that in the case that as elliptic operator L we take the Laplace operator, Δ, the Levi function of second order becomes

$$H(x, \xi) = \frac{1}{(n-2)\omega_n} \frac{1}{r^{n-2}}, \quad r = \sqrt{\sum_{i=1}^{n} (x_i - \xi_i)^2}.$$

Then, according to Theorem 2.2.3, as a Levi function (of first order) we can take

$$\Lambda(x, \xi) = H(\xi, x) = \frac{1}{(n-2)\omega_n} \frac{1}{r^{n-2}}.$$

Based on Theorem 2.2.1, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 H(x, \xi)}{\partial x_i \partial x_j} = 0.$$

Because in the case $L = \Delta$ we have $a_{ij} = \delta_{ij}$, we deduce that the above equation becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \frac{\partial^2 H(x, \xi)}{\partial x_i \partial x_j} = \sum_{i=1}^{n} \frac{\partial^2 H(x, \xi)}{\partial x_i^2} = 0,$$

that is,

$$\Delta_x H(x, \xi) = 0.$$

This means that the function $H(x, \xi)$ is a Levi function and is also a fundamental solution for the Laplace operator.

Theorem 2.2.4 *Let L be an elliptic operator which satisfies standard assumptions (on its coefficients). Then, if $\Lambda(x, \xi)$ is a Levi function of exponent α, attached to the operator L and the domain Ω, we have the estimates*

$$L_x \Lambda(x, \xi) = O(r^{\alpha-n}), \tag{2.2.13}$$

$$M_x \Lambda(x, \xi) = O(r^{\alpha-n}), \tag{2.2.14}$$

for $x \to \xi$, uniformly on compact sets from Ω.

Proof If we take into account that the elliptic operator L is linear, we have

$$L_x \Lambda(x, \xi) = L_x [\Lambda(x, \xi) - H(x, \xi)] + L_x H(x, \xi).$$

In what follows, we use the effective expression of the operator L

$$\begin{aligned} L_x \Lambda(x, \xi) &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 [\Lambda(x, \xi) - H(x, \xi)]}{\partial x_i \partial x_j} \\ &+ \sum_{i=1}^{n} b_i(x) \frac{\partial [\Lambda(x, \xi) - H(x, \xi)]}{\partial x_i} + c(x) [\Lambda(x, \xi) - H(x, \xi)] \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 H(x, \xi)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial H(x, \xi)}{\partial x_i} + c(x) H(x, \xi). \end{aligned}$$

For the first three terms on the right-hand side of this relation, we use the estimates from Definition 2.2.3, and for the last three terms on the right-hand side of the last relation, we use the evaluations from Theorem 2.2.2. Thus, we obtain

$$\begin{aligned} L_x \Lambda(x, \xi) &= O(r^{\alpha-n}) + O(r^{\alpha+1-n}) + O(r^{\alpha+2-n}) \\ &+ O(r^{1-n}) + O(r^{2-n}) + r O(r^{-n}) = O(r^{\alpha-n}) + O(r^{1-n}) = O(r^{\alpha-n}), \end{aligned}$$

because $\alpha \leq 1$. Therefore, we proved the estimate (2.2.13). The estimate (2.2.14) can be proven analogously. ■

Recall, in the conclusion of this paragraph, a few considerations on the convergence of improper integrals of volume.

Let Ω be a domain and the functions $f, g : \Omega \to R$, assumed continuous, so that $g(x, \xi) = 0$ for $x = \xi$. Consider the improper integral

$$\int_{\Omega} \frac{f(x, \xi)}{g(x, \xi)} \mathrm{d}x. \tag{2.2.15}$$

If we fix ξ in Ω and eliminate a subset K from Ω, to prevent x from taking the value ξ, we obtain

$$\int_{\Omega\setminus K} \frac{f(x,\xi)}{g(x,\xi)}\mathrm{d}x,$$

which is a volume integral, in the proper sense.

Consider now an ascending string $\{K_n\}$ of subsets of Ω so that

$$K_1 \supset K_2 \supset \ldots \supset K_{n-1} \supset K_n \supset K_{n+1} \supset \ldots$$

so that

$$\bigcap_{n=1}^{\infty} K_n = \xi.$$

Then for each $n = 1, 2, \ldots,$ the integrals

$$\int_{\Omega\setminus K_n} \frac{f(x,\xi)}{g(x,\xi)}\mathrm{d}x,$$

are proper integrals of volume. If the following limit exists and is finite

$$\lim_{n\to\infty} \int_{\Omega\setminus K_n} \frac{f(x,\xi)}{g(x,\xi)}\mathrm{d}x, \tag{2.2.16}$$

then, by definition,

$$\int_{\Omega} \frac{f(x,\xi)}{g(x,\xi)}\mathrm{d}x = \lim_{n\to\infty} \int_{\Omega\setminus K_n} \frac{f(x,\xi)}{g(x,\xi)}\mathrm{d}x.$$

We say that the integral (2.2.15) is conditioning convergent with the help of the string $\{K_n\}$. If the limit (2.2.16) exists and is finite for any string of sets $\{K_n\}$, we say that the improper integral (2.2.15) is unconditioned convergent (or, shorter, convergent).

It is easy to show that if there is a string of balls $\{B_n\}$, $B_n = B(\xi, \varrho_n)$, with $\varrho_n \to 0$ so that the following limit exists and is finite

$$\lim_{n\to\infty} \int_{\Omega\setminus B_n} \frac{f(x,\xi)}{g(x,\xi)}\mathrm{d}x,$$

then the value of this limit is the same as the value of the limit (2.2.16), for any choice of the string $\{K_n\}$ of subsets from Ω, with the above properties.

In other words, the convergence with the help of the string of balls guarantees the (conditioning) convergence of the improper integral (2.2.15).

2.3 Potentials

Let Ω be a bounded domain from $\mathbb{R}^n$ with boundary $\partial\Omega$ which admits a tangent plane continuously varying almost everywhere. On Ω we define the elliptic operator L and its adjoint M, by

$$Lu(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x), \quad (2.3.1)$$

$$Mv(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 (a_{ij}v)}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{n} \frac{\partial (b_i v)}{\partial x_i}(x) + c(x)v(x). \quad (2.3.2)$$

In the whole paragraph, we will use the following standard hypotheses:

$$a_{ij} = a_{ji}(x) \in C^2(\Omega),\ b_i = b_i(x) \in C^1(\Omega),\ c = c(x) \in C^0(\Omega).$$

We defined above the Levi function $\Lambda(x, \xi)$ which for $x \neq \xi$ is of class C^2. We will isolate the point ξ with an ellipsoid (e) centered in the point ξ and the radius ϱ, of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(\xi)\,(x_i - \xi_i)\left(x_j - \xi_j\right) \leq \varrho^2, \quad (2.3.3)$$

which has the boundary of the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(\xi)\,(x_i - \xi_i)\left(x_j - \xi_j\right) = \varrho^2. \quad (2.3.4)$$

The coefficients A_{ij} are determined with the help of the coefficients a_{ij} of the operator L, as in the previous paragraph. Since the operator L is assumed to be elliptic, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\lambda_i\lambda_j \geq 0,\ \forall x \in \Omega,\ \forall(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n,$$

where the equality appears if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. This means that the matrix of the coefficients a_{ij} is positive definite and, consequently, the matrix of the coefficients A_{ij} is also positive definite. This justifies the fact that in (2.3.3) we have effectively an ellipsoid centered in the point ξ and the radius ϱ. We choose the radius ϱ small enough so that the ellipsoid (e) is included fully in Ω. Since $\xi \notin \Omega \setminus (e)$, on the annulus $\Omega \setminus (e)$, which remains by removing the ellipsoid, we can apply Green's formula for a pair of functions $\Lambda(x, \xi)$ and $u(x)$

$$\int_{\Omega\setminus(e)} [\Lambda(x, \xi)Lu(x) - u(x)M(\Lambda(x, \xi))]\, \mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\partial\Omega} \left\{ \gamma \left[\Lambda(x,\xi) \frac{\partial u}{\partial \gamma}(x) L u(x) - u(x) \frac{\partial \Lambda}{\partial \gamma}(x,\xi) \right] + b\Lambda(x,\xi) u(x) \right\} \mathrm{d}\sigma_x \quad (2.3.5) \\
&+ \int_{\partial e} \left\{ \gamma \left[\Lambda(x,\xi) \frac{\partial u}{\partial \gamma}(x) L u(x) - u(x) \frac{\partial \Lambda}{\partial \gamma}(x,\xi) \right] + b\Lambda(x,\xi) u(x) \right\} \mathrm{d}\sigma_x .
\end{aligned}
$$

The left-hand member of the relation (2.3.5) depends on the choice of the ellipsoid (e). The statement is true for the last integral from the right-hand member. We will show that if the ellipsoid (e) tends to the point ξ, that is, (e) is deformed homothetic to ξ (a deformation is called homothetic if during deformation the ratio of axes remains constant), then the integrals which depend on the choice of (e) are convergent. In the same time, the surface integral from the right-hand member of the relation (3.5), extended to the entire surface $\partial\Omega$, remains constant. So,

$$
\begin{aligned}
\left| \int_e [-u(x) M\Lambda(x,\xi)]\, \mathrm{d}x \right| &\le \int_e |u(x)|\, |M\Lambda(x,\xi)|\, \mathrm{d}x \\
&\le c_0 \int_e |M\Lambda(x,\xi)|\, \mathrm{d}x,
\end{aligned}
$$

and this is because u is a continuous function and therefore it effectively takes its maximum value.

We now use the fact that

$$
|M\Lambda(x,\xi)| = O(r^{\alpha-n}),\ 0 < \alpha \le 1
$$

so that the above inequality becomes

$$
\begin{aligned}
&\left| \int_e [-u(x) M\Lambda(x,\xi)]\, \mathrm{d}x \right| \\
&\le c_0 c_1 \int_e r^{\alpha-n} \mathrm{d}x \le c_0 c_1 \int_{B(\xi,r_1)} r^{\alpha-n} \mathrm{d}x, \qquad (2.3.6)
\end{aligned}
$$

where the ball $B(\xi, r_1)$ includes the ellipsoid (e) and r_1 is the maximum radius for which the inclusion of (e) holds true so that the ball $B(\xi, r_1)$ is entirely in Ω. If we pass to generalized polar coordinates, we obtain

$$
\int_{B(\xi,r_1)} r^{\alpha-n} \mathrm{d}x = \omega_n \int_0^{r_1} r^{\alpha-n} r^{n-1} \mathrm{d}x = \omega_n \frac{r^\alpha}{\alpha},
$$

in which ω_n is the area of the unit sphere in the space $\mathbb{R}^n$. Based on these evaluations, the equality (2.3.6) becomes

$$
\left| \int_e [-u(x) M\Lambda(x,\xi)]\, \mathrm{d}x \right| \le c_0 c_1 \omega_n \frac{r^\alpha}{\alpha},
$$

which proves that the integral from the left-hand member becomes null if (e) is deformed homothetic to the point ξ.

On the other hand, the last integral from (2.3.5) can be written in the form

$$\int_{\partial e}\left\{\gamma\left[\Lambda(x,\xi)\frac{\partial u}{\partial\gamma}u(x)-u(x)\frac{\partial\Lambda(x,\xi)}{\partial\gamma}\right]+b\Lambda(x,\xi)u(x)\right\}\mathrm{d}\sigma_x$$
$$=\int_{\partial e}\Lambda(x,\xi)\left[\gamma\frac{\partial u}{\partial\gamma}u(x)+bu(x)\right]\mathrm{d}\sigma_x-\int_{\partial e}\gamma u(x)\frac{\partial\Lambda(x,\xi)}{\partial\gamma}\mathrm{d}\sigma_x. \quad (2.3.7)$$

Here, we can bound the following quantities from above:

$$\left|\int_{\partial e}\Lambda(x,\xi)\left[\gamma\frac{\partial u}{\partial\gamma}u(x)+bu(x)\right]\mathrm{d}\sigma_x\right]$$
$$\leq\int_{\partial e}|\Lambda(x,\xi)|\left|\gamma\frac{\partial u}{\partial\gamma}u(x)+bu(x)\right|\mathrm{d}\sigma_x \quad (2.3.8)$$
$$\leq c_2\int_{\partial e}|\Lambda(x,\xi)|\,\mathrm{d}\sigma_x,$$

in which we used the continuity on compact sets of the function u.

If we write

$$|\Lambda(x,\xi)|=|\Lambda(x,\xi)-H(x,\xi)+H(x,\xi)|$$
$$\leq|\Lambda(x,\xi)-H(x,\xi)|+|H(x,\xi)|=O(r^{\alpha+2-n})+O(r^{2-n}),$$

in which the last evaluations are based on Definition 2.2.3 and, respectively, on Theorem 2.2.2. With these estimates, the right-hand member from (2.3.8) can be bounded from above by

$$c_2\int_{\partial e}|\Lambda(x,\xi)-H(x,\xi)|\,\mathrm{d}\sigma_x+c_2\int_{\partial e}|H(x,\xi)|\,\mathrm{d}\sigma_x$$
$$\leq c_2M_1\int_{\partial e}\frac{1}{r^{n-2-\alpha}}\mathrm{d}\sigma_x+c_2M_2\int_{\partial e}\frac{1}{r^{n-2}}\mathrm{d}\sigma_x$$
$$\leq\frac{c_3}{r_{\min}^{n-2-\alpha}}\int_{\partial e}\mathrm{d}\sigma_x+\frac{c_4}{r_{\min}^{n-2}}\int_{\partial e}\mathrm{d}\sigma_x,$$

in which c_3 and c_4 are constants obtained by coupling the constant c_2 with the constants M_1 and M_2, respectively.

Also, in the last bounds, the integral

$$\int_{\partial e}\mathrm{d}\sigma_x$$

has the value equal to area of the ellipsoid which can be bounded from above by the area of the sphere of maximum radius which includes the ellipsoid (e) and then the above bounds may be continued as

$$c_3\omega_n\left(\frac{r_{\max}}{r_{\min}}\right)^{n-2-\alpha} r_{\max}^{\alpha+1} + c_4\omega_n\left(\frac{r_{\max}}{r_{\min}}\right)^{n-2} r_{\max},$$

and this expression tends to zero as the ellipsoid is deformed homothetic to the point ξ (because the deformation takes place homothetic, the ratio $r_{\max}/r_{\min}$ is constant). Therefore, the integral from the left-hand member of relation (2.3.8) (and, consequently, the first integral from the right-hand member of the relation (2.3.7)) tends to zero.

We consider now the last integral from (2.3.7)

$$\begin{aligned}
&-\int_{\partial e} u(x)\gamma u(x)\frac{\partial\Lambda(x,\xi)}{\partial\gamma}\mathrm{d}\sigma_x \\
&\qquad -\int_{\partial e} u(x)\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\cos\alpha_i\frac{\partial\Lambda(x,\xi)}{\partial x_j}\mathrm{d}\sigma_x \\
&-\int_{\partial e}[u(x)-u(\xi)]\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\cos\alpha_i\frac{\partial\Lambda(x,\xi)}{\partial x_j}\mathrm{d}\sigma_x \\
&-u(\xi)\int_{\partial e}\sum_{i=1}^{n}\sum_{j=1}^{n}[a_{ij}(x)-a_{ij}(\xi)\cos\alpha_i\frac{\partial\Lambda(x,\xi)}{\partial x_j}\mathrm{d}\sigma_x \\
&\qquad -u(\xi)\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\xi)\int_{\partial e}\cos\alpha_i\frac{\partial\Lambda(x,\xi)}{\partial x_j}\mathrm{d}\sigma_x \\
&\qquad = I_1+I_2+I_3,
\end{aligned}$$

in which

$$\begin{aligned}
I_1 &= -\int_{\partial e}[u(x)-u(\xi)]\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\cos\alpha_i\frac{\partial\Lambda(x,\xi)}{\partial x_j}\mathrm{d}\sigma_x, \\
I_2 &= -u(\xi)\int_{\partial e}\sum_{i=1}^{n}\sum_{j=1}^{n}[a_{ij}(x)-a_{ij}(\xi)\cos\alpha_i\frac{\partial\Lambda(x,\xi)}{\partial x_j}\mathrm{d}\sigma_x, \\
I_3 &= -u(\xi)\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\xi)\int_{\partial e}\cos\alpha_i\frac{\partial\Lambda(x,\xi)}{\partial x_j}\mathrm{d}\sigma_x.
\end{aligned} \tag{2.3.9}$$

We will prove that if (e) is deformed homothetic to the point ξ, then

$$I_1 \to 0,\ I_2 \to 0,\ I_3 \to -u(\xi).$$

First, due to the fact that the functions a_{ij} are continuous on compact set (∂e), and $|\cos\alpha_i| \leq 1$, we have

$$|I_1| \leq c_1 \int_{\partial e} |u(x) - u(\xi)| \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\frac{\partial\Lambda(x,\xi)}{\partial x_j}\right| \mathrm{d}\sigma_x$$
$$= nc_1 \int_{\partial e} |u(x) - u(\xi)| \sum_{j=1}^{n} \left|\frac{\partial\Lambda(x,\xi)}{\partial x_j}\right| \mathrm{d}\sigma_x.$$

In what follows, in the above inequality we apply the theorem of finite increases and then we use the bound $|x_i - \xi_i| \leq r$

$$|I_1| \leq nc_1 \int_{\partial e} \sum_{k=1}^{n} \left|\frac{\partial u(\xi^*)}{\partial x_k}\right| |x_k - \xi_k| \sum_{j=1}^{n} \left|\frac{\partial\Lambda(x,\xi)}{\partial x_j}\right| \mathrm{d}\sigma_x$$
$$\leq c_2 \int_{\partial e} r \sum_{j=1}^{n} \left|\frac{\partial\Lambda(x,\xi)}{\partial x_j}\right| \mathrm{d}\sigma_x \leq c_2 \int_{\partial e} r \sum_{j=1}^{n} \left|\frac{\partial[\Lambda(x,\xi) - H(x,\xi)]}{\partial x_j}\right| \mathrm{d}\sigma_x$$
$$+ c_2 \int_{\partial e} r \sum_{j=1}^{n} \left|\frac{\partial H(x,\xi)}{\partial x_j}\right| \mathrm{d}\sigma_x,$$

where $\xi^* \in (x, \xi)$, and $c_2 = nc_1$.

We will use now the evaluations from Definition 2.2.3 and Theorem 2.2.2

$$|I_1| \leq nc_2 \int_{\partial e} r_{\alpha+2-n} \mathrm{d}\sigma_x + nc_2 \int_{\partial e} r_{2-n} \mathrm{d}\sigma_x$$
$$\leq \frac{c_3}{r_{\min}^{n-2-\alpha}} \int_{\partial e} \mathrm{d}\sigma_x + \frac{c_4}{r_{\min}^{n-2}} \int_{\partial e} \mathrm{d}\sigma_x,$$

in which

$$\int_{\partial e} \mathrm{d}\sigma_x$$

is the area of the ellipsoid which can be bounded from above by the area of the sphere of maximum radius which can include the ellipsoid and be entirely in Ω

$$|I_1| \leq c_3 \frac{\omega_n r_{\max}^{n-1}}{r_{\min}^{n-2-\alpha}} + c_3 \frac{\omega_n r_{\max}^{n-1}}{r_{\min}^{n-2}}$$
$$= c_3\omega_n \left(\frac{r_{\max}}{r_{\min}}\right)^{n-2-\alpha} r_{\max}^{1+\alpha} + c_3\omega_n \left(\frac{r_{\max}}{r_{\min}}\right)^{n-2} r_{\max}$$

now it is clear that if (e) is deformed homothetic to the point ξ (during the transformation the ration $r_{\max}/r_{\min}$ is constant), then $I_1 \to 0$.

With regard to the integral I_2, the bounds are obtained by analogy

$$\begin{aligned}|I_2| &\le \int_{\partial e} \sum_{i=1}^{n} \sum_{j=1}^{n} \left|a_{ij}(x) - a_{ji}(\xi)\right| \left|\frac{\partial \Lambda(x,\xi)}{\partial x_j}\right| \mathrm{d}\sigma_x \\ &\le \int_{\partial e} \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left|\frac{\partial a_{ij}(\xi^*)}{\partial x_k}\right| |x_k - \xi_k| \left|\frac{\partial \Lambda(x,\xi)}{\partial x_j}\right| \mathrm{d}\sigma_x,\end{aligned}$$

in which we used the theorem of finite increases, with $\xi^* \in (x, \xi)$.

We use then the bound $|x_k - \xi_k| \le r$ and use the evaluations from Definition 2.2.3 and Theorem 2.2.2

$$|I_2| \le c_3 \omega_n \left[\left(\frac{r_{\max}}{r_{\min}}\right)^{n-2-\alpha} r_{\max}^{1+\alpha} + \left(\frac{r_{\max}}{r_{\min}}\right)^{n-2} r_{\max}\right],$$

and, therefore, $I_2 \to 0$, when (e) is deformed homothetic to the point ξ.

For the last integral from (2.3.9), we have

$$\begin{aligned}I_3 = &-u(\xi) \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\xi) \int_{\partial e} \cos \alpha_i \frac{\partial[\Lambda(x,\xi) - H(x,\xi)]}{\partial x_j} \mathrm{d}\sigma_x \\ &-u(\xi) \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\xi) \int_{\partial e} \cos \alpha_i \frac{\partial H(x,\xi)}{\partial x_j} \mathrm{d}\sigma_x = I_4 + I_5,\end{aligned}$$

in which the meaning of I_4 and I_5 is clear. Then

$$\begin{aligned}|I_4| &\le c_1 \int_{\partial e} \left|\frac{\partial[\Lambda(x,\xi) - H(x,\xi)]}{\partial x_j}\right| \mathrm{d}\sigma_x \le c_2 \int_{\partial e} \frac{1}{r^{n-\alpha-1}} \mathrm{d}\sigma_x \\ &\le \frac{c_2}{r_{\min}^{n-\alpha-1}} \int_{\partial e} \mathrm{d}\sigma_x \le c_2 \frac{r_{\max}^{n-1}}{r_{\min}^{n-\alpha-1}} = c_2 \left(\frac{r_{\max}}{r_{\min}}\right)^{n-\alpha-1} r_{\max},\end{aligned}$$

and, therefore, $I_4 \to 0$, when (e) is deformed homothetic to the point ξ.

The integral I_5 can be written in the form

$$I_5 = \frac{u(\xi)}{\omega_n \sqrt{|A(\xi)|}} \int_{\partial e} \frac{\sum_{i=1}^{n} \cos \alpha_i \left(x_i - \xi_j\right)}{\varrho^n} \mathrm{d}\sigma_x.$$

We can here apply Gauss–Ostrogradsky's formula, with change of the sign because the outside normal to the ellipsoid (e) is inside with regard to the boundary of the domain Ω

$$I_5 = -\frac{u(\xi)}{\varrho^n \omega_n \sqrt{|A(\xi)|}} \int_{\partial(e)} \sum_{i=1}^{n} \frac{\partial \left(x_i - \xi_j\right)}{\partial x_i} \mathrm{d}x$$
$$= -\frac{n u(\xi)}{\varrho^n \omega_n \sqrt{|A(\xi)|}} \int_{\partial(e)} \mathrm{d}x = -u(\xi),$$

where we used the fact that the volume of the ellipsoid (e) is

$$vol(e) = \int_{\partial(e)} \mathrm{d}x = \varrho^n \frac{\omega_n}{n} \sqrt{|A(\xi)|}.$$

Thus, $I_5 \to -u(\xi)$, when (e) is deformed homothetic to the point ξ.

In conclusion, if in (2.3.7) we pass to the limit with $(e) \to \xi$, homothetic, and we use the notation

$$\int_{\Omega} [\Lambda(x,\xi) L u(x) - u(x) M \Lambda(x,\xi)] \mathrm{d}x$$
$$= \lim_{(e)\to\xi} \int_{\Omega\setminus(e)} [\Lambda(x,\xi) L u(x) - u(x) M \Lambda(x,\xi)] \mathrm{d}x,$$

then we proved the following theorem.

Theorem 2.3.1 *If L is an elliptic operator, M is the adjoint operator of L, and $\Lambda(x,\xi)$ is a Levi function attached to the operator L and the domain Ω, then the following formula holds true:*

$$u(\xi) = -\int_{\Omega} [\Lambda(x,\xi) L u(x) - u(x) M \Lambda(x,\xi)] \mathrm{d}x$$
$$+ \int_{\partial\Omega} \left\{ \gamma \left[\Lambda(x,\xi) \frac{\partial u}{\partial \gamma}(x) - u(x) \frac{\partial \Lambda}{\partial \gamma}(x,\xi) \right] + b \Lambda(x,\xi) u(x) \right\} \mathrm{d}\sigma_x, \qquad (2.3.10)$$

called Riemann–Green's formula.

Observation 2.3.1 *1°. If the Levi function $\Lambda(x,\xi)$ is a fundamental solution of the elliptic operator L, then we can obtain a simplification of Riemann–Green's formula.*

2°. Let $\Lambda(x,\xi)$ be a Levi function which in addition is a fundamental solution of the elliptic operator L. Consider the problem

$$L u(x) = f(x), \ \forall x \in \Omega,$$
$$u(y) = \varphi(y), \ \forall y \in \partial\Omega,$$
$$\frac{\partial u}{\partial \gamma}(x) = \psi(x), \ \forall x \in \partial\Omega,$$

in which the prescribed functions f, φ, and ψ satisfy suitable conditions of regularity. Then Riemann–Green's formula (2.3.10) becomes

$$u(\xi) = -\int_\Omega \Lambda(x,\xi) f(x)\mathrm{d}x + \int_{\partial\Omega} \Lambda(x,\xi)[\gamma\psi(x) + b(x)\varphi(x)]\mathrm{d}\sigma_x$$
$$- \int_{\partial\Omega} \gamma\varphi(x)\frac{\partial\Lambda}{\partial\gamma}(x,\xi)\mathrm{d}\sigma_x. \tag{2.3.11}$$

Formula (2.3.11) is also called the formula of three potentials because the integrals

$$\begin{aligned} &-\int_\Omega \Lambda(x,\xi) f(x)\mathrm{d}x, \\ &\int_{\partial\Omega} \Lambda(x,\xi)[\gamma\psi(x) + b(x)\varphi(x)]\mathrm{d}\sigma_x, \\ &-\int_{\partial\Omega} \gamma\varphi(x)\frac{\partial\Lambda}{\partial\gamma}(x,\xi)\mathrm{d}\sigma_x, \end{aligned} \tag{2.3.12}$$

are called, respectively, the

- *generalized potential of volume,*
- *generalized potential of surface of single layer,*
- *generalized potential of surface of double layer.*

In the particular case when $L = \Delta$, where Δ is the Laplace operator, as we have already seen, the Levi function becomes

$$\Lambda(x,\xi) = \frac{1}{(n-2)\omega_n r^{n-2}},$$

and the formula of three potentials (2.3.11) receives the form

$$u(\xi) = -\frac{1}{(n-2)\omega_n}\int_\Omega \frac{f(x)}{r^{n-2}}\mathrm{d}x + \frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\frac{\psi(x)}{r^{n-2}}\mathrm{d}\sigma_x$$
$$-\frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\varphi(x)\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x.$$

Accordingly, generalized potentials from (2.3.12) receive the form

$$\begin{aligned} &-\frac{1}{(n-2)\omega_n}\int_\Omega \frac{f(x)}{r^{n-2}}\mathrm{d}x, \\ &\frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\frac{\psi(x)}{r^{n-2}}\mathrm{d}\sigma_x, \\ &-\frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\varphi(x)\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x, \end{aligned} \tag{2.3.13}$$

and they are called classical potentials, of volume, of surface of single layer, and of surface of double layer, respectively.

2.4 Boundary Value Problems

In this paragraph, we consider the particular case when the elliptic operator L is the Laplace operator, Δ, defined, as everybody knows, by

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}.$$

If Ω is a domain from the space $\mathbb{R}^n$, we call Poisson's equation the following linear partial differential equation of second order:

$$\Delta u(x) = f(x), \ \forall x \in \Omega, \tag{2.4.1}$$

where $f(x)$ is a given function, and $u(x)$ is the unknown function. In the particular case when $f(x) \equiv 0$, Poisson's equation becomes the Laplace equation

$$\Delta u(x) = 0, \ \forall x \in \Omega. \tag{2.4.2}$$

Both in the case of Poisson's equation (2.4.1) and in the case of the Laplace equation (2.4.2), the solution $u = u(x)$ has a high degree of indeterminacy.

To eliminate the arbitrary elements from the solution, Eqs. (2.4.1) and (2.4.2), respectively, are accompanied by boundary conditions with concrete physical meaning. The most common types of boundary conditions are

- Dirichlet's condition

$$u(y) = \varphi(y), \ \forall y \in \partial\Omega;$$

- Neumann's condition

$$\frac{\partial u(y)}{\partial \nu} = \psi(y), \ \forall y \in \partial\Omega;$$

- mixed condition

$$\alpha u(y) + \beta \frac{\partial u(y)}{\partial \nu} = \chi(y), \ \forall y \in \partial\Omega,$$

with α and β being given constants.

The functions φ, ψ, and χ are prescribed and are assumed to be continuous on $\partial\Omega$. We want to outline that instead of the derivative in the normal direction ν, we can take the derivative in an arbitrary direction $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

We have the following boundary value problems with regard to the above boundary conditions.

(i) The inside problem of Dirichlet, for Poisson's equation

$$\Delta u(x) = f(x),\ \forall x \in \Omega,$$
$$u(y) = \varphi(y),\ \forall y \in \partial\Omega. \tag{2.4.3}$$

If the function f is given and continuous on Ω and the function φ is given and continuous on $\partial\Omega$, then we call the *classical solution* of the inside Dirichlet's problem, a function $u = u(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$, which verifies Eq. $(2.4.3)_1$ and satisfies the condition $(2.4.3)_2$.

(ii) The outside problem of Dirichlet, for Poisson's equation

$$\Delta u(x) = f(x),\ \forall x \in \mathbb{R}^n \setminus \overline{\Omega},$$
$$u(y) = \varphi(y),\ \forall y \in \partial\Omega.$$

The classical solution can be defined analogously as in the case of the inside Dirichlet's problem and we must add a condition to characterize the behavior of the solution to infinity.

(iii) The inside Neumann's problem, for Poisson's equation

$$\Delta u(x) = f(x),\ \forall x \in \Omega,$$
$$\frac{\partial u}{\partial \nu}(y) = \psi(y),\ \forall y \in \partial\Omega.$$

(iv) The outside Neumann's problem, for Poisson's equation

$$\Delta u(x) = f(x),\ \forall x \in \mathbb{R}^n \setminus \overline{\Omega},$$
$$\frac{\partial u}{\partial \nu}(y) = \psi(y),\ \forall y \in \partial\Omega.$$

The classical solution for the inside Neumann's problem can be defined analogously as in the case of Dirichlet's problem, and the classical solution for the outside Neumann's problem can be defined analogously as in the case of Dirichlet's problem.

(v) The inside mixed problem, for Poisson's equation

$$\Delta u(x) = f(x),\ \forall x \in \Omega,$$
$$\alpha u(y) + \beta\frac{\partial u}{\partial \nu}(y) = \chi(y),\ \forall y \in \partial\Omega.$$

(vi) The outside mixed problem, for Poisson's equation

$$\Delta u(x) = f(x),\ \forall x \in \mathbb{R}^n \setminus \overline{\Omega},$$
$$\alpha u(y) + \beta\frac{\partial u}{\partial \nu}(y) = \chi(y),\ \forall y \in \partial\Omega.$$

The classical solution for the mixed problems, inside and outside, can be defined analogously for the case of inside Dirichlet's problems and outside Dirichlet's problems, respectively.

In the case in which $f(x) = 0, \forall x \in \Omega$ and $f(x) = 0,\ \forall x \in \mathbb{R}^n \setminus \overline{\Omega}$, respectively, the above boundary value problems become the boundary value problems for the Laplace equation.

In the particular case when $n = 2$, we can proceed as in the case $n \geq 3$. So, we obtain that the form of the fundamental solution is

$$\Lambda(x, \xi) = \frac{1}{2\pi} \log \frac{1}{r},\ r = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}.$$

After analogous reasoning to those from above, we obtain that Riemann–Green's formula becomes for $n = 2$

$$u(\xi) = -\frac{1}{2\pi} \int_{\Omega} \Delta u \log \frac{1}{r} \mathrm{d}x + \frac{1}{2\pi} \int_{\partial\Omega} \log \frac{1}{r} \frac{\partial u}{\partial \nu} \mathrm{d}\sigma_x \\ - \frac{1}{2\pi} \int_{\partial\Omega} u(x) \frac{\partial}{\partial \nu} \left(\log \frac{1}{r} \right) \mathrm{d}\sigma_x,$$

and the classical potentials (2.3.13) from Sect. 2.3 become

$$-\frac{1}{2\pi} \int_{\Omega} f(x) u \log \frac{1}{r} \mathrm{d}x, \\ \frac{1}{2\pi} \int_{\partial\Omega} \log \frac{1}{r} \frac{\partial u}{\partial \nu} \mathrm{d}\sigma_x, \\ -\frac{1}{2\pi} \int_{\partial\Omega} u(x) \frac{\partial}{\partial \nu} \left(\log \frac{1}{r} \right) \mathrm{d}\sigma_x,$$

and are called Newtonian logarithmic potential, logarithmic single-layer potential and logarithmic potential of double layer, respectively.

Definition 2.4.1 The function $u = u(x) = u(x_1, x_2, \ldots, x_n)$ is called a harmonic function on the domain $\Omega \subset \mathbb{R}^n$ if it satisfies the Laplace equation

$$\Delta u(x) = 0,\ \forall x \in \Omega.$$

If all derivatives of second order of the function u are continuous functions on Ω, then u is called a regular harmonic function.

Proposition 2.4.1 *Let u be a numerical function, $u : \overline{\Omega} \to R$, with the properties $u \in C^1(\overline{\Omega})$ and u harmonic on Ω.*

Then

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(y) \mathrm{d}\sigma_y = 0. \tag{2.4.4}$$

Proof The result is obtained immediately if we write Green's formula in the case $L = \Delta$,

$$\int_{\Omega} [v(x)\Delta u(x) - u(x)\Delta v(x)]\mathrm{d}x = \int_{\partial\Omega} \left[v(x)\frac{\partial u}{\partial \nu_x}(x) - u(x)\frac{\partial v}{\partial \nu_x}(x) \right] \mathrm{d}\sigma_x$$

and take $v(x) \equiv 1$. ■

The result which follows is known as the mean value theorem for harmonic functions.

Theorem 2.4.1 (Gauss) *Let Ω be a bounded domain and the function u which is harmonic on Ω. Then for any ball $B(\xi, \varrho)$ so that*

$$\overline{B(\xi,\varrho)} \subset \Omega, \ \overline{B(\xi,\varrho)} = B(\xi,\varrho) \cup \partial B(\xi,\varrho),$$

we have

$$u(\xi) = \frac{1}{\varrho^{n-2}\omega_n} \int_{\partial B(\xi,\varrho)} u(x)\mathrm{d}\sigma_x. \tag{2.4.5}$$

Proof We start by using Riemann–Green's formula, written for $L = \Delta$ and $\Omega = B(\xi, \varrho)$. We take into account that $\Delta u(x) = 0$ and $r = \varrho$ and then we obtain

$$\begin{aligned} u(\xi) = & \frac{1}{(n-2)\omega_n} \int_{\partial B(\xi,\varrho)} \frac{1}{\varrho^{n-2}} \frac{\partial u}{\partial \nu_x}(y)\mathrm{d}\sigma_x \\ & - \frac{1}{(n-2)\omega_n} \int_{\partial B(\xi,\varrho)} u(x) \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x. \end{aligned} \tag{2.4.6}$$

The first integral from the right-hand member of the relation (2.4.6) is null, based on result (2.4.4) from Proposition 2.4.1 and then

$$\begin{aligned} u(\xi) &= \frac{1}{\omega_n} \int_{\partial B(\xi,\varrho)} u(x)\frac{\partial r}{\partial \nu_x}\mathrm{d}\sigma_x \\ &= \frac{1}{\varrho^{n-1}\omega_n} \int_{\partial B(\xi,\varrho)} u(x)\frac{\partial r}{\partial \nu_x}\mathrm{d}\sigma_x. \end{aligned} \tag{2.4.7}$$

But

$$\frac{\partial r}{\partial \nu_x} = \sum_{i=1}^{n} \frac{\partial r}{\partial x_i} \cos\alpha_i = \sum_{i=1}^{n} \frac{x_i - \xi_i}{r} \frac{x_i - \xi_i}{r} = 1,$$

because

$$\overline{n} = \frac{\overline{r}}{r}, \ \text{si } r = \sqrt{\sum_{i=1}^{n} (x_i - \xi_i)^2} = 1.$$

With the help of these calculations, the representation (2.4.5) is obtained immediately from (2.4.7). ■

The theorem which follows is known as the theorem of extreme values (or min-max principle) for harmonic functions.

Theorem 2.4.2 *Let Ω be a bounded domain from $\mathbb{R}^n$, with $\overline{\Omega} = \Omega \cup \partial\Omega$. If the function $u : \overline{\Omega} \to R$ is a harmonic function on Ω, then*

(i) $u(x) = C$ *=constant,* $\forall x \in \overline{\Omega}$,
or
(ii) the extreme values of the function u

$$\sup_{x\in\overline{\Omega}} u(x), \;\; \inf_{x\in\overline{\Omega}} u(x),$$

are reached on boundary $\partial\Omega$.

Proof Let us first observe that the extreme values of the function u are effectively reached, based on the classical Theorem of Weierstrass. Also, we will write the proof only for the supremum, because the function $-u$ satisfies the hypotheses which are satisfied by u and, in addition, $-u$ takes the maximum value where u takes the minimum value and conversely.

Assume by contradiction that

$$\sup_{x\in\overline{\Omega}} u(x) = M \tag{2.4.8}$$

is reached in a point x^0 from inside of the domain Ω. Then we will show that the function u is constant for any x from the biggest ball with center in x^0, contained in Ω.

Consider $B(x^0, r_{\max})$ the biggest ball with center in x^0, contained fully in Ω. We have two possibilities

(a) $u(x) = u(x^0) \equiv M, \;\; \forall x \in B(x^0, r_{\max})$;
(b) there is a point $x^1 \in B(x^0, r_{\max})$, so that $u(x^1) < u(x^0)$.

In the case (a), the proof is complete. Regarding the case (b), we consider the ball $B(x^0, \overline{x^0x^1})$. Because u is a continuous function, we deduce that there is a neighborhood $U_{x^1} \in \mathcal{V}_{x^1}$, so that

$$u(x) < u(x^0), \;\; \forall x \in U_{x^1}.$$

Denote by σ the set

$$\sigma = U_{x^1} \cap \partial B\left(x^0, \overline{x^0x^1}\right).$$

Obviously, the measure of the set σ, $meas(\sigma) > 0$. We denote by $\varrho = \overline{x^0x^1}$ and write the result from Theorem 2.4.1 in the point x^0 for the ball $B(x^0, \varrho)$

$$
\begin{aligned}
u(x^0) &= \frac{1}{\varrho^{n-1}\omega_n}\int_{\partial B(x^0,\varrho)} u(x)\mathrm{d}\sigma_x \\
&= \frac{1}{\varrho^{n-1}\omega_n}\int_{\sigma} u(x)\mathrm{d}\sigma_x + \frac{1}{\varrho^{n-1}\omega_n}\int_{\partial B(x^0,\varrho)\setminus\sigma} u(x)\mathrm{d}\sigma_x \\
&\le \frac{u(x^0)}{\varrho^{n-1}\omega_n}\int_{\sigma} \mathrm{d}\sigma_x + \frac{u(x^0)}{\varrho^{n-1}\omega_n}\int_{\partial B(x^0,\varrho)\setminus\sigma} \mathrm{d}\sigma_x \\
&= \frac{u(x^0)}{\varrho^{n-1}\omega_n}\int_{\partial B(x^0,\varrho)} \mathrm{d}\sigma_x = u(x^0).
\end{aligned}
$$

This contradiction proves that the above case (b) cannot hold true and then we deduce that the function u is constant in the ball $B(x^0, r_{\max})$.

We now arbitrarily fix a point x^2 in Ω. Because Ω is a domain, we deduce that Ω is a convex set and then there is a continuous path (which is homomorph with the segment that connects the points x^0 and x^2) contained fully in Ω. Consider all balls with centers on the segment $\overline{x^0x^2}$, with maximum radius contained fully in Ω.

These balls constitute a covering of the path $\overline{x^0x^2}$. The intersection of these balls with the arch $\overline{x^0x^2}$ constitutes a linear covering. But the arch $\overline{x^0x^2}$ is a compact set and then from the respective covering we can extract a finite covering. In the area in which the balls intersect each other in pairs, the value of the function is a constant, the same constant for two balls which intersect. Therefore, the constant value, M, obtained in the first part of the proof, for the first ball, starting, for instance, from x^0 to x^2, is transferred from ball to ball (for those from the finite covering) such as we reach the last ball (therefore, which has center in the point x^2) with the same constant.

This reasoning is correct, since we have a finite number of balls in the covering, we have a finite number of steps.

We proved that $u(x^0) = u(x^2)$ and x^2 was chosen arbitrarily in Ω. So, we can deduce that the function u is constant in the entire domain Ω and because u was assumed to be continuous on $\overline{\Omega}$, we deduce that u is constant on $\overline{\Omega}$, and this ends the proof of the theorem. ■

As an immediate consequence of the min-max principle for harmonic functions, we can prove the uniqueness of the solution in the case of inside Dirichlet's problem, attached to the equation of Poisson.

Theorem 2.4.3 *Consider the inside Dirichlet's problem attached to the Poisson's equation*

$$
\begin{aligned}
\Delta u(x) &= f(x), \ \forall x \in \Omega, \\
u(y) &= \varphi(y), \ \forall y \in \partial\Omega.
\end{aligned}
$$

If the functions f and φ are given and are continuous on Ω and on $\partial\Omega$, respectively, then the problem admits at most a classical solution.

Proof We mention that the result does not refer to the existence of the solution. That is, if the inside Dirichlet's problem admits solutions, then it can admit only one solution.

Suppose, by absurd, that there are two solutions, denoted by $u_1(x)$ and $u_2(x)$. Then, we define the function v by

$$v(x) = u_1(x) - u_2(x),\ \forall x \in \overline{\Omega}.$$

Since $u_1(x),\ u_2(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$, we deduce that $v \in C(\overline{\Omega}) \cap C^2(\Omega)$. Then, based on the linearity, we have

$$\Delta v(x) = \Delta u_1(x) - \Delta u_2(x) = f(x) - f(x) = 0,\ \forall x \in \Omega,$$
$$v(y) = u_1(y) - u_2(y) = \varphi(y) - \varphi(y) = 0,\ \forall y \in \partial\Omega.$$

Because v is a harmonic function on the domain Ω, we can apply the theorem of extreme values for harmonic functions (Theorem 2.4.2), according to which we have two possibilities

(i) $v(x) \equiv C =$ constant on $\overline{\Omega}$;
(ii) the extreme values of v,

$$\sup_{x\in\overline{\Omega}} v(x),\ \inf_{x\in\overline{\Omega}} v(x),$$

are reached only on the boundary $\partial\Omega$.

In the case (i), because $v(y) = 0,\ \forall y \in \partial\Omega$ and $v(x) = C =$ constant on $\overline{\Omega}$, we deduce that this constant is null and therefore $v(x) = 0,\ \forall x \in \overline{\Omega}$. In conclusion, we have $u_1(x) = u_2(x),\ \forall x \in \overline{\Omega}$.

In the case (ii), because the extreme values of the function v are reached on the boundary $\partial\Omega$, and $v(y) = 0,\ \forall y \in \partial\Omega$, we deduce that

$$\sup_{x\in\overline{\Omega}} u(x) = \inf_{x\in\overline{\Omega}} u(x) = 0,$$

and then $v(x) = 0\ \forall x \in \overline{\Omega}$, that is, $u_1(x) = u_2(x),\ \forall x \in \overline{\Omega}$. ■

Using again the min-max principle for harmonic functions, we prove the result of stability for the inside Dirichlet's problem.

Theorem 2.4.4 *Consider functions* $u_1,\ u_2 : \overline{\Omega} \to R,\ u_1,\ u_2 \in C(\overline{\Omega}) \cap C^2(\Omega)$*, which are solutions for the following inside Dirichlet's problems:*

$$\Delta u_i(x) = f(x),\ \forall x \in \Omega,\ i = 1, 2,$$
$$u_i(y) = \varphi_i(y),\ \forall y \in \partial\Omega,\ i = 1, 2.$$

If $\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon)$ *so that*

$$|\varphi_1(y) - \varphi_2(y)| < \delta, \ \forall y \in \partial\Omega,$$

then

$$|u_1(x) - u_2(x)| < \varepsilon, \ \forall x \in \overline{\Omega}.$$

Proof We define the function v by

$$v(x) = u_1(x) - u_2(x), \ \forall x \in \overline{\Omega}.$$

Because $u_1, \ u_2 \in C(\overline{\Omega}) \cap C^2(\Omega)$ we deduce that $v \in C(\overline{\Omega}) \cap C^2(\Omega)$. Also, v satisfies the following inside Dirichlet's problem, attached to the Laplace equation:

$$\Delta v(x) = \Delta u_1(x) - \Delta u_2(x) = f(x) - f(x) = 0, \ \forall x \in \Omega, \ i = 1, 2,$$
$$v(y) = u_1(y) - u_2(y) = \varphi_1(y) - \varphi_2(y), \ \forall y \in \partial\Omega.$$

Because the function v is harmonic on Ω and $v \in C(\overline{\Omega})$, we deduce that we can apply the min-max principle for harmonic functions. Then, the extreme values of the function v,

$$\sup_{x\in\overline{\Omega}} v(x), \ \inf_{x\in\overline{\Omega}} v(x),$$

are reached only on the boundary $\partial\Omega$. But on $\partial\Omega$, we have

$$v(y) = u_1(y) - u_2(y) = \varphi_1(y) - \varphi_2(y) \Rightarrow$$
$$\Rightarrow |v(y)| = |\varphi_1(y) - \varphi_2(y)| < \delta.$$

if we take $\delta = \varepsilon$, we deduce that

$$\sup_{x\in\overline{\Omega}} v(x) < \varepsilon, \ \inf_{x\in\overline{\Omega}} v(x) > -\varepsilon \Rightarrow |v(x)| < \delta(\varepsilon) = \varepsilon.$$

In conclusion, if v is not identically equal to a constant, then the proof is complete. If v is a constant function, then this constant is the value of v also on the boundary. But on the boundary $\partial\Omega$, we have

$$|v(y)| = |\varphi_1(y) - \varphi_2(y)| < \delta(\varepsilon) = \varepsilon,$$

and this ends the proof of the theorem. ■

Because for the inside Dirichlet's problem, we have a theorem of uniqueness and a theorem of stability in the class of functions f and φ which are continuous functions, we say that this problem is *a correctly formulated problem* and the class of continuous functions is *the class of correctness* for the inside Dirichlet's problem.

If there is a solution of the inside Dirichlet's problem, which corresponds to a member in the right-hand side f, which is given and continuous, and to a function to the limit, φ, which is given and continuous, then this solution is unique and is a *particular solution* of the problem. For all possibilities of choosing the continuous functions f and φ, all the corresponding (unique) solutions form a family which is called *the general integral* of the Poisson's equation for Dirichlet's boundary conditions.

We want to now make some considerations on the concept of *correctly formulated problem*.

Following the idea of Hadamard, a problem is correctly formulated if the following conditions are satisfied.

(i) The solution of the problem must exist in a certain class of functions;

(ii) the solution must be unique in a certain class of functions;

(iii) the solution must continuous depending on the data of the problem (that is, the right member of the equation, boundary conditions, initial conditions, etc.), in a certain class of functions.

The considerations on the correctness of a mixed problem started from a famous theorem, due to Sofia Kovalevskaia, which addresses the correctness of a Cauchy problem in the context of analytic functions.

But Hadamard proved by a concrete example that the problem of correctness was not completely solved by S. Kovalevskaia, because her results were related to a local solution.

In addition, in a Cauchy problem in general the initial conditions and the right member of the equation are not analytic functions. Moreover, in his example of a Cauchy problem, Hadamard shows that the solution is not continuously depending on the initial conditions. Hadamard considers the following Cauchy's problem:

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial x^2},$$
$$u(0,x) = 0, \quad \frac{\partial u}{\partial t}(0,x) = \frac{1}{k}\sin kx,$$

which admits the solution

$$u(t,x) = \frac{\sinh kt}{k^2}\sin kx.$$

It is clear that

$$\frac{1}{k}\sin kx \to 0, \ \text{for } k \to +\infty.$$

However, for $x \neq n\pi,\ n = 0, \pm 1, \ldots,$ we have

$$u(t,x) = \frac{\sinh kt}{k^2}\sin kx \not\to 0, \ \text{for } k \to +\infty,$$

that is, the problem Cauchy is not correctly formulated.

We approach now the outside problem of Dirichlet. For this, consider the bounded domain $\Omega \subset \mathbb{R}^n$, with boundary $\partial\Omega$ that admits tangent plane, continuously varying almost everywhere and $\overline{\Omega} = \Omega \cup \partial\Omega$.

On the set $\mathbb{R}^n \setminus \overline{\Omega}$, we define the problem

$$\begin{aligned} &\Delta u(x) = f(x), \ \forall x \in \mathbb{R}^n \setminus \overline{\Omega}, \\ &u(y) = \varphi(y), \ \forall y \in \partial\Omega. \end{aligned} \tag{2.4.9}$$

We call the *classical solution* for the outside Dirichlet's problem, a function $u : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$, $u \in C(\mathbb{R}^n \setminus \Omega) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega})$ which satisfies Eq. $(2.4.9)_1$ and verifies the boundary condition $(2.4.9)_2$. In addition, we consider the condition of behavior to infinity

$$(c) \qquad \forall \varepsilon > 0, \ \exists R_0 = R_0(\varepsilon), \ \text{such that if } |ox| > R_0 \Rightarrow |u(x)| < \varepsilon.$$

In the formulation of the outside Dirichlet's problem and in the definition of its classical solution, it is assumed that the functions $f : \mathbb{R}^n \setminus \overline{\Omega} \to \mathbb{R}$ and $\varphi : \partial\Omega \to \mathbb{R}$ are continuous.

We prove now that for the outside Dirichlet's problem we have a result of uniqueness for the classical solution.

Theorem 2.4.5 *The outside Dirichlet's problem admits at most a classical solution.*

Proof We emphasize that this theorem does not consider the effective existence of the solution. The statement can be reformulated more completely in this way: if the problem admits solutions, then it cannot admit more than one solution.

Suppose, by absurd, that the problem admits two solutions $u_1(x)$ and $u_2(x)$. Then, we define the function v by

$$v : \mathbb{R}^n \setminus \overline{\Omega} \to \mathbb{R}, \ v(x) = u_1(x) - u_2(x), \ \forall x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

It is clear that $v \in C(\mathbb{R}^n \setminus \Omega) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega})$ and satisfies the problem

$$\begin{aligned} &\Delta v(x) = \Delta u_1(x) - \Delta u_2(x) = 0, \ \forall x \in \mathbb{R}^n \setminus \overline{\Omega}, \\ &v(y) = u_1(y) - u_2(y) = \varphi_1(y) - \varphi_2(y) = 0, \ \forall y \in \partial\Omega. \end{aligned} \tag{2.4.10}$$

Taking into account the behavior to infinity of a classical solution, from the condition (c), we deduce

$$\begin{aligned} &\forall \varepsilon > 0, \ \exists R_0 = R_0(\varepsilon), \ \text{such that if } |ox| > R_0 \Rightarrow \\ &|v(x)| = |u_1(x) - u_2(x)| \le |u_1(x)| + |u_2(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We arbitrarily fix a point x^0 in $\mathbb{R}^n \setminus \overline{\Omega}$ and we take a ball $B(0, R)$, with its center at the origin and the radius R big enough so that the ball includes the point x^0 and the domain Ω. Also, the radius R is taken so that $R \geq R_0(\varepsilon/2)$. Consider the corona determined by $\partial\Omega$ and $\partial B(0, R) = S(0, R)$. From $(2.4.10)_1$, we deduce that v is a harmonic function in the corona, and from $(2.4.10)_2$ we have that v becomes null on boundary $\partial\Omega$. We can apply now the min-max principle for harmonic functions, and then the extreme values of the function v are reached, either on $\partial\Omega$ or on $S(0, R)$. In the first case, the minimum value and also the maximum value are null and then $v \equiv 0$ in the corona. In the second case, we have

$$\sup_{x \in \mathbb{R}^n \setminus \Omega} v(x) < \varepsilon, \quad \inf_{x \in \mathbb{R}^n \setminus \Omega} v(x) > -\varepsilon \Rightarrow |v(x)| < \varepsilon$$

for any x from the closed corona.

In particular, we have $|v(x^0)| < \varepsilon \Rightarrow v(x^0) = 0$. From the arbitrariness of x^0, we deduce that $v(x) = 0$, $\forall x \in \mathbb{R}^n \setminus \overline{\Omega}$, that is, $u_1(x) = u_2(x)$, $\forall x \in \mathbb{R}^n \setminus \overline{\Omega}$. ■

By analogy with the case of the inside Dirichlet's problem, a result of stability can be proved, with respect to boundary data, also for the outside Dirichlet's problem.

We approach now the inside Neumann's problem. First, in next the proposition we prove an auxiliary result.

Proposition 2.4.2 *Let Ω be a bounded domain from $\mathbb{R}^n$, having boundary $\partial\Omega$ which admits a tangent plane continuously varying almost everywhere and consider the functions $g \in C^1(\Omega)$ and $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$.*

Then, we have the equality

$$\int_\Omega (\text{grad}\, g \; \text{grad}\, h + g\Delta h)\, \mathrm{d}x = \int_{\partial\Omega} g \frac{\partial h}{\partial \nu} \mathrm{d}\sigma_x, \tag{2.4.11}$$

where ν is unit normal to $\partial\Omega$, oriented to outside of Ω.

Proof The following equality is obvious:

$$\begin{aligned}\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(g(x)\frac{\partial h}{\partial x_i}(x)\right) &= \sum_{i=1}^n \frac{\partial g}{\partial x_i}(x)\frac{\partial h}{\partial x_i}(x) + g(x)\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(x)\\ &= (\text{grad}\, g(x))(\text{grad}\, h(x)) + g(x)\Delta h(x).\end{aligned}$$

Integrating this equality on Ω and, using Gauss–Ostrogradsky's formula, we obtain

$$\begin{aligned}&\int_\Omega [(\text{grad}\, g(x))(\text{grad}\, h(x)) + g(x)\Delta h(x)]\mathrm{d}x\\ &= \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(g(x)\frac{\partial h}{\partial x_i}(x)\right)\mathrm{d}x = \int_{\partial\Omega}\sum_{i=1}^n g(x)\frac{\partial h}{\partial x_i}(x)\cos\alpha_i \mathrm{d}\sigma_x\end{aligned}$$

$$= \int_{\partial\Omega} g(x) \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x) \cos \alpha_i \mathrm{d}\sigma_x = \int_{\partial\Omega} \sum_{i=1}^{n} g(x) \frac{\partial h}{\partial \nu}(x) \mathrm{d}\sigma_x,$$

and this ends the proof. ■

Corollary 2.4.1 *In the same conditions as in Proposition 2.4.2, the following equality holds true:*

$$\int_{\Omega} \left(\mathrm{grad}^2 h + h\Delta h\right) \mathrm{d}x = \int_{\partial\Omega} h \frac{\partial h}{\partial \nu} \mathrm{d}\sigma_x. \tag{2.4.12}$$

Proof The result is obtained immediately from (2.4.11), by taking $g = h$. ■

On the domain Ω, which satisfies the conditions from Proposition 2.4.2, we define the inside Neumann's problem by

$$\begin{aligned} &\Delta u(x) = f(x),\ \forall x \in \Omega, \\ &\frac{\partial u}{\partial \nu}(y) = \psi(y),\ \forall y \in \partial\Omega, \end{aligned} \tag{2.4.13}$$

where $f : \Omega \to \mathbb{R},\ f \in C(\Omega)$ and $\psi : \partial\Omega \to \mathbb{R},\ \psi \in C(\partial\Omega)$.

A *classical solution* for the inside Neumann's problem (2.4.13) is a function $f : \overline{\Omega} \to \mathbb{R},\ u \in C(\overline{\Omega}) \cap C^2(\Omega)$ which verifies Eq. $(2.4.13)_1$ and satisfies the boundary condition $(2.4.13)_2$. We have the following result about the uniqueness of a classic solution of the inside Neumann's problem.

Theorem 2.4.6 *The classical solution of the inside Neumann's problem is determined until an additive constant.*

Proof The statement of the theorem can be reformulated in this form: any two classical solutions of the inside Neumann's problem differ by a constant, that is, if u_1 and u_2 are two classical solutions, then their difference is a constant on $\overline{\Omega}$. We define the function v by

$$v : \overline{\Omega} \to \mathbb{R},\ v(x) = u_1(x) - u_2(x),$$

where $u_1(x)$ and $u_2(x)$ are two classical solutions of the inside Neumann's problem. Then $v \in C(\overline{\Omega}) \cap C^2(\Omega)$ and

$$\begin{aligned} &\Delta v(x) = 0,\ \forall x \in \Omega, \\ &\frac{\partial v}{\partial \nu}(y) = 0,\ \forall y \in \partial\Omega. \end{aligned} \tag{2.4.14}$$

If we write formula (2.4.12) in which $h(x) = v(x)$ and we take into account $(2.4.13)_1$ and $(2.4.13)_2$, we obtain

$$\int_{\Omega} \mathrm{grad}^2 v(x) \mathrm{d}x = 0,$$

and therefore

$$\text{grad}^2 v(x) = 0 \Rightarrow \frac{\partial v(x)}{\partial x_i} = 0,\ \forall x \in \Omega,\ \forall i = 1, 2, \ldots, n,$$

which proves that v is constant on Ω and because v is continuous on $\overline{\Omega}$ we deduce that v is constant on $\overline{\Omega}$. ■

Observation 2.4.1 *(i) If in the formulation of the inside Neumann's problem we add the condition $u(x^0) = u^0$, where x^0 is an arbitrarily fixed point in Ω and u^0 is given, then we have assured the uniqueness of a classic solution. Indeed, in the proof of the Theorem 4.6 we have $v(x^0) = u_1(x^0) - u_2(x^0) = u^0 - u^0 = 0$ and because v is constant (as we already proved) on $\overline{\Omega}$, we deduce that the value of the constant is zero.*

(ii) Taking into account the previous comments and Theorem 2.4.6, we can observe that the inside Neumann's problem is incorrectly formulated and therefore we will not have a theorem of stability.

In the standard conditions imposed to the domain Ω, we now formulate the outside Neumann's problem

$$\begin{aligned} &\Delta u(x) = f(x),\ \forall x \in \mathbb{R}^n \setminus \overline{\Omega}, \\ &\frac{\partial u}{\partial \nu}(y) = \psi(y),\ \forall y \in \partial\Omega. \end{aligned} \tag{2.4.15}$$

A *classical solution* for the outside Neumann's problem (2.4.15) is a function $u : \mathbb{R}^n \setminus \Omega \to \mathbb{R},\ u \in C(\mathbb{R}^n \setminus \Omega) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega})$ which satisfies Eq. $(2.4.15)_1$, verifies the boundary condition $(2.4.15)_2$, and satisfies the conditions of behavior to infinity that follow

$$\begin{aligned} &|u(x)| \le \frac{A}{|\overline{ox}|^{(n+\alpha)/2}}, \\ &\left|\frac{\partial u(x)}{\partial x_i}\right| \le \frac{B}{|\overline{ox}|^{(n+\alpha)/2-1}},\ i = 1, 2, \ldots, n, \end{aligned} \tag{2.4.16}$$

in which α, A, and B are constants and the distance $|\overline{ox}|$ is sufficient big.

Due to the conditions (2.4.16), we will be able to prove the uniqueness of classical solution for the problem (2.4.15).

Theorem 2.4.7 *The outside Neumann's problem admits at most a classical solution.*

Proof We fix an arbitrary point x^0 outside of Ω and consider the ball $B(0, R)$ with R big enough so that the ball contains the point x^0, and the domain Ω is wholly contained in the ball. Also, R is taken big enough so that the conditions (2.4.16) are satisfied.

Let u_1 and u_2 be two classical solutions of the problem (2.4.15) and we define the function v by

$$v : \mathbb{R}^n \setminus \Omega \to \mathbb{R},\ v(x) = u_1(x) - u_2(x),\ \forall x \in R^n \setminus \Omega.$$

Then, $v \in C(\mathbb{R}^n \setminus \Omega) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega})$ and v satisfies the homogeneous problem

$$\begin{aligned} &\Delta v(x) = 0,\ \forall x \in \mathbb{R}^n \setminus \overline{\Omega}, \\ &\frac{\partial v}{\partial \nu}(y) = 0,\ \forall y \in \partial\Omega. \end{aligned} \tag{2.4.17}$$

As far as the conditions (2.4.16) are concerned, we have

$$\begin{aligned} &|v(x)| \leq |u_1(x)| + |u_2(x)| \leq \frac{2A}{|\overline{ox}|^{(n+\alpha)/2}}, \\ &\left|\frac{\partial v(x)}{\partial x_i}\right| \leq \left|\frac{\partial u_1(x)}{\partial x_i}\right| + \left|\frac{\partial u_2(x)}{\partial x_i}\right| \leq \frac{2B}{|\overline{ox}|^{(n+\alpha)/2-1}},\ i = 1, 2, \ldots, n. \end{aligned} \tag{2.4.18}$$

We denote by K the corona $\mathbb{R}^n \setminus \overline{\Omega}$ and we apply formula (2.4.12) for the corona K and $h = v$

$$\int_K \left(\text{grad}^2 v + v\Delta v\right) \mathrm{d}x = \int_{\partial\Omega} v\frac{\partial v}{\partial \nu}\mathrm{d}\sigma_x + \int_{\partial B(0,R)} v\frac{\partial v}{\partial \nu}\mathrm{d}\sigma_x.$$

Taking into account $(2.4.1)_1$ and $(2.4.1)_2$, the above equality becomes

$$\int_K \text{grad}^2 v \mathrm{d}x = \int_{\partial B(0,R)} v\frac{\partial v}{\partial \nu}\mathrm{d}\sigma_x.$$

Considering the evaluations (2.4.18), we obtain the bounds

$$\begin{aligned} \left|\int_K \text{grad}^2 v \mathrm{d}x\right| &\leq \int_{\partial B(0,R)} |v| \left|\frac{\partial v}{\partial \nu}\right| \mathrm{d}\sigma_x \\ &\leq c \int_{\partial B(0,R)} \frac{1}{\overline{ox}^{n+\alpha-1}} \mathrm{d}\sigma_x \leq \frac{c\omega_n}{R^\alpha}, \end{aligned}$$

and therefore

$$\int_K \text{grad}^2 v \mathrm{d}x \to 0,\ R \to \infty.$$

It is clear then that grad $v = 0 \Rightarrow \frac{\partial v}{\partial x_i} = 0,\ i = 1, 2, \ldots, n$, that is, v is constant in the corona. But from $(2.4.18)_1$ we obtain that $v \to 0,\ x \to \infty$ and then it is necessary that $v = 0$ is in the corona. ■

At the end of this chapter, we will make some considerations on the boundary value problems in the case in which the Laplace operator is replaced with an arbitrary elliptic operator.

On the bounded domain $\Omega \subset \mathbb{R}^n$, we define the elliptic operator L by

$$Lu(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i} + c(x)u(x), \tag{2.4.19}$$

where $a_{ij} = a_{ji} \in C^2(\Omega),\ b_i \in C^1(\Omega), and\ c \in C^0(\Omega).$

If the function $f : \Omega \to \mathbb{R}$ is continuous, then we call the *regular solution* of the equation

$$Lu(x) = f(x),\ \forall x \in \Omega \tag{2.4.20}$$

a function $u : \Omega \to \mathbb{R},\ u \in C^2(\Omega)$ which replaced in (2.4.20), transforms it into an identity.

In the following theorem, we prove the non-existence of a positive relative maximum (of a negative relative minimum respectively) for a regular solution of Eq. (2.4.20).

Theorem 2.4.8 *Suppose that the domain Ω and the elliptic operator L satisfy the standard hypotheses and consider a regular solution $u(x)$ of Eq. (2.4.20). Then, we have two alternatives*

(i) If

$(a)\ \ c(x) < 0,\ and\ f(x) \leq 0,\ \forall x \in \Omega$

or

$(b)\ \ c(x) \leq 0,\ and\ f(x) < 0,\ \forall x \in \Omega$

then u cannot have a negative relative minimum in Ω.

(ii) If

$(c)\ \ c(x) < 0,\ and\ f(x) \geq 0,\ \forall x \in \Omega$

or

$(d)\ \ c(x) \leq 0,\ si\ f(x) > 0,\ \forall x \in \Omega$

then u cannot have a positive relative maximum in Ω.

Proof (i) Taking into account (2.4.19), we can write

$$\begin{aligned} Lu(x) - c(x)u(x) &= f(x) - c(x)u(x) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i}. \end{aligned} \tag{2.4.21}$$

Suppose, by contradiction, that although the conditions from (a) or (b) are satisfied, the function u, which is a solution of Eq. (2.4.20), has a negative relative minimum, which is reached in the point $x^0 \in \Omega$.

Then, according to the conditions of the minimum, we have

$$\begin{aligned} &u(x^0) < 0, \\ &\frac{\partial u}{\partial x_k}(x^0) = 0,\ k = 1, 2, \ldots, n, \end{aligned} \tag{2.4.22}$$

$$\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 u}{\partial x_k \partial x_s}(x^0)\lambda_k\lambda_s \geq 0,\ \forall\lambda_k \in \mathbb{R},\ k=1,2,\ldots,n.$$

Using conditions from (a) or (b) and the relations (2.4.21) and $(2.4.22)_1$, we obtain

$$(Lu(x) - c(x)u(x))_{x=x^0} = f(x^0) - c(x^0)u(x^0) < 0. \tag{2.4.23}$$

We define the matrices $A = [a_{ij}]$ and $P = [p_{ij}]$, where

$$p_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}(x^0).$$

Taking into account $(2.4.22)_2$, we deduce that the matrix P is positive definite. We can now write

$$(Lu(x) - c(x)u(x))_{x=x^0} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x^0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x^0)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x^0)p_{ij} = \mathrm{tr}(AP).$$

Denote by H the orthogonal matrix, $H^{-1} = H^T$, for which $H^{-1}AH = I$.

It is well known that the trace of a matrix is the sum of eigenvalues of the respective matrix.

Also, it is known that the self-values of a matrix are preserved if we multiply the respective matrix, on the left side by H^{-1} and on the right side by H, where H is a non-degenerate matrix. Therefore, we can write

$$\mathrm{tr}(AP) = \mathrm{tr}(H^{-1}APH) = \mathrm{tr}(H^{-1}AHH^{-1}PH).$$

Because $H^{-1}AH = I$ and $H^{-1} = H^T$, we have

$$\mathrm{tr}(AP) = \mathrm{tr}(H^{-1}PH) = \mathrm{tr}(H^T PH)$$

and if we denote by q_{ij} the elements of the matrix H^{-1}, we deduce

$$\mathrm{tr}(AP) = \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}q_{ik}q_{jk}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}q_{i1}q_{j1} + \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}q_{i2}q_{j2} + \cdots + \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}q_{in}q_{jn}.$$

Each term of the sum on the right-hand side is positive, based on the fact that the matrix P is positive definite. Finally, we obtain

$$(Lu(x) - c(x)u(x))_{x=x^0} = \mathrm{tr}(AP) \geq 0,$$

and this is in contradiction with relation (2.4.23). The contradiction has appeared because we assumed the existence of a negative relative minimum.

(ii) Suppose now, by contradiction, that although the conditions (c) or (d) hold true, u admits a positive relative maximum and this maximum is reached in the point $x^1 \in \Omega$. Then, based on the definition of a point of relative maximum, we have

$$\begin{aligned} &u(x^1) > 0, \\ &\frac{\partial u}{\partial x_k}(x^1) = 0, \ k = 1, 2, \ldots, n, \\ &\sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial^2 u}{\partial x_k \partial x_s}(x^1)\lambda_k\lambda_s \leq 0, \ \forall \lambda_k \in \mathbb{R}, \ k = 1, 2, \ldots, n. \end{aligned} \tag{2.4.24}$$

Taking into account (2.4.19), we deduce

$$(Lu(x) - c(x)u(x))_{x=x^1} = f(x^0) - c(x^0)u(x^0) > 0, \tag{2.4.25}$$

the last inequality is due to the conditions (c) or (d) and to condition $(2.4.24)_1$. On the other hand, with the help of condition $(2.4.24)_2$, we obtain

$$(Lu(x) - c(x)u(x))_{x=x^1} = \sum_{k=1}^{n}\sum_{s=1}^{n} a_{ij}(x^1)q_{ij} = \mathrm{tr}(AQ), \tag{2.4.26}$$

where we used the matrix notations $A = [a_{ij}]$, $Q = [q_{ij}]$, with

$$q_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}(x^1).$$

From $(2.4.24)_3$, it is deduced that the matrix Q is positive definite. Because the matrix A is positive definite in any point from Ω, we deduce that there is a orthogonal matrix $H, \ H^{-1} = H^T$, so that $H^{-1}AH = I$.

Then, based on the considerations made at the point (i) on trace of a matrix, we can write

$$\begin{aligned} \mathrm{tr}(AQ) &= \mathrm{tr}(H^{-1}AQH) = \mathrm{tr}(H^{-1}AHH^{-1}QH) \\ &= \mathrm{tr}(H^{-1}QH) = \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}h_{ik}h_{jk}, \end{aligned}$$

where h_{ij} are the components of the matrix $H^{-1} = H^T$. Therefore,

$$\mathrm{tr}(AQ) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}q_{i1}q_{j1} + \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}q_{i2}q_{j2} + \cdots + \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}q_{in}q_{jn},$$

where each term of the sum on the right-hand side is positive, based on the fact that the matrix P is positive definite. Finally, we get that

$$(Lu(x) - c(x)u(x))_{x=x^1} = \mathrm{tr}(AQ) \leq 0,$$

and this is in contradiction with relation (2.4.25).

The contradiction has appeared because we assumed the existence of a positive relative maximum. ■

It is clear that any absolute maximum is also a relative maximum and then a condition of non-existence of a relative maximum is surely a condition of non-existence of an absolute maximum. We have an analogous comment in the case of a minimum.

As an immediate application of Theorem 2.4.8, we will prove a theorem of uniqueness of the solution for the inside Dirichlet's problem, in the general case of an elliptic arbitrary operator.

$$\begin{aligned} Lu(x) &= f(x), \ \forall x \in \Omega, \\ u(y) &= \varphi(y), \ \forall y \in \partial\Omega, \end{aligned} \tag{2.4.27}$$

in which Ω is a bounded domain from $\mathbb{R}^n$ and the elliptic operator L is defined in (2.4.19). The functions f and φ are definite and are continuous on Ω and on $\partial\Omega$, respectively. As usual, the classical solution for the inside Dirichlet's problem (2.4.27) is a function

$$u : \overline{\Omega} \to \mathbb{R}, \ u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$$

which verifies Eq. $(2.4.27)_1$ and satisfies the boundary condition $(2.4.27)_2$. In the case of an outside Dirichlet's problem, we must add in addition a condition of behavior to infinity.

Theorem 2.4.9 *If* $c(x) < 0, \ \forall x \in \Omega$, *then the inside Dirichlet's problem admits at most a classical solution.*

Proof Suppose by absurd that the problem (2.4.27) would admit two classical solutions u_1 and u_2. We define the function v by

$$v : \overline{\Omega} \to \mathbb{R}, \ v(x) = u_1(x) - u_2(x), \ \forall x \in \Omega.$$

Based on the properties of the solutions u_1 and u_2, we deduce that $v \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and v satisfies the homogeneous Dirichlet's problem

$$Lv(x) = Lu_1(x) - Lu_2(x) = f(x) - f(x) = 0, \ \forall x \in \Omega,$$
$$v(y) = u_1(y) - u_2(y) = \varphi(y) - \varphi(y) = 0, \ \forall y \in \partial\Omega.$$

Taking into account the hypothesis $c(x) < 0$, we deduce that we have the same conditions as in the cases (a) and (c) from Theorem 2.4.8, and therefore v does not admits neither a negative relative minimum nor a positive relative maximum. Thus,

$$\sup_{x \in \overline{\Omega}} v(x) < 0, \ \inf_{x \in \overline{\Omega}} v(x) > 0$$

and because $v(y) = 0, \ \forall y \in \partial\Omega$, we deduce that $v \equiv 0$ on $\overline{\Omega}$. ∎

Chapter 3
The Theory of Potential

3.1 The Newtonian Potential

The Newtonian potential, or the potential of volume, associated to the Laplace equation $\Delta u = 0$ is, by definition, the following improper integral:

$$U(\xi) = -\frac{1}{(n-2)\omega_n} \int_\Omega \frac{f(x)}{r^{n-2}} \mathrm{d}x, \tag{3.1.1}$$

where $r = r_{\xi x} = \left|\overline{\xi x}\right| = \sqrt{\sum_{i=1}^{n} (x_i - \xi_i)^2}$, and ω_n is the area of the unit sphere from the n-dimensional space $\mathbb{R}^n$.

For simplicity of writing, we use the notation

$$\varrho(x) = -\frac{f(x)}{(n-2)\omega_n}. \tag{3.1.2}$$

Then, the Newtonian potential can be written in the form

$$U(\xi) = \int_\Omega \frac{\varrho(x)}{r^{n-2}} \mathrm{d}x. \tag{3.1.3}$$

The function $\varrho(x)$ defined in (3.1.2) will be called in the following, *the density* of the potential of volume.

In the particular case when $n = 2$, we have the so-called logarithmic Newtonian potential

$$U(\xi_1, \xi_2) = \frac{1}{2\pi} \int_\Omega f(x_1, x_2) \log \frac{1}{r} \mathrm{d}x_1 \mathrm{d}x_2$$

or

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_3

$$U(\xi_1, \xi_2) = \int_\Omega \varrho(x_1, x_2) \log \frac{1}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} \mathrm{d}x_1 \mathrm{d}x_2,$$

where

$$\varrho(\xi_1, \xi_2) = \frac{1}{2\pi} f(x_1, x_2)$$

is *the density* of the logarithmic Newtonian potential.

In the proposition which follows, we prove a first property of the Newtonian potential.

Proposition 3.1.1 *For* $\forall \xi \in \mathbb{R}^n \setminus \overline{\Omega}$, *we have that* $U(\xi) \in C^\infty$ *and*

$$\Delta U(\xi) = 0,$$

that is, the Newtonian potential is a harmonic function outside of the domain Ω.

Proof If we can derive under the integral sign, then we apply the Laplacian in (3.1.3), and we obtain

$$\Delta_\xi U(\xi) = \int_\Omega \varrho(x) \Delta_\xi \left(\frac{1}{r^{n-2}} \right) \mathrm{d}x = 0,$$

because, as is well known

$$\Delta_\xi \left(\frac{1}{r^{n-2}} \right) = 0.$$

Since the set $\mathbb{R}^n \setminus \overline{\Omega}$ is open, we deduce that any point ξ is inside, and therefore, we can consider a compact set K which contains ξ and which is located at a distance d from $\overline{\Omega}$. Obviously, we have $d(\xi, x) > d$, and then

$$\frac{1}{r^{n-2}} \le \frac{1}{d^{n-2}} \Rightarrow \int_\Omega \varrho(x) \frac{1}{r^{n-2}} \mathrm{d}x \le \int_\Omega \varrho(x) \frac{1}{d^{n-2}} \mathrm{d}x.$$

We can now derive an infinity of times under the integral sign. Each time we obtain convergent integrals. Obviously, $U(\xi)$ is continuous and therefore integrable in the Riemann sense. This is the justification for the fact that we can derive under the integral sign in (3.1.3) and then, taking into account the above considerations, the proof is complete. ■

In the following, we will have the following goals:

- we will find sufficient conditions for a better definition of the potential of volume;
- we will identify the conditions in which the derivatives of the Newtonian potential are continuous functions;
- we will determine the conditions in which the second derivatives of the Newtonian potential are continuous functions;

- we will establish the partial differential equation satisfied by the potential of volume and under what conditions it is.

  We should note that all these conditions will be imposed to the density ϱ.

Theorem 3.1.1 *Assume that Ω is a bounded domain and the density ϱ is a bounded function on Ω. Then, the Newtonian potential $U(\xi)$ and the derivatives of first order are continuous functions in the whole space $\mathbb{R}^n$,*

$$\frac{\partial U}{\partial \xi_k}(\xi) \in C(\mathbb{R}^n),\ k = 1, 2, \ldots, n.$$

In addition, we have

$$\frac{\partial U}{\partial \xi_k}(\xi) = \int_\Omega \varrho(x) \frac{\partial}{\partial \xi_k}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}x,\ k = 1, 2, \ldots, n.$$

Proof Clearly, the potential of volume $U(\xi)$ becomes infinite if $\xi = x$. Together with the function $U(\xi)$, we consider another function, defined everywhere on Ω, which will be called *the regularization* of the function $1/r^{n-2}$. Thus, for $\forall \delta > 0$ we define the regularization g_δ by

$$g_\delta = \begin{cases} \frac{1}{\delta^{n-2}}\left[\frac{n+p-2}{p} - \frac{n-2}{p}\frac{r^p}{\delta^p}\right], & \text{if } r \le \delta \\ \frac{1}{r^{n-2}} & , \text{if } r > \delta. \end{cases} \tag{3.1.4}$$

As we can see from the definition, the regularization g_δ is a function of class C^1 for $r < \delta$ and also for $r > \delta$. We intend to prove the continuity and continuous differentiation of the regularization g_δ for $r = \delta$.

Obvious for the limit on the right-hand side, we have

$$g_\delta(\delta + 0) = \frac{1}{\delta^{n-2}}.$$

Then, the limit on the left-hand side is

$$g_\delta(\delta - 0) = \frac{1}{\delta^{n-2}}\left[\frac{n+p-2}{p} - \frac{n-2}{p}\frac{\delta^p}{\delta^p}\right] = \frac{1}{\delta^{n-2}},$$

and these two lateral limits ensure the continuity of the function g_δ for $r = \delta$.

We now compute the lateral derivatives

$$\frac{\partial g_\delta(\delta + 0)}{\partial \xi_k} = \frac{\partial}{\partial \xi_k}\left(\frac{1}{r^{n-2}}\right)|_{r=\delta} = (2-n)\frac{x_k - \xi_k}{r^n}|_{r=\delta},$$

and, respectively,

$$\frac{\partial g_\delta(\delta - 0)}{\partial \xi_k} = -\frac{(n-2)r^{p-1}}{\delta^{n+p-2}} \frac{x_k - \xi_k}{r}|_{r=\delta} = (2-n)\frac{x_k - \xi_k}{\delta^n}|_{r=\delta},$$

which proves that the function g_δ is differentiable for $r = \delta$.

Using the procedure by which we associate a Newtonian potential $U(\xi)$ for the function $1/r^{n-2}$, we will associate to the regularization g_δ "the potential" U_δ, by

$$U_\delta(\xi) = \int_\Omega \varrho(x) g_\delta(r) \mathrm{d}x.$$

We will prove that $U_\delta(\xi)$ is uniformly convergent, with respect to ξ and x, to $U(\xi)$ for $\delta \to 0$. Because the function g_δ is continuous, as a function of ξ, and ϱ is, by the hypothesis, a bounded function, we deduce that $U_\delta(\xi)$ is a continuous function, as a function of ξ. If we prove that $U_\delta(\xi)$ is uniformly convergent to $U(\xi)$, we obtain a sequence of continuous functions $\{U_\delta\}$, indexed on values of δ, which is uniformly convergent, and therefore, its limit is also a continuous function. In this way, we deduce that the function $U(\xi)$ is continuous.

So, we need to prove the uniform convergence of the sequence $\{U_\delta\}$. To this end, we have

$$\begin{aligned}
|U_\delta(\xi) - U(\xi)| &= \left| \int_\Omega \varrho(x) \left[g_\delta(r) - \frac{1}{r^{n-2}} \right] \mathrm{d}x \right| \\
&= \left| \int_{B(\xi,\delta)} \varrho(x) \left[g_\delta(r) - \frac{1}{r^{n-2}} \right] \mathrm{d}x \right| \\
&= \left| \int_{B(\xi,\delta)} \varrho(x) \left[\frac{1}{\delta^{n-2}} \frac{n+p-2}{p} - \frac{n-2}{p} \frac{r^p}{r^{n+p-2}} - \frac{1}{r^{n-2}} \right] \mathrm{d}x \right| \\
&\le \frac{c_0}{\delta^{n-2}} \int_{B(\xi,\delta)} \mathrm{d}x + \frac{c_1}{\delta^{n-2}} \int_{B(\xi,\delta)} \mathrm{d}x + c_2 \int_{B(\xi,\delta)} \frac{1}{r^{n-2}} \mathrm{d}x \\
&\le \frac{c_0}{\delta^{n-2}} \frac{\delta^n \omega_n}{n} + \frac{c_1}{\delta^{n-2}} \frac{\delta^n \omega_n}{n} + \omega_n \int_0^\delta \frac{r^{n-1}}{r^{n-2}} \mathrm{d}r \\
&= \frac{c_0 \omega_n}{n} \delta^2 + \frac{c_1 \omega_n}{n} \delta^2 + \frac{c_2 \omega_n}{n} \delta^2,
\end{aligned}$$

Here, we pass to the generalized polar coordinates and we use the fact that the area of the unit sphere from the space $\mathbb{R}^n$ is denoted by ω_n.

It is clear that the last three terms do not depend on ξ and, also, do not depend on x. Then, the above evaluations lead to the conclusion that

$$|U_\delta(\xi) - U(\xi)| \to 0, \text{ for } \delta \to 0,$$

and this ensures the uniform convergence of $U_\delta(\xi) \to U(\xi)$.

We now want to prove that

$$\frac{\partial U_\delta(\xi)}{\partial \xi_i} \to \int_\Omega \varrho(x) \frac{\partial}{\partial \xi_i}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}x = \chi_i, \ i = 1, 2, \ldots, n,$$

where the convergence to χ_i is uniform with respect to ξ. We can see immediately that χ_i is in fact the derivative of the function $U(\xi)$ with respect to ξ_i. For the moment, we suppose that this uniform convergence is proven. Because the functions $\partial g_\delta/\partial \xi_i$ are continuous on $\mathbb{R}^n$, we deduce that the functions $\partial U_\delta/\partial \xi_i$ are continuous on $\mathbb{R}^n$. Thus, we deduce that the functions χ_i are continuous, as a uniform limit of a sequence of continuous functions. Moreover, using a classical theorem of mathematical analysis (namely, the closing property of the operator of differentiation), we know that if the sequence $\{\psi_\delta(x)\}_k$ is uniformly convergent with respect to x to $\psi(x)$, for $\delta \to 0$ and the sequence $\{\partial \psi_\delta(x)/\partial x_k\}_k$ is uniformly convergent with respect to x to the function $\varphi_k(x)$, then the sequence $\{\psi_\delta(x)\}_k$ is differentiable term by term and $\varphi_k(x) = \partial \psi(x)/\partial x_k$.

In our case, if we prove that the sequence $\{\partial U_\delta(\xi)/\partial \xi_i\}_\delta$ is uniformly convergent to χ_i, for each $i = 1, 2, \ldots, n$, we will deduce, according to the above considerations, that the functions χ_i are continuous and $\chi_i = \partial U(\xi)/\partial \xi_i$.

The uniform convergence is deduced from the evaluations

$$\begin{aligned}
\left|\frac{\partial U_\delta(\xi)}{\partial \xi_i} - \chi_i(\xi)\right| &= \left|\int_\Omega \varrho(x)\left[\frac{\partial g_\delta(\xi)}{\partial \xi_i} - \frac{\partial}{\partial \xi_i}\left(\frac{1}{r^{n-2}}\right)\right]\mathrm{d}x\right| \\
&= \left|\int_{B(\xi,\delta)} \varrho(x)\left[\frac{\partial g_\delta(\xi)}{\partial \xi_i} - \frac{\partial}{\partial \xi_i}\left(\frac{1}{r^{n-2}}\right)\right]\mathrm{d}x\right| \\
&\le c\int_{B(\xi,\delta)} \left|\frac{\partial g_\delta(\xi)}{\partial \xi_i}\right| \mathrm{d}x + c\int_{B(\xi,\delta)} \left|\frac{\partial}{\partial \xi_i}\left(\frac{1}{r^{n-2}}\right)\right| \mathrm{d}x \\
&\le \frac{c(n-2)}{\delta^{n-2}}\int_{B(\xi,\delta)} \frac{r^{p-1}}{\delta^p}\mathrm{d}x + c\int_{B(\xi,\delta)} \frac{(2-n)|x_i-\xi_i|}{r^n}\mathrm{d}x \\
&\le \frac{c(n-2)}{\delta^{n-1}}\int_{B(\xi,\delta)} \mathrm{d}x + (n-2)c\int_{B(\xi,\delta)} \frac{1}{r^{n-1}}\mathrm{d}x \\
&\le \frac{c(n-2)\omega_n}{n}\delta + c(n-2)\omega_n\delta,
\end{aligned}$$

where c is a constant which bounds the following quantity from above:

$$\sup_{x\in B(\xi,\delta)} \varrho(x).$$

The other bounds from above are obvious. Based on the last inequality, it is deduced immediately that if $\delta \to 0$, we have

$$\frac{\partial U_\delta(\xi)}{\partial \xi_i} \to \chi_i(\xi),$$

the convergence being uniform with respect to ξ, because the last terms from the above evaluations do not depend on choice of ξ. With this, the theorem is proven. ■

According to the outline from the beginning of the paragraph, it only remains to determine the sufficient conditions for which the second derivatives of the Newtonian potential exist, under what conditions these second derivatives are continuous functions and which is the equation with partial derivatives of second order which is satisfied by the potential of volume. All these results are obtained in the following theorem.

Theorem 3.1.2 *Suppose that Ω is a bounded domain from $\mathbb{R}^n$, with the boundary $\partial\Omega$ having a tangent plane continuously varying almost everywhere. If the density ϱ is a function of class $C^0(\overline{\Omega}) \cap C^1(\Omega)$, and the derivatives $\partial\varrho/\partial x_i$ are bounded functions on Ω, for all indices $i = 1, 2, \ldots, n$, then*

(i) all derivatives $\partial^2 U(\xi)/\partial\xi_i^2$, $i = 1, 2, \ldots, n$ exist and are continuous;
(ii) $U(\xi)$ verifies the equation

$$\Delta_\xi U(\xi) = f(\xi) = -(n-2)\omega_n \varrho(\xi), \ \ \forall \xi \in \Omega.$$

Proof (i) We are in the hypotheses of Theorem 3.1.1 and thus have

$$\frac{\partial U(\xi)}{\partial \xi_i} = \int_\Omega \varrho(x) \frac{\partial}{\partial \xi_i}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}x.$$

Taking into account that

$$r = \sqrt{\sum_{i=1}^{n} (x_i - \xi_i)^2},$$

we take into account the symmetry in x and ξ of r and therefore write

$$\frac{\partial U(\xi)}{\partial \xi_i} = -\int_\Omega \varrho(x) \frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}x. \tag{3.1.5}$$

Because the density ϱ is of class C^1, we can write

$$\frac{\partial U(\xi)}{\partial \xi_i} = -\int_\Omega \frac{\partial}{\partial x_i}\left[\varrho(x)\frac{1}{r^{n-2}}\right] \mathrm{d}x + \int_\Omega \frac{\partial \varrho(x)}{\partial x_i}\frac{1}{r^{n-2}} \mathrm{d}x,$$

and after application of the Gauss–Ostrogradsky formula (which is allowed by the hypothesis on the boundary $\partial\Omega$)

$$\frac{\partial U(\xi)}{\partial \xi_i} = -\int_{\partial\Omega} \varrho(x)\frac{1}{r^{n-2}} \cos\alpha_i \mathrm{d}\sigma_x + \int_\Omega \frac{\partial \varrho(x)}{\partial x_i}\frac{1}{r^{n-2}} \mathrm{d}x. \tag{3.1.6}$$

The first term from the right-hand member of the formula (3.1.6) is a function of class C^∞ because it only depends on ξ only by means of $1/r^{n-2}$, and $r^{n-2} \neq 0$, because $x \in \partial\Omega$ and $\xi \in Int\Omega$. The second term from the right-hand member of the formula (3.1.6) is an improper integral, of the same type as the Newtonian potential, but instead of the density ϱ, $\partial\varrho/\partial x_i$ appears. Based on the hypotheses of the theorem, we deduce that $\partial\varrho/\partial x_i$ can play the role of the density ϱ, that is, we are in the hypotheses of Theorem 3.1.1. Therefore is allowed to differentiate with respect to ξ_i in the second integral from the right-hand side of the formula (3.1.6). Thus, we deduce that $\partial^2 U(\xi)/\partial\xi_i^2$ exists, and these derivatives are continuous functions.

(ii) By differentiating in (3.1.6), we obtain

$$\frac{\partial^2 U(\xi)}{\partial\xi_i^2} = -\int_{\partial\Omega} \varrho(x)\frac{\partial}{\partial\xi_i}\left(\frac{1}{r^{n-2}}\right)\cos\alpha_i \mathrm{d}\sigma_x + \int_\Omega \frac{\partial\varrho(x)}{\partial x_i}\frac{\partial}{\partial\xi_i}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x.$$

We now take into account the considerations on which the formula (3.1.5) is based, and then the above relation becomes

$$\frac{\partial^2 U(\xi)}{\partial\xi_i^2} = \int_{\partial\Omega} \varrho(x)\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\cos\alpha_i \mathrm{d}\sigma_x - \int_\Omega \frac{\partial\varrho(x)}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x. \quad (3.1.7)$$

We sum up the derivatives from (3.1.7), by all values of $i = 1, 2, \ldots, n$ and we take into account the definition of the derivative in the normal direction and the definition of the gradient

$$\Delta_\xi U(\xi) = \int_{\partial\Omega} \varrho(x)\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x - \int_\Omega \mathrm{grad}_x\ \varrho(x)\ \mathrm{grad}_x\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x. \quad (3.1.8)$$

The second integral from (3.1.8) is an improper integral, having a singular point for $x = \xi$. We will isolate the point ξ with a ball $B(\xi,\delta)$ and we work on the corona $\Omega \setminus B(\xi,\delta)$. So, we have

$$\begin{aligned}&\int_\Omega \mathrm{grad}_x\ \varrho(x)\ \mathrm{grad}_x\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x \\ &= \lim_{\delta\to 0}\int_{\Omega\setminus B(\xi,\delta)} \mathrm{grad}_x\ \varrho(x)\ \mathrm{grad}_x\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x \\ &= \lim_{\delta\to 0}\left\{\int_{\Omega\setminus B(\xi,\delta)} \varrho(x)\Delta_x\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x + \int_{\partial\Omega}\varrho(x)\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x + \right. \\ &\left. + \int_{\partial B(\xi,\delta)}\varrho(x)\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x\right\},\end{aligned}$$

in which we take into account the formula (2.4.11), from Chap. 2.

But in the corona $\Omega \setminus B(\xi, \delta)$, we have

$$\Delta_x \left(\frac{1}{r^{n-2}} \right) = 0,$$

and then the above formula becomes

$$\begin{aligned} &\int_{\Omega} \text{grad}_x \; \varrho(x) \; \text{grad}_x \left(\frac{1}{r^{n-2}} \right) \mathrm{d}x \\ &= \int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x + \lim_{\delta \to 0} \int_{\partial B(\xi,\delta)} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x, \end{aligned}$$

so that if we replace in (3.1.8), we obtain

$$\begin{aligned} \Delta_\xi U(\xi) &= \int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x - \int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x \\ &= - \lim_{\delta \to 0} \int_{\partial B(\xi,\delta)} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x, \end{aligned}$$

that is

$$\Delta_\xi U(\xi) = - \lim_{\delta \to 0} \int_{\partial B(\xi,\delta)} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x. \tag{3.1.9}$$

But

$$\begin{aligned} &\lim_{\delta \to 0} \int_{\partial B(\xi,\delta)} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x \\ &\quad = (n-2)\omega_n \lim_{\delta \to 0} \int_{\partial B(\xi,\delta)} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{(n-2)\omega_n} \frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x \\ &\quad = (n-2)\omega_n \varrho(\xi), \end{aligned}$$

the last part is obtained from the fact that

$$\frac{1}{(n-2)\omega_n} \frac{1}{r^{n-2}}$$

is a fundamental solution for the Laplace operator.

Also, we take into account the fact that the normal ν is outside to the corona $\Omega \setminus B(\xi, \delta)$, therefore inside with respect to the boundary $\partial\Omega$.

Then from (3.1.9), we deduce that

$$\Delta_\xi U(\xi) = -(n-2)\omega_n \varrho(\xi) = f(\xi), \;\; \forall \xi \in \Omega,$$

and the proof of the theorem is complete. ■

We now intend to prove that the results from Theorem 3.1.2 remain valid if we replace the condition that the density ϱ is continuous differentiable with the weaker condition that the density ϱ is a local Hőlder function on Ω.

We recall that if Ω is an open set from $\mathbb{R}^n$, a function $h : \Omega \to \mathbb{R}$ is *local Hőlder* on Ω if the constants $K > 0$ and $\alpha \in (0, 1]$ exist so that

$$|h(x) - h(\tilde{x})| \leq K\, |x\tilde{x}|^{\alpha}\,, \tag{3.1.10}$$

for any two points $x, \tilde{x} \in \Omega$. We denote by $|x\tilde{x}|$ the distance between the points x and $\tilde{x}$. The constant α is called *the Hőlder exponent* of the function h, and K is the *Hőlder constant* of the function h. We must outline that in the particular case when $\alpha = 1$ in (3.1.10) we have the Lipschitz condition.

The function h is *local uniformly Hőlder* on Ω if it satisfies the condition of Hőlder on any compact set from Ω. Therefore, α from (3.1.10) is the same for any compact set from Ω, and the constant K is the same irrespective of compact set from Ω.

Observation 3.1.1 *1°. If a function h is locally Hőlder on a compact set, then h is continuous on the respective compact set.*

2°. The condition of Hőlder is of interest for points x and $\tilde{x}$ close enough, and in this case, it can be seen that the restriction that a function h be Hőlder is weaker than the restriction that h be a Lipschitz function.

A classical result from mathematical analysis proves that any function h which is locally uniform Hőlder can be approximated, uniform on compact sets, with Hőlder functions h_p having the same Hőlder exponent as h, with functions h_p continuously differentiable. Usually, the approximate functions, h_p, are polynomial functions. For instance, if h is a local Hőlder function on Ω and in addition h is continuous on Ω, then h can be approximated by a sequence of polynomials $\{h_p\}_p$ of the form

$$h_p(x)=\frac{1}{c^n}\int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1}h(x)\prod_{i=1}^{n}\left[1-(x_i-\xi_i)^2\right]^p \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n, \tag{3.1.11}$$

where p is an integer number, $p \in \mathbb{N}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n)$ and the constant c is given by

$$c = \int_{-1}^{1} \left(1 - t^2\right) \mathrm{d}t.$$

We can verify, without difficulty, that the polynomials h_p from (3.1.11) have the properties:

(i) h_p are locally Hőlder functions on Ω with the same Hőlder exponent as the function h;

(ii) if $p \to \infty$, then the sequence $\{h_p\}_p$ converges to h, uniformly on compact sets from Ω;

(iii) the functions h_p are of classC^∞.

In the following theorem, we prove that the results from Theorems 3.1.1 and 3.1.2 remain valid even if we give up the very restrictive hypothesis that the density ϱ is a function of class C^1 and we introduce the hypothesis that ϱ is a locally Hőlder function.

Theorem 3.1.3 *Assume that Ω is a bounded domain from $\mathbb{R}^n$ with the boundary $\partial\Omega$ having a tangent plane continuously varying almost everywhere.*

If the density ϱ is a continuous function on $\overline{\Omega}$ and ϱ is locally uniform Hőlder on Ω, then

(i) the derivatives $\frac{\partial U(\xi)}{\partial \xi_i}$ exist and $\frac{\partial^2 U(\xi)}{\partial \xi_i^2}$ for $i = 1, 2, \ldots, n$ and these derivatives are continuous on Ω;

(ii) the Newtonian potential $U(\xi)$ satisfies the following Poisson equation:

$$\Delta_\xi U(\xi) = f(\xi) = -(n-2)\omega_n \varrho(\xi), \ \forall \xi \in \Omega.$$

Proof Let us observe, first, that if the density ϱ would be a continuously differentiable function, then the relation (3.1.7) could be written in the form

$$\begin{aligned}\frac{\partial^2 U(\xi)}{\partial \xi_i^2} &= \int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\cos\alpha_i \mathrm{d}\sigma_x \\ &- \int_\Omega \frac{\partial}{\partial x_i}\left[\varrho(x) - \varrho(\xi)\right]\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x,\end{aligned} \tag{3.1.12}$$

in which we take into account that $\varrho(\xi)$ is constant with respect to x.

The last integral from the right-hand side of the relation (3.1.12) can be transformed, by applying Gauss–Ostrogradsky's formula, as follows:

$$\begin{aligned}&\int_\Omega \frac{\partial}{\partial x_i}\left[\varrho(x) - \varrho(\xi)\right]\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x \\ &= \int_\Omega \frac{\partial}{\partial x_i}\left[\left[\varrho(x) - \varrho(\xi)\right]\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\right]\mathrm{d}x \\ &\qquad - \int_\Omega \left[\varrho(x) - \varrho(\xi)\right]\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x \\ &= \int_{\partial\Omega}\left[\varrho(x) - \varrho(\xi)\right]\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\cos\alpha_i \mathrm{d}\sigma_x \\ &\qquad - \int_\Omega \left[\varrho(x) - \varrho(\xi)\right]\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}x.\end{aligned}$$

We introduce this evaluation in (3.1.12) and we obtain

$$\frac{\partial^2 U(\xi)}{\partial \xi_i^2} = \varrho(\xi) \int_{\partial\Omega} \frac{\partial}{\partial x_i} \left(\frac{1}{r^{n-2}} \right) \cos \alpha_i \mathrm{d}\sigma_x + \int_{\Omega} [\varrho(x) - \varrho(\xi)] \frac{\partial^2}{\partial x_i^2} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}x. \tag{3.1.13}$$

We want to outline that (3.1.13) is, in fact, just an equivalent formulation of the formula (3.1.7), obtained using the hypothesis that the density ϱ is a function of class C^1, because at a given moment, in the deduction of the formula (3.1.7), it is essential that the density ϱ is of class C^1. On the other hand, at least formally, in (3.1.13), which is equivalent to (3.1.7), the condition that ϱ is of class C^1 is not needed. From here comes the idea to approximate the density ϱ, which was assumed to be locally uniform Hőlder on Ω, with polynomial functions ϱ_p, as in the preamble of the statement of Theorem 3.1.3.

In this sense, we define some Newtonian potentials, which are analogs of the potential $U(\xi)$, having as density the functions ϱ_p

$$U_p(\xi) = \int_{\Omega} \varrho_p(x) \frac{1}{r^{n-2}} \mathrm{d}x. \tag{3.1.14}$$

If we take into account that ϱ_p are polynomial functions, therefore at least of class C^1, it means that we are in the hypotheses of Theorem 3.1.2 and then for the Newtonian potentials U_p, we have a formula analogous to (3.1.13)

$$\frac{\partial^2 U_p(\xi)}{\partial \xi_i^2} = \varrho_p(\xi) \int_{\partial\Omega} \frac{\partial}{\partial x_i} \left(\frac{1}{r^{n-2}} \right) \cos \alpha_i \mathrm{d}\sigma_x + \int_{\Omega} \left[\varrho_p(x) - \varrho_p(\xi)\right] \frac{\partial^2}{\partial x_i^2} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}x, \; i = 1, 2, \ldots, n. \tag{3.1.15}$$

By analogy with the result from Theorem 3.1.1, we have

$$\frac{\partial U_p(\xi)}{\partial \xi_i} = \int_{\Omega} \varrho_p(\xi) \frac{\partial}{\partial \xi_i} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}x.$$

Now, we introduce the notation

$$v_i(\xi) = \varrho_p(\xi) \int_{\partial\Omega} \frac{\partial}{\partial x_i} \left(\frac{1}{r^{n-2}} \right) \cos \alpha_i \mathrm{d}\sigma_x + \int_{\Omega} \left[\varrho_p(x) - \varrho_p(\xi)\right] \frac{\partial^2}{\partial x_i^2} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}x, \; i = 1, 2, \ldots, n. \tag{3.1.16}$$

If we prove that

$$\lim_{p \to \infty} \frac{\partial^2 U_p(\xi)}{\partial \xi_i^2} = v_i(\xi), \tag{3.1.17}$$

the limit taking place uniformly with respect to ξ, on compact sets, we get that on any compact set from Ω, there is

$$\frac{\partial^2 U(\xi)}{\partial \xi_i^2}$$

and this derivative coincides with $v_i(\xi)$, for each i in part, $i = 1, 2, \ldots, n$, because in agreement with a theorem of classical analysis, the operator of differentiation is a closed operator. Thus, we prove that the uniform convergence from the relation (3.1.17) holds true.

From (3.1.15) and (3.1.16), we have

$$\left|\frac{\partial^2 U_p(\xi)}{\partial \xi_i^2} - v_i(\xi)\right| \leq \left|\varrho_p(\xi) - \varrho(\xi)\right| \int_{\partial\Omega} \left|\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\cos\alpha_i\right| \mathrm{d}\sigma_x$$
$$+ \int_{\Omega} \left|\varrho_p(x) - \varrho_p(\xi) - \varrho(x) + \varrho(x)\right| \left|\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\right| \mathrm{d}x. \quad (3.1.18)$$

It is clear that the surface integral from this formula is convergent because the integrand is a function of class C^∞, and therefore, we can write $\forall \varepsilon > 0,\ \exists p_0 = p_0(\varepsilon)$ so that for any $p > p_0(\varepsilon)$ we have

$$\left|\varrho_p(\xi) - \varrho(\xi)\right| \int_{\partial\Omega} \left|\frac{\partial}{\partial x_i}\left(\frac{1}{r^{n-2}}\right)\cos\alpha_i\right| \mathrm{d}\sigma_x \leq \frac{\varepsilon}{2}, \quad (3.1.19)$$

uniformly with respect to ξ, on compact sets from Ω.

For the volume integral from (3.1.18), we consider a ball $B(\xi, \delta)$, with δ small enough, and write the integral in the form

$$\int_{\Omega} \left|\varrho_p(x) - \varrho_p(\xi) - \varrho(x) + \varrho(x)\right| \left|\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\right| \mathrm{d}x = I_1 + I_2, \quad (3.1.20)$$

where

$$I_1 = \int_{\Omega \setminus B(\xi,\delta)} \left|\varrho_p(x) - \varrho_p(\xi) - \varrho(x) + \varrho(x)\right| \left|\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\right| \mathrm{d}x,$$
$$I_2 = \int_{B(\xi,\delta)} \left|\varrho_p(x) - \varrho_p(\xi) - \varrho(x) + \varrho(x)\right| \left|\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\right| \mathrm{d}x.$$

We have the estimate

$$I_1 \leq \int_{\Omega \setminus B(\xi,\delta)} \left[\left|\varrho_p(x) - \varrho(\xi)\right| + \left|\varrho_p(\xi) - \varrho(\xi)\right|\right] \left|\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\right| \mathrm{d}x \leq \frac{\varepsilon}{4},$$

because p is big enough and $\varrho_p \to \varrho$, for $p \to \infty$.

Also, the quantity

$$\left|\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\right|,$$

is bounded, because on the corona $\Omega \setminus B(\xi, \delta)$ we cannot have $x = \xi$.

Taking into account that the functions ϱ_p and ϱ are local Hőlder, for I_2 we have the evaluations

$$\begin{aligned} I_2 &\le \int_{B(\xi,\delta)} \left[\left|\varrho_p(x) - \varrho_p(\xi)\right| + |\varrho(x) - \varrho(\xi)|\right] \left|\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r^{n-2}}\right)\right| \mathrm{d}x \\ &\le K_0 \int_{B(\xi,\delta)} r^\alpha \left|\frac{(2-n)n}{r^n}\right| \mathrm{d}x \le K_1 \int_{B(\xi,\delta)} r^\alpha \frac{1}{\delta^n} \mathrm{d}x \\ &= K_1 \omega_n \int_0^\delta r^{\alpha-1} \mathrm{d}r = K_1 \omega_n \delta^\alpha. \end{aligned}$$

This inequality leads to the conclusion that $I_2 \to 0$, as $\delta \to 0$, because $\alpha \in (0, 1]$. We can thus, write that $I_2 \le \varepsilon/4$ and, because we also have $I_1 \le \varepsilon/4$, we deduce that the volume integral from (3.1.20) is not greater than $\varepsilon/2$. With this observation and taking into account (3.1.19), from (3.1.18) we deduce that the uniform convergence (3.1.17) holds true. ■

3.2 The Solid Angle

Consider a surface S with two faces which, conventionally, are called the outside face and the inside face, respectively. If the surface S is closed, then the part contained inside of the surface will be called the inside face and the other one, the outside face.

If S is a closed Jordan surface, that is, S is the homeomorphic image of a sphere, then we can apply Jordan's theorem, according to which the open set $\mathbb{R}^n \setminus S$ is a reunion of open disjoint components.

If S_1 is the sphere which corresponds to the closed Jordan surface S, then the open sets $\mathbb{R}^n \setminus S$ and $\mathbb{R}^n \setminus S_1$ have the same number of connected components.

In general, if K_1 and K_2 are two compact sets and homeomorphic sets, then the sets $\mathbb{R}^n \setminus K_1$ and $\mathbb{R}^n \setminus K_2$ can be decomposed in the same number of connected components. In the case of the sphere S_1, the set $\mathbb{R}^n \setminus S_1$ is a reunion of two connected components: the inside and the outside of the sphere, respectively. Therefore, using Theorem of Jordan, if S is a closed Jordan surface, then $\mathbb{R}^n \setminus S$ are two connected components which will be the two faces of the surface.

We can therefore reformulate Jordan's theorem: any closed Jordan surface has the property that after its elimination from the space $\mathbb{R}^n$, we obtain two connected components. The inside face will be the homeomorphic correspondent of the inside of the sphere, and the outside face is the component which "looks" to the unbounded component of the set $\mathbb{R}^n \setminus S_1$.

When the surface S is not closed, the two faces are established conventionally and will also be called the inside face and the outside face, respectively.

In the following, consider those surfaces S which have the following properties which we will call standard properties:

1^o. S has two faces;
2^o. S admits continuously varying tangent plane, in any of its points;
3^o. S is a union of the form

$$S = \bigcup_{k=1}^{m} S_k,$$

having the property that for $\xi \notin S$, the vector $\vec{\xi x}$ cuts the surface S in a single point, or S_k belongs fully to a cone with the top in ξ. Also, the surface S_k admits a continuously varying tangent plane in any its points.

Observation 3.2.1 *(i) If ξ "looks" at the inside face of the surface S, supposed be closed, then*

$$\angle\left(\vec{\xi x}, \vec{\eta x}\right) < \frac{\pi}{2}. \tag{3.2.1}$$

Now, we can make the convention that for a given surface, its inside face is the face for which (3.2.1) holds true. Consequently,

$$\cos\left(\vec{\xi x}, \vec{\eta x}\right) > 0 \Rightarrow \text{sign}\cos\left(\vec{\xi x}, \vec{\eta x}\right) = +1.$$

In these considerations, $\vec{\eta x}$ represents the outside normal to the surface S, in the point x.

(ii) If ξ "looks" at the outside face of the surface S, supposed to be closed, then

$$\angle\left(\vec{\xi x}, \vec{\eta x}\right) > \frac{\pi}{2} \Rightarrow \cos\left(\vec{\xi x}, \vec{\eta x}\right) < 0 \Rightarrow \text{sign}\cos\left(\vec{\xi x}, \vec{\eta x}\right) = -1.$$

Definition 3.2.1 Let S be a surface with the above standard properties and, in addition,

$$S = \bigcup_{l=1}^{m} S_l.$$

Consider a point $\xi \notin S$. We will call a *solid angle* under which a surface S_l can be seen from the point ξ, the area of the portion from the unit sphere $\partial B(\xi, 1)$ intercepted by the vector radius $\vec{\xi x}$, when x travels through the surface S_l.

The solid angle is denoted by $\omega(\xi, S_l)$ and the area from its definition is taken with the sign "+" in the case in which ξ looks at the inside face of the surface S_l and with the sign "−" when ξ looks at the outside face.

Observation 3.2.2 *Because $\xi \notin S$, we deduce that ξ can see the portion S_l of the surface S only either from the inside of the portion S_l, or from its outside.*

Denote by Σ_l the projection on the sphere $\partial B(\xi, 1)$ of the portion S_l, obtained with the help of the vector radius $\vec{\xi x}$,

$$\Sigma_l = Pr_{\partial B(\xi,1)} S_l, \ l = 1, 2, \ldots, m.$$

Then, the solid angle can be written in the form

$$\omega(\xi, S_l) = \int_{\Sigma_l} \operatorname{sign} \cos\left(\vec{\xi x}, \vec{\eta x}\right) \mathrm{d}\sigma_x. \tag{3.2.2}$$

Because the portions S_l only have surrounding curves in common, which are sets of null measure, we deduce that the solid angle receives the form

$$\omega(\xi, S_l) = \sum_{l=1}^{m} \int_{\Sigma_l} \operatorname{sign} \cos\left(\vec{\xi x}, \vec{\eta x}\right) \mathrm{d}\sigma_x = \int_{\Sigma} \operatorname{sign} \cos\left(\vec{\xi x}, \vec{\eta x}\right) \mathrm{d}\sigma_x, \tag{3.2.3}$$

where Σ is the projection on the sphere $\partial B(\xi, 1)$ of the whole surface S and the projection is carried out with the vector radius $\vec{\xi x}$.

Theorem 3.2.1 *If the surface S has the above standard properties and ξ is a point so that $\xi \notin S$, then*

$$\omega(\xi, S) = -\frac{1}{n-2} \int_S \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x. \tag{3.2.4}$$

Proof We connect the point ξ with an arbitrary point x of the surface S. Denote by K the lateral surface of the cone determined by the vector radius $\vec{\xi x}$. Also, we denote by Σ_R the portion intercept by this cone on the sphere $\partial B(\xi, R)$, with the center in the point ξ and having the radius R. We obtain a body delimited by the surfaces S, Σ_R and K. We denote this body by Ω and we have $\xi \notin \Omega$. According to one consequence of the formula of Green, if the function u is a harmonic function on the closed domain Ω, then

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma = 0,$$

where ν is the outside unit normal to the surface $\partial\Omega$.

Because $\xi \notin \Omega$, we deduce that $\xi \neq x, \ \forall x \in \Omega$, and then the function $1/r^{n-2}$ is harmonic on the domain Ω and according to the above reminded consequence of the formula of Green, we have

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x = 0,$$

that is

$$\int_{\Sigma_R} \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x + \int_S \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x + \int_K \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x = 0. \tag{3.2.5}$$

Taking into account the definition of the surface K, we have

$$\int_K \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x = 0,$$

because

$$\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) = (2-n)\frac{1}{r^{n-1}} \sum_{i=1}^{n} \frac{x_i - \xi_i}{r} \cos \alpha_i = (2-n)\frac{1}{r^{n-1}}\left(\vec{\xi x}, \vec{\nu_x}\right) = 0.$$

Indeed, on the surface K we have $\vec{\xi x} \perp \vec{\nu_x}$, and then the scalar product $\left(\vec{\xi x}, \vec{\nu_x}\right)$ is null.

The relation (3.2.5) holds true for any R and then we can take $R = 1$ and with the above considerations, (3.2.5) becomes

$$\int_{\Sigma_1} \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x + \int_S \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x = 0. \tag{3.2.6}$$

On the other hand,

$$\int_{\Sigma_1} \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x = (2-n) \int_{\Sigma_1} \frac{1}{r^{n-2}} \sum_{i=1}^{n} \frac{x_i - \xi_i}{r} \cos \alpha_i \mathrm{d}\sigma_x$$

$$-(2-n) \int_{\Sigma_1} \frac{1}{r^{n-2}} \sum_{i=1}^{n} \frac{x_i - \xi_i}{r} \frac{x_i - \xi_i}{r} \mathrm{d}\sigma_x = -(2-n) \int_{\Sigma_1} \frac{1}{r^{n-1}} \mathrm{d}\sigma_x,$$

because the normal is a unit vector and therefore

$$\sum_{i=1}^{n} \frac{x_i - \xi_i}{r} \frac{x_i - \xi_i}{r} = 1.$$

Using a convenient change of variables, we deduce that

$$\int_{\Sigma_1} \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x = (n-2) \int_{\Sigma_1} \frac{1}{r^{n-1}} \mathrm{d}\sigma_x$$

$$= (n-2) \int_{\Sigma_1} d\sigma_1 = (n-2)\omega(\xi, \Sigma_1) = (n-2)\omega(\xi, S). \tag{3.2.7}$$

If we replace (3.2.7) in (3.2.6), we obtain

$$(n-2)\omega(\xi, S) + \int_S \frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) d\sigma_x = 0,$$

from where the formula (3.2.4) is derived and the proof of the theorem is complete. ■

3.3 The Double Layer Potential

Definition 3.3.1 We call the following integral the double layer potential of the surface:

$$W(\xi) = -\frac{1}{(n-2)\omega_n}\int_{\partial\Omega} u(x)\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) d\sigma_x, \tag{3.3.1}$$

where ω_n is the area of the unit sphere from the space $\mathbb{R}^n$.

We use the notation

$$\frac{1}{\omega_n}u(x) = \varrho(x)$$

and the function $\varrho(x)$ will be called the density of the double layer potential. We can therefore rewrite the double layer potential in the form

$$W(\xi) = -\frac{1}{n-2}\int_{\partial\Omega} \varrho(x)\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) d\sigma_x. \tag{3.3.2}$$

If we avoid the conditions imposed on the surface $\partial\Omega$ and we differentiate formally under the integral sign in (3.3.2), we obtain

$$\begin{aligned} \Delta_\xi W(\xi) &= -\frac{1}{n-2}\int_{\partial\Omega} \varrho(x)\Delta_\xi\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right) d\sigma_x \\ &= -\frac{1}{n-2}\int_{\partial\Omega} \varrho(x)\frac{\partial}{\partial \nu_x}\left(\Delta_\xi\left(\frac{1}{r^{n-2}}\right)\right) d\sigma_x = 0, \end{aligned}$$

and this relation holds true for $\forall\xi \in \Omega$ and for $\forall\xi \in \mathbb{R}^n \setminus \overline{\Omega}$.

Therefore, the double layer potential satisfies the Laplace equation, and therefore, it is a harmonic function, on the inside and also on the outside of Ω.

Let us analyze what conditions must be imposed on the boundary $\partial\Omega$ so that the differentiation under the previous integral is rigorous.

Definition 3.3.2 The surface S is called a Lyapunov surface if it satisfies the following conditions:

(i) S admits the tangent plane for any of its points;

(ii) $\exists a_0 > 0$ so that for $\forall x \in S$, the ball $B(x, a_0)$ has the property that the parallel in any of its point to the normal ν intersects the portion $B(x, a_0) \cap S$ in a single point;

(iii) The versor of the normal in any point of the ball $B(x, a_0)$ is a uniformly Hőlder function, that is

$$\forall x_1, x_2 \in B(x, a_0)\ \exists A > 0,\ \alpha \in (0, 1) \text{ so that } \left|\vec{\nu_{x_1}} - \vec{\nu_{x_2}}\right| \le A|\vec{x_1 x_2}|^{\alpha};$$

(iv) The solid angle under which any portion from the surface S is seen, in an arbitrary point, is a bounded function.

Proposition 3.3.1 *The double layer potential can be written in the form*

$$W(\xi) = \int_{\partial\Omega} \varrho(x) \frac{\cos\varphi}{r^{n-1}} \mathrm{d}\sigma_x, \tag{3.3.3}$$

where we denote by φ the angle between the vector radius $\vec{\xi x}$ and the outside normal $\vec{\nu_x}$, in the point x, to the surface $\partial\Omega$.

Proof We take into account the definition of the derivative in the direction of the normal and we obtain

$$-\frac{1}{n-2}\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right) = \frac{1}{r^{n-2}}\sum_{i=1}^{n}\frac{x_i - \xi_i}{r}\cos\alpha_i = \frac{1}{r^{n-1}}\cos\varphi$$

because

$$\sum_{i=1}^{n}\frac{x_i - \xi_i}{r}\cos\alpha_i$$

is the scalar product between the unit vectors $\vec{\xi x}$ and $\vec{\nu_x}$. ■

Theorem 3.3.1 *Let Ω be a bounded domain from the space $\mathbb{R}^n$ whose boundary $\partial\Omega$ is a closed Lyapunov surface. If the density $\varrho(x)$ is a bounded function on $\partial\Omega$, then the double layer potential from (3.3.1) is well defined.*

Proof We must show that the integral from the definition of the double layer potential is convergent.

We arbitrarily fix the point $\xi \in \partial\Omega$ which will be considered the origin O of a system of coordinates. We then take the tangent plane to the boundary $\partial\Omega$ in the point $\xi \equiv O$.

The following considerations are related to the ball $B(0, a_1) = B(\xi, a_1)$, where

$$a_1 = \min\left\{a_0, \left(\frac{1}{nA}\right)^{1/\alpha}\right\}, \tag{3.3.4}$$

where A and α are the constants occurring in the property (iii) of a Lyapunov surface, and n is the dimension of the space, $n \geq 3$.

Based on property (ii) of a Lyapunov surface, we can represent our surface in a neighborhood of the point ξ in the form

$$x_n = x_n(x_1, x_2, \ldots, x_{n-1}). \tag{3.3.5}$$

Given the fact that the inside normal is $0\vec{x}_n$, we have

$$\cos\varphi = \sum_{i=1}^{n} \frac{x_i - \xi_i}{r} \cos\left(\vec{\nu}_x, 0\vec{x}_i\right) = \sum_{i=1}^{n} \frac{x_i}{r} \cos\left(\vec{\nu}_x, 0\vec{x}_i\right),$$

because $\xi \equiv O$ and then $\xi_i = 0,\ i = 1, 2, \ldots, n.$

But

$$\left|\cos\left(\vec{\nu}_x, 0\vec{x}_i\right)\right| = |(\vec{\nu}_x, \vec{e}_i)| = \left|\left(\vec{\nu}_x - \vec{\nu}_\xi, \vec{e}_i\right) + \left(\vec{\nu}_\xi, \vec{e}_i\right)\right|$$

$$= \left|\left(\vec{\nu}_x - \vec{\nu}_\xi, \vec{e}_i\right)\right| \leq \left|\vec{\nu}_x - \vec{\nu}_\xi\right| |\vec{e}_i| = \left|\vec{\nu}_x - \vec{\nu}_\xi\right| \leq A \left|\vec{\xi x}\right|^\alpha = Ar^\alpha, \tag{3.3.6}$$

in which we used the inequality of Cauchy–Schwartz and the fact that the normal is a Hőlder function. We denoted by $\vec{e}_i$ the versors of the axes Ox_i.

On the other hand, we have

$$\left|\cos\left(\vec{\nu}_x, 0\vec{x}_n\right)\right| = |(\vec{\nu}_x, \vec{e}_n)| = \left|\left(\vec{\nu}_x - \vec{\nu}_\xi, \vec{e}_n\right) + \left(\vec{\nu}_\xi, \vec{e}_n\right)\right|$$

$$= \left|\left(\vec{\nu}_x - \vec{\nu}_\xi, \vec{e}_n\right) - 1\right| \geq 1 - \left|\left(\vec{\nu}_x - \vec{\nu}_\xi, \vec{e}_n\right)\right| \tag{3.3.7}$$

$$\geq 1 - \left|\vec{\nu}_x - \vec{\nu}_\xi\right| |\vec{e}_n| = 1 - \left|\vec{\nu}_x - \vec{\nu}_\xi\right| \geq Ar^\alpha,$$

in which we used, again, the inequality of Cauchy–Schwartz and the fact that the normal is a Hőlder function. Also, $\vec{e}_n$ is versor and therefore $|\vec{e}_n| = 1$.

We should note the fact that

$$r \leq a_1 < \left(\frac{1}{nA}\right)^{1/\alpha},$$

and this involves

$$r^\alpha \leq \frac{1}{nA} \Rightarrow Ar^\alpha \leq \frac{1}{n}.$$

From (3.3.6), we deduce

$$\left|\cos\left(\vec{\nu}_x, \vec{0x_i}\right)\right| \le Ar^{\alpha},\ i = 1, 2, \ldots, n,$$

and from (3.3.7)

$$\left|\cos\left(\vec{\nu}_x, \vec{0x_n}\right)\right| \ge 1 - Ar^{\alpha} \ge 1 - \frac{1}{n}. \tag{3.3.8}$$

Taking into account that the surface has a Cartesian representation, we have

$$\left|\frac{\partial x_n(x_1, x_2, \ldots, x_{n-1})}{\partial x_i}\right| = \left|-\frac{\cos\left(\vec{\nu}_x, \vec{0x_i}\right)}{\cos\left(\vec{\nu}_x, \vec{0x_n}\right)}\right| \le \frac{Ar^{\alpha}}{1 - Ar^{\alpha}} \le \frac{nAr^{\alpha}}{n-1}, \tag{3.3.9}$$

in which we used the equality (3.3.8).

We now introduce the notation

$$r_1 = \sqrt{\sum_{i=1}^{n-1}(x_i - \xi_i)^2} = \sqrt{\sum_{i=1}^{n-1} x_i^2},$$

where we take into account the fact that $\xi \equiv 0$.

Considering that

$$r = \sqrt{\sum_{i=1}^{n}(x_i - \xi_i)^2} = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{r_1^2 + x_n^2},$$

we deduce that $r_1 < r$.

We intend to prove that in the ball $B(\xi, 1) = B(0, 1)$, we have $r < 2r_1$. With the help of the formula of finite increases, we obtain the evaluations

$$|x_n(x_1, x_2, \ldots, x_{n-1})| = \left|x_n(0, 0, \ldots, 0) + \sum_{i=1}^{n-1}\frac{\partial x_n(x_1^*, x_2^*, \ldots, x_{n-1}^*)}{\partial x_i}x_i\right|$$
$$\le \sum_{i=1}^{n-1}\left|\frac{\partial x_n(x_1^*, x_2^*, \ldots, x_{n-1}^*)}{\partial x_i}\right||x_i|,$$

because $x_n(0, 0, \ldots, 0) = 0$.

Furthermore, based on inequality (3.3.9), we have

$$|x_n(x_1, x_2, \ldots, x_{n-1})| \le \frac{n}{n-1}Ar^{\alpha}\sum_{i=1}^{n-1}|x_i| \le nAr^{\alpha}r_1 \le r_1. \tag{3.3.10}$$

With the help of equality (3.3.10), we can strengthen the inequalities (3.3.6) and (3.3.8). Thus, from (3.3.6) and (3.3.10), we deduce

$$\left|\cos\left(\vec{\nu_x}, 0\vec{x}_n\right)\right| \le 2Ar_1^{\alpha},\ i = 1, 2, \ldots, n, \tag{3.3.11}$$

and from (3.3.8) and (3.3.10)

$$\left|\cos\left(\vec{\nu_x}, 0\vec{x}_n\right)\right| \ge 1 - 2Ar_1^{\alpha}. \tag{3.3.12}$$

We take σ a portion of the surface around of the point ξ, comprised in the ball $B(\xi, a_1)$ and write the double layer potential as follows:

$$W(\xi) = \int_{\partial\Omega} \varrho(x)\frac{\cos\varphi}{r^{n-1}}\mathrm{d}\sigma_x$$

$$= \int_{\partial\Omega\setminus\sigma} \varrho(x)\frac{\cos\varphi}{r^{n-1}}\mathrm{d}\sigma_x + \int_{\sigma} \varrho(x)\frac{\cos\varphi}{r^{n-1}}\mathrm{d}\sigma_x, \tag{3.3.13}$$

in which we take into account formula (3.3.3) from Proposition 3.3.1. Denote by $I(\xi)$ the last integral from (3.3.13). We intend to show that $I(\xi) \to 0$. Because we used the Cartesian representation, we can transform the surface integral into a $(n-1)$-multiple integral. Above, we considered the tangent plane in the origin to the surface $\partial\Omega$. Now, we consider the geometric projection of the surface σ on this plane, denote by D this projection and we will obtain the multiple integral on D

$$I(\xi) = \int_{\sigma} \varrho(x)\frac{\cos\varphi}{r^{n-1}}\mathrm{d}\sigma_x = \int_{D} \varrho(x)\frac{\cos\varphi}{r^{n-1}}\frac{\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_{n-1}}{\cos\left(\vec{\nu_x}, 0\vec{x}_n\right)},$$

so that we have the evaluations

$$|I(\xi)| \le c_0\int_D \frac{|\cos\varphi|}{r^{n-1}}\frac{\mathrm{d}x_1\ldots\mathrm{d}x_{n-1}}{|\cos(\vec{\nu_x}, 0\vec{x}_n)|} \le \frac{nc_0}{n-1}\int_D \frac{|\cos\varphi|}{r^{n-1}}\mathrm{d}x_1\ldots\mathrm{d}x_{n-1}, \tag{3.3.14}$$

in which we take into account the equality (3.3.8).

Because

$$\cos\varphi = \sum_{i=1}^{n}\frac{x_i}{r}\cos\left(\vec{\nu_x}, 0\vec{x}_i\right) =$$

$$= \sum_{i=1}^{n-1}\frac{x_i}{r}\cos\left(\vec{\nu_x}, 0\vec{x}_i\right) + \frac{x_n}{r}\cos\left(\vec{\nu_x}, 0\vec{x}_n\right),$$

we can use the inequalities (3.3.10) and (3.3.11) so that we get

$$|\cos\varphi| \le \sum_{i=1}^{n-1}\left|\cos\left(\vec{\nu}_x, 0\vec{x}_i\right)\right| + \frac{|x_n|}{r}$$
$$\le 2(n-1)Ar_1^{\alpha} + \frac{2n}{r}Ar^{\alpha} \le 2(n-1)Ar_1^{\alpha} + \frac{n}{r_1}Ar^{\alpha+1} \le c_3 r_1^{\alpha}. \quad (3.3.15)$$

If we substitute (3.3.15) into (3.3.14) and keep in mind that $r_1 < r$, we are led to the inequality

$$|I(\xi)| \le c_4 \int_D \frac{r_1^{\alpha}}{r^{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \le c_4 \int_D \frac{r_1^{\alpha}}{r_1^{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1}$$

$$\le c_4\omega_n \int_0^{a_1} \frac{r_1^{\alpha} r_1^{n-2}}{r_1^{n}} \mathrm{d}r_1 = c_4\omega_{n-1} \int_0^{a_1} r_1^{\alpha-1} \mathrm{d}r_1 = c_4 \frac{a_1^{\alpha}}{\alpha},$$

and then, clearly if $a_1 \to 0$, we have $I(\xi) \to 0$.

In conclusion, if in (3.3.13) we take $\sigma = \partial B(\xi, a_1)$ and we pass to the limit with $a_1 \to 0$ (that is, the surface $\partial B(\xi, a_1)$ is deformed homothetic to 0), taking into account that $I(\xi) \to 0$, we obtain that the improper integral, which is defined as the double layer potential, is convergent. ■

In the sequel, we will address the problem of the values of the double layer potential in points from outside the domain Ω and in points from the inside of Ω, in the sense of boundary values.

Consider, as usual, the bounded domain Ω whose boundary $\partial\Omega$ is supposed to be a closed Lyapunov surface.

We have seen that if the density ϱ is a bounded function, then the values of the double layer potential are defined in any point $\xi_0 \in \partial\Omega$. Consider now a sequence $\{\xi_e\}$ of points from $\mathbb{R}^n \setminus \overline{\Omega}$ and a sequence $\{\xi_i\}$ of points from inside the domain Ω.

If the limit below exists and is finite

$$\lim_{\xi_e \to \xi_0} W(\xi_e),$$

for any sequence $\{\xi_e\} \subset \mathbb{R}^n \setminus \overline{\Omega}$, then we denote the value of this limit by $W_e(\xi_0)$, that is

$$W_e(\xi_0) = \lim_{\xi_e \to \xi_0} W(\xi_e).$$

Analogously, if the limit below exists and is finite

$$\lim_{\xi_i \to \xi_0} W(\xi_i),$$

for any sequence $\{\xi_i\} \subset \Omega$, then we denote the value of this limit with $W_i(\xi_0)$, that is

$$W_i(\xi_0) = \lim_{\xi_i \to \xi_0} W(\xi_i).$$

We now approach the problem of finding the conditions in which the two limits exist and which is their relation to the value of the double layer potential in the point $\xi_0 \in \partial\Omega$.

Theorem 3.3.2 *If Ω is a bounded domain from $\mathbb{R}^n$ whose boundary $\partial\Omega$ is a closed Lyapunov surface and the density ϱ is a continuous function on $\partial\Omega$, then the boundary values $W_e(\xi_0)$ and $W_i(\xi_0)$ exists, for any point $\xi_0 \in \partial\Omega$.*

Moreover, the following jump formulas hold true:

$$\begin{aligned} W_i(\xi_0) &= W(\xi_0) + \frac{\omega_n}{2}\varrho(\xi_0), \\ W_e(\xi_0) &= W(\xi_0) - \frac{\omega_n}{2}\varrho(\xi_0). \end{aligned} \tag{3.3.16}$$

Proof We arbitrarily fix $\xi_0 \in \partial\Omega$ and write the double layer potential in the form (3.3.2)

$$\begin{aligned} W(\xi) = &-\frac{1}{n-2}\int_{\partial\Omega}[\varrho(x) - \varrho(\xi)]\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x \\ &-\frac{\varrho(\xi_0)}{n-2}\int_{\partial\Omega}\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x. \end{aligned} \tag{3.3.17}$$

For $\xi \notin \partial\Omega$, taking into account Theorem 3.2.1 from the solid angle, we can write

$$-\frac{1}{n-2}\int_{\partial\Omega}\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x = \omega(\xi, \partial\Omega)$$

and then we can write (3.3.17) in the form

$$W(\xi) = I(\xi, \xi_0) + \varrho(\xi_0)\omega(\xi, \partial\Omega), \tag{3.3.18}$$

in which we used the notation

$$I(\xi, \xi_0) = -\frac{1}{n-2}\int_{\partial\Omega}[\varrho(x) - \varrho(\xi_0)]\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x.$$

We intend to show that $I(\xi, \xi_0)$ is a continuous function, as a function of ξ, both for points from outside of Ω and also for points inside of Ω. When ξ takes values from the outside of Ω, the statement is obvious.

Consider the ball $B(\xi, a_1)$, where the radius a_1 is given by

$$a_1 = \min\left\{a_0, \left(\frac{1}{nA}\right)^{1/\alpha}\right\},$$

where the constants a_0, A and α are those from the definition of the Lyapunov surface, and n is the dimension of the space ($\Omega \subset \mathbb{R}^n, n \geq 3$).

Then for the surface σ, given by

$$\sigma = \partial\Omega \cap B(\xi, a_1)$$

we will benefit from the Cartesian representation of the surface and, also, from the evaluations of Theorem 3.3.1.

We write $I(\xi, \xi_0)$ in the form

$$I(\xi, \xi_0) = -\frac{1}{n-2}\int_{\partial\Omega\setminus\sigma} [\varrho(x) - \varrho(\xi_0)]\,\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x$$

$$-\frac{1}{n-2}\int_{\sigma} [\varrho(x) - \varrho(\xi_0)]\,\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x. \tag{3.3.19}$$

For ξ close enough to ξ_0, ξ is situated in σ, and therefore, the first integral from the right-hand side of the formula is of class C^∞. Then, to prove the continuity of the function $I(\xi, \xi_0)$, it is sufficient to prove that the last integral from (3.3.19) is uniformly convergent to zero, when $a_1 \to 0$.

To this end, we use the evaluations

$$\left|-\frac{1}{n-2}\int_{\sigma} [\varrho(x) - \varrho(\xi_0)]\,\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x\right|$$

$$\leq \frac{1}{n-2}\int_{\sigma} |\varrho(x) - \varrho(\xi_0)|\left|\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\right|\mathrm{d}\sigma_x$$

$$\leq \frac{\varepsilon}{n-2}\int_{\sigma}\left|\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\right|\mathrm{d}\sigma_x$$

$$= \varepsilon\left[-\frac{1}{n-2}\int_{\sigma_N}\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x - \left(-\frac{1}{n-2}\right)\int_{\sigma_P}\frac{\partial}{\partial\nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x\right],$$

in which we use the notations

$$\sigma_N = \left\{ y \in \sigma : \ \text{sign} \frac{\partial}{\partial \nu_y} \left(\frac{1}{r^{n-2}} \right) = +1 \right\},$$
$$\sigma_P = \left\{ y \in \sigma : \ \text{sign} \frac{\partial}{\partial \nu_y} \left(\frac{1}{r^{n-2}} \right) = -1 \right\}.$$

The last evaluations together with the last condition from the definition of a Lyapunov surface ensures the conclusion

$$\left| -\frac{1}{n-2} \int_\sigma [\varrho(x) - \varrho(\xi_0)] \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}} \right) \mathrm{d}\sigma_x \right| \le 2K\varepsilon,$$

and this leads to the conclusion that the last integral from (3.3.19) is uniformly convergent to zero, and therefore, $I(\xi, \xi_0)$ is a continuous function, as a function of variable ξ.

We will come back to (3.3.18) and write the value of the double layer potential for the sequences of points $\{\xi_i\}$ and $\{\xi_e\}$, respectively,

$$\begin{aligned} W(\xi_i) &= I(\xi_i, \xi_0) + \varrho(\xi_0)\omega(\xi_i, \partial\Omega) = I(\xi_i, \xi_0) + \varrho(\xi_0)\omega_n, \\ W(\xi_e) &= I(\xi_e, \xi_0) + \varrho(\xi_0)\omega(\xi_e, \partial\Omega) = I(\xi_i, \xi_0), \end{aligned} \tag{3.3.20}$$

in which we take into account that the solid angle under which the surface $\partial\Omega$ is seen from the outside is null.

Considering the continuity of the function $I(\xi_i, \xi_0)$, we deduce that there exist the limits

$$\lim_{\xi_i \to \xi_0} I(\xi_i, \xi_0), \ \lim_{\xi_e \to \xi_0} I(\xi_e, \xi_0)$$

and then we can pass to the limit in $(3.3.20)_1$:

$$W_i(\xi_0) = \lim_{\xi_i \to \xi_0} W(\xi_i) = \lim_{\xi_i \to \xi_0} I(\xi_i, \xi_0) + \varrho(\xi_0)\omega_n. \tag{3.3.21}$$

Analogously, by passing to the limit in $(3.3.20)_2$:

$$W_e(\xi_0) = \lim_{\xi_e \to \xi_0} W(\xi_e) = \lim_{\xi_e \to \xi_0} I(\xi_e, \xi_0). \tag{3.3.22}$$

We intend to prove the formula

$$W(\xi_0) = I(\xi_0, \xi_0) + \varrho(\xi_0)\frac{\omega_n}{2}. \tag{3.3.23}$$

Essentially, the proof of the formula (3.3.23) is reduced to proving that

$$\omega(\xi_0, \partial\Omega) = \frac{\omega_n}{2}, \ \xi_0 \in \partial\Omega.$$

Denote by Γ the curve given by the intersection of the surfaces $\partial\Omega$ and $\partial B(\xi_0, a_1)$

$$\Gamma = \partial\Omega \cap \partial B(\xi_0, a_1).$$

We denote by σ_1 the spherical calotte from $\partial B(\xi_0, a_1)$ delimited by Γ and located in the inside of Ω and also we denote by σ_2 the spherical calotte of the sphere $\partial B(\xi_0, a_1)$ delimited by Γ and located outside of Ω.

Now, we denote by Σ_1 the portion from the surface $\partial\Omega$ located outside of the ball $\partial B(\xi_0, a_1)$.

Then, it is obvious that

$$\omega(\xi_0, \Sigma_1) = \omega(\xi_0, \sigma_1). \tag{3.3.24}$$

Moreover, it is normal to consider by definition

$$\omega(\xi_0, \partial\Omega) = \lim_{a_1 \to 0} \omega(\xi_0, \Sigma_1) = \lim_{a_1 \to 0} \omega(\xi_0, \sigma_1). \tag{3.3.25}$$

The altitude of the curve Γ in relation to the equator of the ball $B(\xi, a_1)$ is exactly x_n from the surface $x_n = x_n(x_1, x_2, \ldots, x_{n-1})$.

As we have already seen in the evaluations from the proof of Theorem 3.3.1, we have

$$|x_n| \le c_0 r_1^{\alpha+1}. \tag{3.3.26}$$

It is known that the ratio between the area of a sphere and the area of a spherical area is equal to the ratio between the diameter of the sphere and the height of the respective spherical area. In our case, we have the ratio $x_n/(2r)$.

In Theorem 3.3.1, we proved that $r < 2r_1$ and taking into account (3.3.26), we get

$$\frac{|x_n|}{2r} \le c_1 r_1^{\alpha+1},$$

and because $\alpha \in (0, 1)$, we deduce that the ratio tends to zero, for $r_1 \to 0$, and this ensures the convergence $B(\xi_0, a_1) \to \xi_0$.

Also, the curve Γ tends to the equator, the last term from (3.3.25) tends to the area of a hemisphere, because σ_1 tends to a half sphere. Thus, the last term from (3.3.25) tends to $\omega_n/2$, and therefore, we get

$$\omega(\xi_0, \partial\Omega) = \lim_{a_1 \to 0} \omega(\xi_0, \Sigma_1) = \frac{\omega_n}{2}.$$

This formula together with (3.3.17) leads to formula (3.3.23). If we subtract term by term the formulas (3.3.23) and (3.3.21), we obtain formula $(3.3.16)_1$, and if we subtract term by term the formulas (3.3.23) and (3.3.22), we obtain formula $(3.3.16)_2$ and the proof of the theorem is complete. ∎

From the evaluations made in the proofs of Theorems 3.3.1 and 3.3.1, it is certified that we cannot define independently the notion of the solid angle under which a closed surface is seen from a point located on the respective surface. This impossibility can be explained by the fact that all the considerations are based on evaluations which are only valid for Lyapunov surfaces.

We finish this paragraph with some considerations on the behavior to infinity of the double layer potential.

Consider again that the boundary $\partial\Omega$ of the domain Ω is a closed Lyapunov surface. Clearly, $\overline{\Omega}$ is a compact set. For x arbitrarily fixed in $\overline{\Omega}$ and for any $\xi \notin \overline{\Omega}$, based on the triangle inequality we have

$$r = \left|\vec{\xi x}\right| \geq \left|\left|\vec{0\xi}\right| - \left|\vec{0x}\right|\right| = \left|\vec{0\xi}\right| \left|1 - \frac{\left|\vec{0x}\right|}{\left|\vec{0\xi}\right|}\right|. \tag{3.3.27}$$

We can choose ξ so that

$$\left|\vec{0\xi}\right| \geq 2 \sup_{x \in \overline{\Omega}} \left|\vec{0x}\right|,$$

so that from (3.3.27) we deduce

$$r \geq \frac{\left|\vec{0\xi}\right|}{2}.$$

Then, based on the definition of the double layer potential, we have

$$|W(\xi)| = \left|-\frac{1}{n-2} \int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r^{n-2}}\right) \mathrm{d}\sigma_x\right|$$

$$\leq c_0 \int_{\partial\Omega} \frac{1}{r^{n-1}} \mathrm{d}\sigma_x \leq \frac{2^{n-1}}{\left|\vec{0\xi}\right|^{n-1}} c_0 \int_{\partial\Omega} \mathrm{d}\sigma_x = \frac{c_1}{\left|\vec{0\xi}\right|^{n-1}}.$$

In conclusion, as far as the behavior to infinity of the potential of double layer is concerned we have the evaluation

$$|W(\xi)| \leq \frac{c_1}{\left|\vec{0\xi}\right|^{n-1}}.$$

3.4 The Single Layer Potential

We want to outline, in the beginning of this section, that in the approach of the single layer potential of surface, we will use analogous techniques to those used in the study of the double layer potential.

Definition 3.4.1 By definition, the following integral is called the potential of the surface of a single layer:

$$V(\xi) = \frac{1}{(n-2)\omega_n} \int_{\partial\Omega} \frac{1}{r^{n-2}} \frac{\partial u(x)}{\partial \nu_x} \mathrm{d}\sigma_x. \tag{3.4.1}$$

We introduce the notation

$$\varrho(x) = \frac{1}{\omega_n} \frac{\partial u(x)}{\partial \nu_x},$$

where the function $\varrho(x)$ will be called the density of the single layer potential.

Then, the single layer potential can be written in the form

$$V(\xi) = \frac{1}{n-2} \int_{\partial\Omega} \frac{\varrho(x)}{r^{n-2}} \mathrm{d}\sigma_x. \tag{3.4.2}$$

Proposition 3.4.1 *Suppose that the density $\varrho(x)$ is an integrable function on $\partial\Omega$. Then, the single layer potential satisfies the Laplace equation (therefore, $\varrho(x)$ is a harmonic function), both outside and also inside the domain Ω*

$$\Delta_\xi V(\xi) = 0, \ \forall \xi \in \Omega, \ \forall \xi \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Proof The result is obtained immediately if we take into account that for points $\xi \in \Omega$ and also for points $\xi \in \mathbb{R}^n \setminus \overline{\Omega}$, the function $1/r^{n-2}$ is of class C^∞. We can work on an arbitrary compact set containing ξ and derive formally under the integral sign and take into account that the function $1/r^{n-2}$ is harmonic and the singularity for $x = \xi$ is avoided because $x \in \partial\Omega$ and $\xi \notin \partial\Omega$. ■

In the following theorem, we prove the well definiteness of the potential of single layer, in the sense that the improper integral which defines this potential is convergent.

Theorem 3.4.1 *Let Ω be a bounded domain from $\mathbb{R}^n$ whose boundary $\partial\Omega$ is a closed Lyapunov surface. If the density ϱ is a bounded function on $\partial\Omega$, then the single layer potential is well defined and is a continuous function on the whole space $\mathbb{R}^n$.*

Proof We arbitrarily fix $\xi \in \partial\Omega$ and we take a system of coordinates having the origin in the point ξ. Consider the ball $B(\xi, a_1)$ with radius a_1 given by

$$a_1 = \min\left\{ a_0, \left(\frac{1}{nA}\right)^{1/\alpha} \right\},$$

in which the constants a_0, A and α appear in the definition of the Lyapunov surface, and n is the dimension of the space, $n \geq 3$.

From the properties of a Lyapunov surface, inside the ball $B(\xi, a_1)$ we can represent our surface in a Cartesian way, in the form

$$x_n = x_n(x_1, x_2, \ldots, x_{n-1}).$$

We also take the tangent plane to the surface $\partial\Omega$ in the point $\xi \equiv O$ and $0\vec{x}_n$, the normal inside $\partial\Omega$ in $\xi \equiv O$.

Denote by σ the portion from the surface $\partial\Omega$ intercepted by the ball $B(\xi, a_1)$

$$\sigma = \partial\Omega \cap B(\xi, a_1),$$

and write the single layer potential in the form

$$V(\xi) = \frac{1}{n-2}\int_{\partial\Omega\setminus\sigma} \frac{\varrho(x)}{r^{n-2}}\mathrm{d}\sigma_x + \frac{1}{n-2}\int_{\sigma} \frac{\varrho(x)}{r^{n-2}}\mathrm{d}\sigma_x. \tag{3.4.3}$$

Then, the proof that the improper integral which defines the single layer potential is convergent is reduced to proving that the last integral from (3.4.3) is convergent, uniform with respect to ξ, to zero, when $a_1 \to 0$.

To this end, we have the evaluations which follow. Due to Cartesian representation, we can transform the surface integral in a $(n-1)$- multiple integral on the domain D which is the projection of the surface σ of the tangent plane in $\xi \equiv O$ to the surface $\partial\Omega$. We have

$$\left|\frac{1}{n-2}\int_{\sigma} \frac{\varrho(x)}{r^{n-2}}\mathrm{d}\sigma_x\right| = \left|\frac{1}{n-2}\int_{D} \frac{\varrho(x)}{r^{n-2}}\frac{\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n}{\cos(\vec{\nu}_x, 0\vec{x}_n)}\right|$$

$$\leq \frac{nc_0}{(n-2)(n-1)}\int_{D} \frac{1}{r^{n-2}}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n,$$

where we used the bounds from above $|\varrho(x)| \leq c_0$ and $\left|\cos\left(\vec{\nu}_x, 0\vec{x}_n\right)\right| \geq (n-1)/n$. Taking into account that

$$r \geq r_1 = \sqrt{\sum_{i=1}^{n-1} x_i^2}$$

we deduce that

$$\left|\frac{1}{n-2}\int_{\sigma} \frac{\varrho(x)}{r^{n-2}}\mathrm{d}\sigma_x\right| \leq c_1\int_{D} \frac{\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n}{r^{n-2}}$$

$$= c_1\omega_{n-1}\int_0^{a_1}\frac{r^{n-2}}{r^{n-2}} = c_1\omega_{n-1}a_1,$$

in which we passed to generalized polar coordinates.

It is clear then that

$$\lim_{a_1\to 0}\frac{1}{n-2}\int_\sigma \frac{\varrho(x)}{r^{n-2}}\mathrm{d}\sigma_x = 0,$$

and then the function $V(\xi)$ is well defined, taking into account (3.4.2).

The first integral from (3.4.3) is a continuous function and its limit is just $V(\xi)$, and therefore, $V(\xi)$ is a continuous function. ■

In the sequel, we are interested in the values of the single layer potential in points from inside the domain Ω and in points from outside Ω, as well as the relation between these values and the values of the single layer potential in points of the boundary $\partial\Omega$.

We fix an arbitrary point $\xi_0 \in \partial\Omega$ and consider the sequence $\{\xi_i\}$ of points from inside of Ω, so that this sequence is convergent to ξ_0. We denote by

$$\left(\frac{\partial V(\xi_0)}{\partial\nu}\right)_i = \lim_{\xi_i\to\xi_0}\frac{\partial V(\xi_i)}{\partial\nu}, \tag{3.4.4}$$

and this notation is valid if the limit exists and is unique, for any sequence $\{\xi_i\}$ of points from the inside of Ω, convergent to ξ_0.

Analogously, if the sequence $\{\xi_e\}$ of points from outside of Ω is convergent to ξ_0, we use the notation

$$\left(\frac{\partial V(\xi_0)}{\partial\nu}\right)_e = \lim_{\xi_e\to\xi_0}\frac{\partial V(\xi_e)}{\partial\nu}, \tag{3.4.5}$$

if the limit exists and is unique, for any sequence $\{\xi_e\}$ of points from outside of Ω, convergent to ξ_0.

Theorem 3.4.2 *Suppose that the domain Ω satisfies the hypotheses of Theorem 3.4.1. If the density ϱ is a continuous function on $\partial\Omega$, then for ξ_0 arbitrarily fixed in $\partial\Omega$, the following jump formulas hold true:*

$$\left(\frac{\partial V(\xi_0)}{\partial\nu}\right)_i = \frac{\partial V(\xi_0)}{\partial\nu} + \frac{\omega_n}{2}\varrho(\xi_0), \tag{3.4.6}$$

$$\left(\frac{\partial V(\xi_0)}{\partial\nu}\right)_e = \frac{\partial V(\xi_0)}{\partial\nu} - \frac{\omega_n}{2}\varrho(\xi_0). \tag{3.4.7}$$

Proof As in the case of the double layer potential, we take the point ξ_0 as origin of a system of coordinates and the ball with center in ξ_0 and radius a_1, $B(\xi_0, a_1)$, where a_1 is defined above, for instance, before formula (3.3.19). We can thus make considerations only in this ball in which the surface can be represented with Cartesian coordinates and we can benefit from the evaluations already made in the study of the double layer potential. In (3.4.4), we take the sequence of inside points $\{\xi_i\}$ on the normal inside of the surface and in (3.4.5) we take the sequence of outside points $\{\xi_e\}$ on the outside normal to the surface. We define the function $F(\xi)$ by

$$F(\xi) = \frac{\partial V(\xi)}{\partial \nu_x} + \tilde{W}(\xi), \tag{3.4.8}$$

$$\tilde{W}(\xi) = -\frac{1}{n-2}\int_{\partial\Omega} \varrho(x)\frac{\partial}{\partial \nu_x}\left(\frac{1}{r^{n-2}}\right)\mathrm{d}\sigma_x, \tag{3.4.9}$$

that is, $\tilde{W}(\xi)$ is the double layer potential attached to the density $\varrho(x)$ which is the density of the single layer potential $V(\xi)$ from (3.4.2).

We can show without any difficulty that the function $F(\xi)$ is continuous in the point ξ_0. Then, we write the jump formulas for the double layer potential $\tilde{W}(\xi)$, namely, analogs of formulas (3.3.16), from Sect. 3.3.

If we take into account the continuity of the function $F(\xi)$ in formula (3.4.8), the two jump formulas will appear for $\partial V/\partial \nu$ which are only formulas (3.4.6) and (3.4.7). ■

We will make some considerations at the end of this paragraph, with regard to the behavior at infinity of the single layer potential, which are analogous to those made in the case of the double layer potential.

For x arbitrarily fixed in Ω and any $\xi \notin \overline{\Omega}$, we have

$$r = \left|\vec{\xi x}\right| = \left|\left|\vec{0\xi}\right| - \left|\vec{0x}\right|\right| = \left|\vec{0\xi}\right| \left|1 - \frac{\left|\vec{0x}\right|}{\left|\vec{0\xi}\right|}\right|.$$

We choose ξ so that

$$\left|\vec{0\xi}\right| \geq 2\sup_{x\in\overline{\Omega}}\left|\vec{0x}\right|$$

and then we obtain

$$r \geq \frac{\left|\vec{0\xi}\right|}{2}.$$

Then

$$|V(\xi)| = \left|\frac{1}{n-2}\int_{\partial\Omega}\frac{\varrho(x)}{r^{n-2}}\mathrm{d}\sigma_x\right|$$

$$\leq c_1 \int_{\partial\Omega} \frac{1}{r^{n-2}} \mathrm{d}\sigma_x \leq \frac{2^{n-2}c_1}{\overrightarrow{0\xi}^{n-2}} \int_{\partial\Omega} \mathrm{d}\sigma_x = \frac{c_2}{\overrightarrow{0\xi}^{n-2}}.$$

Thus, the relation which characterizes the behavior at infinity of the single layer potential is

$$|V(\xi)| \leq \frac{c_2}{\overrightarrow{0\xi}^{n-2}}.$$

3.5 Reduction of Boundary Value Problems to Fredholm Integral Equations

At the beginning of the paragraph, we will make some considerations on integral equations of the Fredholm type.

Let $\alpha(x)$ and $K_0(x, y, z)$ be two real functions which depend on the points x and y from the domain $D \subset \mathbb{R}^n$ and of real variable z. Suppose that z is a function of $y \in D$, $z = \varphi(y)$ and that the integral

$$\int_D K_0(x, y, \varphi(y)) \mathrm{d}\sigma_y$$

exists in the whole domain D. Then, an equality of the form

$$\alpha(x)\varphi(x) + \int_D K_0(x, y, \varphi(y)) \mathrm{d}\sigma_y = 0, \ \ x \in D,$$

is called an *integral equation* with respect to the unknown function $\varphi(x)$ and is defined for $x \in D$.

This integral equation will be called *linear*, if the function $K_0(x, y, z)$ depends linear on z, that is, $K_0(x, y, z)$ has the form

$$K_0(x, y, z) = K(x, y)z + K^0(x, y).$$

A linear integral equation can be written in the form

$$\alpha(x)\varphi(x) + \int_D K(x, y)\varphi(y) \mathrm{d}\sigma_y = f(x), \ \ x \in D,$$

where the function

$$f(x) = -\int_D K^0(x, y) \mathrm{d}\sigma_y, \ \ x \in D,$$

is a given function.

A linear integral equation is called homogeneous if $f(x) = 0, \ \forall x \in D$ and non-homogeneous otherwise.

The function $K(x, y)$ is called *kernel* of the integral equation, and the integral

$$\int_D K(x, y)\varphi(y)\mathrm{d}\sigma_y, \ x \in D,$$

which appears in the left-hand member of the above linear integral equation, is called *integral operator*, defined on the set of functions to which the unknown function $\varphi(x)$ belongs.

If the domain D is bounded, with boundary $S = \partial D$, and the functions $\alpha(x)$ and $K(x, y)$ are continuous on $D \cup S$ (the function $K(x, y)$ can be only bounded and integrable on $D \cup S$), then we say that the above integral equation is an *integral equation of Fredholm type*.

The integral equations of Fredholm type are of first, second or third order if the function $\alpha(x)$ is identical equal to zero, identical equal to 1 or is not identical equal to zero or to 1, respectively.

In the case that $\alpha(x) \neq 0$ on $D \cup S$, the integral equation of Fredholm of third order can be reduced to an integral equation of second order, by dividing both members of the equation by $\alpha(x)$.

In this paragraph, we will use in the following, in particular, integral equations of second order. Let us mention that many mathematicians accept that the concept of equations of mathematical physics includes both partial differential equations of second order, and also the integral equations. Let Ω be a bounded domain from $\mathbb{R}^n, \ n \geq 3$, having boundary $\partial\Omega$ which is supposed be closed Lyapunov surface and consider the inside Dirichlet's problem

$$\begin{aligned} &\Delta_\xi u(\xi) = 0, \ \forall \xi \in \Omega, \\ &u(y) = \varphi_1(y), \ \forall y \in \partial\Omega, \end{aligned} \tag{3.5.1}$$

in which the function φ_1 is given and continuous on $\partial\Omega$.

A function $u : \overline{\Omega} \to \mathbb{R}$ is a *classical solution* for the inside Dirichlet problem (3.5.1) if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$, u verifies the equation of Laplace $(3.5.1)_1$ and satisfies the Dirichlet boundary condition $(3.5.1)_2$.

We try to solve the problem (3.5.1) with the help of a potential of double layer, that is, we are searching for the solution of the problem in the form

$$u(\xi) = -\int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r_{\xi x}^{n-2}} \right) \mathrm{d}\sigma_x, \tag{3.5.2}$$

in which $r_{\xi x}$ is the distance between the points $x \in \partial\Omega$ and $\xi \notin \partial\Omega$.

For the density ϱ, we do impose for the moment only the condition to be continuous on $\partial\Omega$. We should determine what additional conditions must be satisfied by ϱ so that u defined in (3.5.2) is effectively a classical solution of the inside Dirichlet's problem (3.5.1).

As we already know from the study of the double layer potential, for $\forall \xi \in \Omega$, the function u from (3.5.2) verifies the Laplace equation $(3.5.1)_1$. From the form of the potential, it is certified that the derivatives $\partial^2 u/\partial \xi^2$ are continuous on Ω. So, we deduce that the restrictions on the density ϱ appear if we impose on u to be a continuous function on $\overline{\Omega}$. Also restrictions on the density ϱ will appear if we impose on the function u to satisfy the boundary condition $(3.5.1)_2$.

For y arbitrarily fixed on the surface $\partial\Omega$, we will satisfy the condition $(3.5.1)_2$ by passing to the limit with points from the inside

$$\varphi_1(y) = \lim_{y_i \to y} u(y_i) = u_i(y).$$

Then, the jump formula from the double layer potential leads to

$$\begin{aligned}\varphi_1(y) &= \lim_{y_i \to y} u(y_i) = u(y) + \frac{n-2}{2}\omega_n \varrho(y)\\ &= -\int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x + \frac{n-2}{2}\omega_n \varrho(y). \end{aligned} \tag{3.5.3}$$

We introduce the notations

$$K(y, x) = -\frac{2}{(n-2)\omega_n} \frac{\partial}{\partial \nu_x}\left(\frac{1}{r_{yx}^{n-2}}\right), \tag{3.5.4}$$

$$g_1(y) = \frac{2}{(n-2)\omega_n}\varphi_1(y). \tag{3.5.5}$$

If we multiply the first term and the last term from (3.5.3) by $\frac{2}{(n-2)\omega_n}$ and take into account the notations (3.5.4) and (3.5.5), we obtain the integral equation

$$\varrho(y) + \int_{\partial\Omega} K(y, x)\varrho(x)\mathrm{d}\sigma_x = g_1(y), \ \forall y \in \partial\Omega. \tag{3.5.6}$$

The proven result is summarized in the following proposition.

Proposition 3.5.1 *The necessary and sufficient condition that the function u, defined in (3.5.2), be a classical solution of the inside Dirichlet's problem (3.5.1) is that the density ϱ be a solution of the integral equation of Fredholm type (3.5.6).*

In the approach of Dirichlet's outside problem, we will use the same procedure as for the inside Neumann's problem and for the outside Neumann problem.

The outside Dirichlet's problem, for the Laplace equation, consists of

$$
\begin{aligned}
&\Delta_{\xi} u(\xi) = 0, \ \forall \xi \in \mathbb{R}^n \setminus \overline{\Omega}, \\
&u(y) = \varphi_2(y), \ \forall y \in \partial\Omega, \\
&|u(\xi)| \leq \frac{A}{|0\xi|^{n-1}}, \forall \xi \text{ so that } |0\xi| \geq R_0,
\end{aligned} \tag{3.5.7}
$$

in which the function φ_2 is given and continuous on $\partial\Omega$, and A and R_0 are given positive constants.

The classical solution for the outside Dirichlet's problem (3.5.7) is a function

$$
u : \mathbb{R}^n \setminus \Omega \to \mathbb{R}, \ u \in C(\mathbb{R}^n \setminus \Omega) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega}),
$$

which verifies the Laplace equation $(3.5.7)_1$, the boundary condition $(3.5.7)_2$, and the condition of behavior to infinity $(3.5.7)_3$.

We are searching for a classical solution of the problem (3.5.7) in the form of a potential of double layer

$$
u(\xi) = -\int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r_{\xi x}^{n-2}} \right) \mathrm{d}\sigma_x. \tag{3.5.8}
$$

The function $1/r_{\xi x}^{n-2}$ is of class C^∞ because it is impossible to achieve the singularity $\xi = x$ since $x \in \partial\Omega$ and $\xi \in \mathbb{R}^n \setminus \overline{\Omega}$.

As we already know, the density ϱ is assumed to be continuous. Let us analyze what additional conditions must be imposed on ϱ so that the function u, defined in (3.5.8), is effectively a classical solution of the problem (3.5.7).

It is clear that the derivatives $\partial^2 u/\partial \xi_i^2$ are continuous functions on $\mathbb{R}^n \setminus \overline{\Omega}$ and the condition $(3.5.7)_3$ is obviously satisfied, taking into account the behavior to infinity of the double layer potential, which is detailed at the end of Sect. 3.3.

Also, from the study of the double layer potential we know that u defined in (3.5.8) satisfies the Laplace equation. It only remains only to impose on u to satisfy the boundary condition $(3.5.7)_2$. We compute the boundary values of the function u by using a sequence $\{y_e\}$ of outside points so that we can use the jump formula for the double layer potential (for outside values)

$$
\begin{aligned}
\varphi_2(y) &= \lim_{y_e \to y} u(y_e) = u(y) + \frac{n-2}{2}\omega_n \varrho(y) \\
&= -\int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r_{yx}^{n-2}} \right) \mathrm{d}\sigma_x - \frac{n-2}{2}\omega_n \varrho(y).
\end{aligned} \tag{3.5.9}
$$

We now introduce the notation

$$
g_2(y) = -\frac{2}{(n-2)\omega_n}\varphi_2(y). \tag{3.5.10}
$$

If we multiply the first and the last term from (3.5.9) by $\frac{2}{(n-2)\omega_n}$ and we take into account the notations (3.5.4) and (3.5.10), we obtain the integral equation

$$\varrho(y) - \int_{\partial\Omega} K(y, x)\varrho(x)\mathrm{d}\sigma_x = g_2(y), \ \forall y \in \partial\Omega. \tag{3.5.11}$$

The proved result can be summarized as in the following Proposition.

Proposition 3.5.2 *The necessary and sufficient condition that the function u, defined in (3.5.8), be a classical solution of the inside Dirichlet's problem (3.5.7) is that the density ϱ be a solution of the integral equation of Fredholm type (3.5.11).*

Observation 3.5.1 *Let us remark the very small difference between Fredholm's integral equation (3.5.6), which is the equivalent to the inside Dirichlet's problem, and Fredholm's integral equation (3.5.11), which is equivalent to the outside Dirichlet's problem. Only the sign of the kernel $K(y, x)$ is different.*

We now approach *the inside Neumann's problem* which consists of

$$\begin{aligned} &\Delta_\xi u(\xi) = 0, \ \forall \xi \in \Omega, \\ &\frac{\partial u}{\partial \nu}(y) = \psi_1(y), \ \forall y \in \partial\Omega, \end{aligned} \tag{3.5.12}$$

in which the function ψ_1 is given and continuous on $\partial\Omega$, and ν is the unit normal to the surface $\partial\Omega$, oriented to the outside.

We call the *classical solution* of the inside Neumann's problem (3.5.12), a function

$$u : \overline{\Omega} \to \mathbb{R}, \ u \in C(\overline{\Omega}) \cap C^2(\Omega),$$

which satisfies Laplace equation $(3.5.12)_1$ and verifies the condition to the limit of Neumann type $(3.5.12)_2$.

We are searching for a classical solution of the problem (3.5.12) in the form of a potential of single layer

$$u(\xi) = \int_{\partial\Omega} \varrho(x) \frac{1}{r_{\xi x}^{n-2}} \mathrm{d}\sigma_x, \tag{3.5.13}$$

in which the density ϱ is a continuous function on $\partial\Omega$. We must find the conditions that must be additionally imposed on the density ϱ so that the function u defined in (3.5.13) be an effective solution of the problem (3.5.12). Obviously, the conditions do not refer to the function $1/r_{\xi x}^{n-2}$ which is of class C^∞, because its singularity at $x = \xi$ cannot be reached, taking into account that $x \in \partial\Omega$ and $\xi \notin \partial\Omega$.

From the theory of the single layer potential, we know that u from (3.5.13) is a harmonic function inside the domain Ω.

To satisfy the boundary condition $(3.5.12)_2$, we will compute the values to the limit for a sequence $\{y_i\}$ of inside points, so that we can use the jump formula from the single layer potential, for inside values

$$\psi_1(y) = \lim_{y_i \to y} \frac{\partial u}{\partial \nu}(y_i) = \left(\frac{\partial u(y)}{\partial \nu}\right)_i = \frac{\partial u(y)}{\partial \nu} + \frac{n-2}{2}\omega_n \varrho(y)$$
$$= \int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_y}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x + \frac{n-2}{2}\omega_n \varrho(y). \quad (3.5.14)$$

We use the notation

$$h_1(y) = \frac{2}{(n-2)\omega_n}\psi_1(y). \quad (3.5.15)$$

If we multiply the first and the last term from (3.5.14) by $\frac{2}{(n-2)\omega_n}$ and we take into account the notations (3.5.4) and (3.5.15), we obtain the following integral equation of Fredholm type:

$$\varrho(y) - \int_{\partial\Omega} K(x, y)\varrho(x)\mathrm{d}\sigma_x = h_1(y), \ \forall y \in \partial\Omega. \quad (3.5.16)$$

Thus, we proved the following result.

Proposition 3.5.3 *The necessary and sufficient condition for the function u, defined in (3.5.13), to be a classical solution of the inside Neumann's problem (3.5.12) is that the density* ϱ *be a solution of the integral equation of Fredholm type (3.5.16).*

We now approach *the outside Neumann's problem* which consists of

$$\begin{aligned} &\Delta_\xi u(\xi) = 0, \ \forall \xi \in \mathbb{R}^n \setminus \overline{\Omega}, \\ &\frac{\partial u}{\partial \nu}(y) = \psi_2(y), \ \forall y \in \partial\Omega, \\ &|u(\xi)| \le \frac{A}{|0\xi|^{n-2}}, \ \forall \xi \text{ so that } |0\xi| > R_0, \\ &\left|\frac{\partial u}{\partial \xi}(\xi)\right| \le \frac{A}{|0\xi|^{n-1}}, \ \forall \xi \text{ so that } |0\xi| > R_0, \end{aligned} \quad (3.5.17)$$

in which the function ψ_2 is given and continuous on $\partial\Omega$, and A and R_0 are given positive constants.

A *classical solution* of the outside Neumann problem (3.5.17) is a function

$$u : \mathbb{R}^n \setminus \Omega \to \mathbb{R}, \ u \in C^1(\mathbb{R}^n \setminus \Omega) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega}),$$

which satisfies Laplace equation $(3.5.17)_1$, verifies the boundary condition of Neumann type $(3.5.17)_2$ and verifies the conditions of behavior to infinity of the form $(3.5.17)_3$ and $(3.5.17)_4$.

We are searching for a classical solution of the problem (3.5.17) in the form of a single layer potential

$$u(\xi) = \int_{\partial\Omega} \varrho(x) \frac{1}{r_{\xi x}^{n-2}} \mathrm{d}\sigma_x, \tag{3.5.18}$$

in which the density ϱ is a continuous function on $\partial\Omega$. We want to find the conditions that must be imposed in addition on the density ϱ so that the function u defined in (3.5.18) is an effective solution of the problem (3.5.17). Obviously, the conditions do not refer to the function $1/r_{\xi x}^{n-2}$ which is of class C^∞, because its singularity at $x = \xi$ cannot be reached, taking into account that $x \in \partial\Omega$ and $\xi \notin \partial\Omega$.

From the theory of the single layer potential, we know that u from (3.5.18) is a harmonic function inside the domain Ω. Also, taking into account the behavior at infinity of the single layer potential, we deduce that u automatically satisfies the conditions $(3.5.17)_3$ and $(3.5.17)_4$. In the following, we will impose on the function u to satisfy the boundary condition $(3.5.17)_2$.

For this, we will compute the values to the limit for a sequence $\{y_e\}$ of outside points, so that we can use jump formula from the single layer potential, for outside values

$$\psi_2(y) = \lim_{y_e \to y} \frac{\partial u}{\partial \nu}(y_e) = \left(\frac{\partial u(y)}{\partial \nu}\right)_e = \frac{\partial u(y)}{\partial \nu} - \frac{n-2}{2}\omega_n \varrho(y)$$

$$= \int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_y}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x - \frac{n-2}{2}\omega_n \varrho(y). \tag{3.5.19}$$

We use the notation

$$h_2(y) = -\frac{2}{(n-2)\omega_n}\psi_2(y). \tag{3.5.20}$$

If we multiply the first and the last term from (3.5.19) by $-\frac{2}{(n-2)\omega_n}$ and we take into account the notations (3.5.4) and (3.5.20), we obtain the following integral equation of Fredholm type:

$$\varrho(y) + \int_{\partial\Omega} K(x, y)\varrho(x)\mathrm{d}\sigma_x = h_2(y), \ \forall y \in \partial\Omega. \tag{3.5.21}$$

Thus, we proved the following result.

Proposition 3.5.4 *The necessary and sufficient condition for the function u, defined in (3.5.18), to be a classical solution of the outside Neumann problem (3.5.17) is that the density ϱ must be a solution of the integral equation of Fredholm type (3.5.21).*

Let us remark the very small difference between the equation of Fredholm (3.5.16) attached to the inside Neumann problem and the equation of Fredholm (3.5.21) attached to the outside Neumann problem. It is just the sign of the kernel $K(x, y)$.

Observation 3.5.2 *We can summarize all the considerations made in this paragraph, in this way: the study of the boundary value problems (3.5.1), (3.5.7), (3.5.12), and (3.5.17) is reduced to the study of the integral equations of Fredholm type (3.5.6), (3.5.11), (3.5.16), and (3.5.21), respectively.*

We intend to study the Fredholm equations, mentioned above, with the help of the so-called *of Theorems of alternative* of Fredholm.

We should note, first, the fact that Eqs. (3.5.6) and (3.5.12) are coupled together and, likewise, Eqs. (3.5.11) and (3.5.16).

Consider therefore the pair of equations

$$\varrho(y) + \int_{\partial\Omega} K(y, x)\varrho(x)\mathrm{d}\sigma_x = g_1(y), \ \forall y \in \partial\Omega,$$
$$\tau(y) + \int_{\partial\Omega} K(x, y)\tau(x)\mathrm{d}\sigma_x = h_2(y), \ \forall y \in \partial\Omega. \tag{3.5.22}$$

The kernel of Eq. $(3.5.22)_2$ is obtained from the kernel of Eq. $(3.5.22)_1$, by changing the arguments. For this reason, we will say that the two kernels are *conjugated kernels*. We attach to the pair of Eq. (3.5.22) the pair of associated homogeneous equations

$$z(y) + \int_{\partial\Omega} K(y, x)z(x)\mathrm{d}\sigma_x = 0, \ \forall y \in \partial\Omega,$$
$$z^*(y) + \int_{\partial\Omega} K(x, y)z^*(x)\mathrm{d}\sigma_x = 0, \ \forall y \in \partial\Omega. \tag{3.5.23}$$

The relation between the pair of Fredholm equations (3.5.22) and the corresponding pair of homogeneous equations (3.5.23) is established by means of the theorems of alternative due to Fredholm.

Theorem 3.5.1 *If Eq. $(3.5.23)_1$ has only the trivial solution, then also Eq. $(3.5.23)_2$ has only the trivial solution and conversely.*

In Theorem 3.5.1, we have *the case I of the Fredholm alternative*, and in this case, the nonhomogeneous equation $(3.5.22)_1$ has only one solution, which is a continuous function, for any function g_1 supposed to be continuous.

Likewise, the nonhomogeneous equation $(3.5.22)_2$ has only one solution, which is continuous, for any arbitrarily fixed function h_2, which is assumed to be continuous.

Theorem 3.5.2 *If Eq. $(3.5.23)_1$ (or Eq. $(3.5.23)_2$) also admits solutions which are different from the trivial solution, then the space of solutions of the equation $(3.5.23)_2$ ($(3.5.23)_1$, respectively) has the same dimension as the space of solutions of the equation $(3.5.23)_1$ ($(3.5.23)_2$, respectively).*

This is *case II of the Fredholm alternative* and can be reformulated as follows: the maximum number of linear independent solutions of a homogeneous equation coincides with the maximum number of linear independent solutions of the homogeneous adjoint equation. In this second case of the alternative, the nonhomogeneous equations $(3.5.22)_1$ and $(3.5.22)_2$ in general have no solutions.

The necessary and sufficient condition for a nonhomogeneous equation has solutions; in case II of the alternative is that its right-hand member is orthogonal on the space of solutions of the homogeneous adjoint equation.

We will show that a pair of homogeneous equations (3.5.23) is situated in the case II of the Fredholm alternative.

Proposition 3.5.5 *Equation* $(3.5.23)_1$ *(Eq.* $(3.5.23)_2$*, respectively) admits only the trivial solution.*

Proof Considering Theorem 3.5.1 we deduce that it is sufficient to prove that Eq. $(3.5.23)_2$ admits only the trivial solution.

Suppose, by absurd, that Eq. $(3.5.23)_2$ also admits a solution $z^* \not\equiv 0$ on $\partial\Omega$. Then z^* is continuous on $\partial\Omega$ and by replacing it in relation $(3.5.23)_2$, this is transformed into an identity. We attach a single layer potential which has as the density just the function z^*

$$V(\xi) = \int_{\partial\Omega} z^*(x) \frac{1}{r_{\xi x}^{n-2}} \mathrm{d}\sigma_x . \tag{3.5.24}$$

In order to prove that

$$\left(\frac{\partial V(y)}{\partial \nu}\right)_e = 0, \ \left(\frac{\partial V(y)}{\partial \nu}\right)_i = 0, \ \forall y \in \partial\Omega \tag{3.5.25}$$

we will rely on the jump formula from the single layer potential, for outside values, so that we deduce

$$\begin{aligned}
\left(\frac{\partial V(y)}{\partial \nu}\right)_e &= \int_{\partial\Omega} z^*(x) \frac{\partial}{\partial \nu_y}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x - \frac{(n-2)\omega_n}{2} z^*(y) \\
&= -\frac{(n-2)\omega_n}{2}\left[z^*(y) - \frac{2}{(n-2)\omega_n} \int_{\partial\Omega} z^*(x) \frac{\partial}{\partial \nu_y}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x\right] \\
&= -\frac{(n-2)\omega_n}{2}\left[z^*(y) + \int_{\partial\Omega} K(x, y) z^*(x) \mathrm{d}\sigma_x\right] = 0,
\end{aligned}$$

in which we use the fact that z^* is a solution of the equation $(3.5.23)_2$. Thus, the relation $(3.5.25)_1$ is proven. We will prove now, in another manner, the relation $(3.5.25)_2$.

Obviously, we have that $\Delta_\xi V(\xi) = 0, \ \forall \xi \in \mathbb{R}^n \setminus \overline{\Omega}$. Also,

$$|V(\xi)| \le \frac{A_1}{|0\xi|^{n-2}}, \quad \left|\frac{\partial V(\xi)}{\partial \nu_\xi}\right| \le \frac{A_2}{|0\xi|^{n-1}},$$

the upper bounds holding true $\forall \xi$ so that $|0\xi| > R_0$. The proof is based on the relations which give the behavior of infinity of the single layer potential.

Based on the uniqueness of the solution of the outside Neumann problem, we immediately obtain that $V(\xi) = 0, \ \forall \xi \in \mathbb{R}^n \setminus \overline{\Omega}$. As we already know, the single layer potential is a harmonic function also inside the domain Ω. We now consider the problem

$$\begin{aligned} &\Delta_\xi V(\xi) = 0, \ \forall \xi \in \Omega, \\ &V(y) = 0, \ \forall y \in \partial\Omega, \end{aligned}$$

that is, the inside Dirichlet's problem with null data on the boundary.

Based on the uniqueness of the solution of the inside Dirichlet's problem, we deduce that $V(\xi) = 0, \ \forall \xi \in \overline{\Omega}$. Then, obviously

$$\left(\frac{\partial V(y)}{\partial \nu}\right)_i = 0, \ \forall y \in \partial\Omega,$$

that is, the relation $(3.5.25)_2$ is proven.

We subtract now member by member the relations $(3.5.25)_1$ and $(3.5.25)_2$ and with the help of the jump formula from the single layer potential, we obtain

$$0 = \left(\frac{\partial V(y)}{\partial \nu}\right)_i = \left(\frac{\partial V(y)}{\partial \nu}\right)_e = (n-2)\omega_n z^*(y), \ \forall y \in \partial\Omega,$$

and therefore $z^*(y) = 0, \ \forall y \in \partial\Omega$, which is in contradiction with the initial assumption that $z^* \not\equiv 0$ on $\partial\Omega$. ∎

From Proposition 3.5.5 and case I of the Fredholm alternative, we deduce that the inside Dirichlet's problem admits only one solution, irrespective of how the function g_1 is fixed, and therefore, irrespective of how the function φ_1 is fixed. Likewise, the outside Neumann's problem admits only one solution, irrespective of how the function h_2 is fixed, and therefore, irrespective of how the function ψ_2 is fixed.

We will now address the pair of Fredholm integral equations attached to the Dirichlet outside problem and to the inside Neumann's problem, respectively, that is, we approach the equations

$$\begin{aligned} &\varrho(y) - \int_{\partial\Omega} K(y,x)\varrho(x)\mathrm{d}\sigma_x = g_2(y), \ \forall y \in \partial\Omega, \\ &\tau(y) - \int_{\partial\Omega} K(x,y)\tau(x)\mathrm{d}\sigma_x = h_1(y), \ \forall y \in \partial\Omega. \end{aligned} \tag{3.5.26}$$

To these two integral equations, we can attach adjoint equations due to the connection between their kernels, which are conjugated kernels. As in the case of the pair of Eq. (3.5.22) we will attach to Eq. (3.5.26), the pair of the associated homogeneous equations

$$z(y) - \int_{\partial\Omega} K(y,x) z(x) \mathrm{d}\sigma_x = 0, \ \forall y \in \partial\Omega,$$
$$z^*(y) - \int_{\partial\Omega} K(x,y) z^*(x) \mathrm{d}\sigma_x = 0, \ \forall y \in \partial\Omega. \tag{3.5.27}$$

We recall that the kernel $K(y,x)$ has the expression

$$K(y,x) = -\frac{\partial}{\partial \nu_x}\left(\frac{1}{r_{yx}^{n-2}}\right).$$

We anticipate that the pair of Eq. (3.5.27) stands in case II of the Fredholm alternative, and then we will put the problem of the dimension of the space of solutions, in the case in which Eq. $(3.5.27)_1$ (or $(3.5.27)_2$) admits nontrivial solutions.

Proposition 3.5.6 *Equation $(3.5.27)_1$ admits the solution $z(x) \equiv 1$.*

Proof We replace in Eq. $(3.5.27)_1$ $z(x)$ with 1 and we must obtain an equality

$$1 - \int_{\partial\Omega} K(y,x).1 \mathrm{d}\sigma_x = 1 + \frac{2}{(n-2)\omega_n} \int_{\partial\Omega} \frac{\partial}{\partial \nu_x}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x$$

$$= 1 - \frac{2}{\omega_n}\left[-\frac{1}{n-2}\int_{\partial\Omega} \frac{\partial}{\partial \nu_x}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x\right]$$

$$= 1 - \frac{2}{\omega_n}\omega(y, \partial\Omega) = 1 - \frac{2}{\omega_n}\frac{\omega_n}{2} = 0,$$

in which we used the definition of the solid angle and the fact that $\partial\Omega$ is a closed Lyapunov surface. Since $y \in \partial\Omega$ we obtain that the solid angle under which $\partial\Omega$ is seen is half of the area of the sphere. ■

Since Eq. $(3.5.27)_1$ admits a nontrivial solution (namely, $z(x) \equiv 1$, according to Proposition 3.5.6), we deduce that the pair of Eq. (3.5.27) stands in case II of the Fredholm alternative. Then, the space of solutions of Eq. $(3.5.27)_1$ has the same dimension as the space of solutions of the equation $(3.5.27)_2$.

We will prove that the dimension of the two spaces of solutions is 1, that is, both Eq. $(3.5.27)_1$ and also Eq. $(3.5.27)_2$ cannot have two linear independent solutions.

Theorem 3.5.3 *The dimension of the space of solutions of the Fredholm integral equation (3.5.27)$_2$ is* 1.

Proof Suppose, by absurd, that Eq. (3.5.27)$_2$ admits two solutions which are not identical null $z_1^* \not\equiv z_1^* \not\equiv 0$, but are linearly independent. Accordingly, to these two solutions, we attach two single layer potentials

$$V_j(\xi) = \int_{\partial\Omega} z_j^*(x) \frac{1}{r_{\xi x}^{n-2}} \mathrm{d}\sigma_x, \;\; j = 1, 2.$$

We write the jump formula for the single layer potential by points from inside the domain

$$\begin{aligned}\left(\frac{\partial V_j(y)}{\partial \nu_y}\right)_i &= \int_{\partial\Omega} z_j^*(x) \frac{\partial}{\partial \nu_y}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x + \frac{(n-2)\omega_n}{2} z_j^*(y) \\ &= \frac{(n-2)\omega_n}{2}\left[z_j^*(y) - \frac{-2}{(n-2)\omega_n} \int_{\partial\Omega} z_j^*(x) \frac{\partial}{\partial \nu_y}\left(\frac{1}{r_{yx}^{n-2}}\right) \mathrm{d}\sigma_x\right] \\ &= \frac{(n-2)\omega_n}{2} z_j^*(y) - \int_{\partial\Omega} K(y,x) z_j^*(x) \mathrm{d}\sigma_x = 0,\end{aligned}$$

the last statement is based on the fact that z_1^* and z_2^* are solutions of the homogeneous equation (3.5.27)$_2$.

Therefore, the derivative in the direction of the normal of the potentials V_j, computed at the limit by inside points, is null. We already know that a single layer potential satisfies the Laplace equation inside the domain Ω. We can therefore consider the following boundary value problem:

$$\begin{aligned}&\Delta_\xi V_j(\xi) = 0, \;\; \forall \xi \in \Omega, \\ &\left(\frac{\partial V_j(y)}{\partial \nu_y}\right)_i = 0, \;\; \forall y \in \partial\Omega,\end{aligned}$$

which is the inside Neumann's problem. As we already proved, the solution of an inside Neumann problem is determined up to an additive constant. In our case, we have

$$V_j(\xi) \equiv c_j, \;\; \forall \xi \in \overline{\Omega}, \;\; j = 1, 2.$$

Only the following two situations are possible:
– 1^o. one of the two constants is null;
– 2^o. both constants are nonzero.
In the second case, we will build the function $\tilde{z}$ by

$$\tilde{z}(\xi) = \frac{z_1^*(\xi)}{c_1} - \frac{z_2^*(\xi)}{c_2}$$

and we attach to this function a single layer potential

$$\tilde{V}(\xi) = \int_{\partial\Omega} \tilde{z}^*(x) \frac{1}{r_{yx}^{n-2}} \mathrm{d}\sigma_x.$$

It is clear then that

$$\tilde{V}(\xi) = \frac{V_1(\xi)}{c_1} - \frac{V_2(\xi)}{c_2} = 0, \ \forall \xi \in \overline{\Omega}.$$

As is well known, the single layer potential is a continuous function in the whole space and satisfies the Laplace equation inside the domain Ω. For the potential $\tilde{V}$, we can consider the boundary value problem

$$\Delta_\xi \tilde{V}(\xi) = 0, \ \forall \xi \in \mathbb{R}^n \setminus \overline{\Omega},$$
$$s\left(\frac{\partial \tilde{V}(y)}{\partial \nu_y}\right)_e = 0, \ \forall y \in \partial\Omega,$$

which is an outside Neumann's problem with null boundary data. From the above definition of the potential $\tilde{V}$, we have

$$\left(\frac{\partial \tilde{V}(y)}{\partial \nu_y}\right)_i = \frac{1}{c_1}\left(\frac{\partial V_1(y)}{\partial \nu_y}\right)_i - \frac{1}{c_2}\left(\frac{\partial V_2(y)}{\partial \nu_y}\right)_i = 0.$$

If we write the formulas of jump for the single layer potential $\tilde{V}$, both by points from the inside and also by points from the outside and we subtract term by term the two relations, then we obtain

$$0 = \left(\frac{\partial \tilde{V}(y)}{\partial \nu_y}\right)_i - \left(\frac{\partial \tilde{V}(y)}{\partial \nu_y}\right)_e = (n-2)\omega_n \tilde{z}(y), \ \forall y \in \partial\Omega,$$

and this leads to the conclusion that

$$\tilde{z}(y) = 0, \ \forall y \in \partial\Omega \Rightarrow \frac{z_1^*}{c_1} = \frac{z_2^*}{c_2},$$

that is, the two solutions are linearly dependent, and this is in contradiction with the initial assumption.

Suppose now that the first case occurs, that is, at least one of the constants c_1 and c_2 is null. If, for instance, $c_1 = 0$, then we can do the same reasonings on the potential V_1 that we have done on $\tilde{V}$ so that we reach the conclusion that $z_1 \equiv 0$, which is in contradiction with the initial assumption.

In conclusion, the dimension of the space of solutions of equation $(3.5.27)_2$ is exactly 1. According to alternative II of Fredholm, we deduce that also the dimension of the space of solutions of the equation $(3.5.27)_1$ is exactly 1. ■

Observation 3.5.3 *(i) Since, as we have already seen, Eq.* $(3.5.27)_1$ *admits the particular solution* $z(x) \equiv 1$ *and since the dimension of the space of solutions of this equation is* 1*, we deduce that the general integral of Eq.* $(3.5.27)_1$ *is* $z_{gen} = c$*, where* c *is an arbitrary constant.*

(ii) We can deduce, based on the alternative II of Fredholm, that the homogeneous equation $(3.5.27)_2$*, has a space of solutions of dimension* 1*. But we do not know a particular solution of this equation, and therefore, we do not know its general integral. If we would know a particular solution, say* z_1*, then its general integral would be* $z_{gen} = cz_1$*, where* c *is an arbitrary constant*

The conclusion of the alternative II of Fredholm is that the outside Dirichlet and the inside Neumann boundary value problems have no unique solutions, for any given boundary data. Moreover, the two boundary value problems admit solutions only under given conditions, called *conditions of compatibility*.

Proposition 3.5.7 *The necessary and sufficient condition of compatibility of the integral equation attached to the inside Neumann's problem is*

$$\int_{\partial\Omega} \psi(y)\mathrm{d}\sigma_y = 0.$$

Proof We have

$$\int_{\partial\Omega} h_1(y)c\mathrm{d}\sigma_y = 0,$$

and this involves

$$0 = \int_{\partial\Omega} h_1(y)\mathrm{d}\sigma_y = \frac{2}{(n-2)\omega_n}\int_{\partial\Omega} \psi(y)\mathrm{d}\sigma_y = 0,$$

from where we obtain the conclusion of the proposition. ■

The above condition of compatibility is obviously satisfied because, for a harmonic function, the surface integral is null, according to a consequence of the Riemann–Green formula.

In the case of the outside Dirichlet's problem and, in fact, in the case of the integral equation attached to this problem, because we are in the case II of the Fredholm alternative, the necessary and sufficient condition of compatibility is that g_2 be a function orthogonal to the space of solutions of the homogeneous adjoint equation, that is $g_2 \perp z^*_{gen}$. Because $z^*_{gen} = c_1 z_1$, where z_1 is a particular solution, the condition of orthogonality can be written in the form

$$0 = c_1 \int_{\partial\Omega} g_2(y) z_1(y) \mathrm{d}\sigma_y = \frac{2c_1}{(n-2)\omega_n} \int_{\partial\Omega} \varphi(y) z_1(y) \mathrm{d}\sigma_y,$$

and therefore

$$\int_{\partial\Omega} \varphi(y) z_1(y) \mathrm{d}\sigma_y = 0.$$

This restriction cannot have any justification. It comes from the method used in the proof. We escape this restriction if we consider again the Dirichlet outside problem and, instead of looking for a solution in the form of the single layer potential, we will look for a solution in the form of a double layer potential, to which we will add a "correction". Therefore, we are searching for the solution of the problem

$$\begin{aligned} &\Delta_\xi u(\xi) = 0, \ \forall \xi \in \mathbb{R}^n \setminus \overline{\Omega}, \\ &u(y) = \varphi(y), \ \forall y \in \partial\Omega, \end{aligned}$$

in the form

$$u(\xi) = -\int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r_{\xi x}^{n-2}} \right) \mathrm{d}\sigma_x + \frac{\alpha}{r_{x_0\xi}^{n-2}}, \tag{3.5.28}$$

where ξ is arbitrarily chosen in $\mathbb{R}^n \setminus \Omega$ and x_0 is fixed inside Ω. According to the general formulation of an outside Dirichlet's problem, we must add a condition of behavior at infinity, which will be taken of the form

$$\forall \varepsilon > 0, \exists N = N(\varepsilon), \ \text{so that } |u(\xi)| < \varepsilon, \ \text{for } |o\xi| > N(\varepsilon).$$

Let us observe that the function u from (3.5.28) satisfies this last condition. Indeed, the term that "corrects" from (3.5.28) is convergent to zero, for ξ enough big, because x_0 is fixed inside Ω. Also, the first term from (3.5.28) tends to zero, as we can deduce from the behavior at infinity of the double layer potential.

Because the distance from the point ξ to any point, in particular also to the point x_0, determine the function $1/r_{x_o\xi}^{n-2}$ to be a harmonic function both inside and also outside the domain Ω, we deduce that the function u from (3.5.28) satisfies the Laplace equation outside the domain.

It remains only to satisfy the boundary condition $u(y) = \varphi(y)$. To this end, we can write

$$\begin{aligned} \varphi(y) = u_e(y = -\int_{\partial\Omega} \varrho(x) \frac{\partial}{\partial \nu_x} \left(\frac{1}{r_{yx}^{n-2}} \right) \mathrm{d}\sigma_x \\ - \frac{(n-2)\omega_n}{2} \varrho(y) + \frac{\alpha}{r_{x_0 y}^{n-2}}, \end{aligned}$$

from where we obtain the equation

$$\varrho(y) - \int_{\partial\Omega} K(y,x)\varrho(x)\mathrm{d}\sigma_x$$
$$- \frac{2}{(n-2)\omega_n}\varphi(y) + \frac{2\alpha}{(n-2)\omega_n r_{x_0 y}^{n-2}} = f_2(y),$$

or

$$\varrho(y) - \int_{\partial\Omega} K(y,x)\varrho(x)\mathrm{d}\sigma_x = g_2(y),$$

where we used the notation

$$g_2(y) = -\frac{2}{(n-2)\omega_n}\varphi(y) + \frac{2\alpha}{(n-2)\omega_n r_{x_0 y}^{n-2}}.$$

We obtained an integral equation as in the previous case of the Dirichlet outside problem, but here we have another "free term" (the function on the right-hand side of the equation). If we impose in this case the condition of compatibility $g_2 \perp z^*_{gen}$, where $z^*_{gen} = c_1 z_1$, z_1 being a particular solution, we obtain the equation

$$0 = \int_{\partial\Omega} g_2(y) z_1(y)\mathrm{d}\sigma_y = -\frac{2}{(n-2)\omega_n}\int_{\partial\Omega}\varphi(y)z_1(y)\mathrm{d}\sigma_y$$
$$+ \frac{2\alpha}{(n-2)\omega_n}\int_{\partial\Omega} z_1(y)\frac{1}{r_{x_0 y}^{n-2}}\mathrm{d}\sigma_y.$$

This can be considered as an equation of first degree with the unknown α. Therefore, we can determine uniquely α, which until now was arbitrary. Thus, u defined in (3.5.28) is a solution for the outside Dirichlet's problem.

If we know a particular solution ϱ_N of the nonhomogeneous integral equation attached to the outside Dirichlet's problem, then the general solution of the nonhomogeneous equation will be

$$z_{gen}(y) = \varrho_N(y) + z^0_{gen}(y),$$

where z^0_{gen} is the general solution of the homogeneous equation. But, as we have already seen, $z^0_{gen} = c.1$ and then we have

$$z_{gen}(y) = \varrho_N(y) + c,$$

so that from (3.5.28) we deduce the general solution of the outside Dirichlet's problem

$$u(\xi) = -\int_{\partial\Omega}\left[\varrho_N(x) + c\right]\frac{\partial}{\partial\nu_x}\left(\frac{1}{r_{\xi x}^{n-2}}\right)\mathrm{d}\sigma_x + \frac{\alpha}{r_{x_0\xi}^{n-2}},$$

with α determined as above.

We will now undertake analogous considerations for the integral equation attached to the inside Neumann's problem. If τ_N is a particular solution of the nonhomogeneous integral equation and τ^0_{gen} is the general solution of the homogeneous equation, then the general solution of the nonhomogeneous equation is

$$\tau_{gen} = \tau_N + \tau^0_{gen}.$$

If z_1 is a particular solution of the homogeneous equation, we know that the general solution of the homogeneous equation is $z^0_{gen} = cz_1$, and then we get

$$\tau_{gen} = \tau_N + cz_1,$$

so that the general solution of the inside Neumann's problem is

$$u(\xi) = \int_{\partial\Omega} [\tau_N(x) + cz_1(x)] \frac{1}{r^{n-2}_{\xi x}} \mathrm{d}\sigma_x.$$

A brief conclusion of Sect. 3.4 and, in general, of this chapter is that the boundary value problems of Dirichlet and Neumann type, attached to the Laplace equation, are solved with the help of the potentials of the surface of double and single layers.

Naturally, the question occurs if perhaps the above considerations also remain valid for boundary value problems attached to other equations with partial derivatives. In the following, we will study the validity of some analogous considerations in the case of Poisson's equation.

Let us consider, for exemplification, the inside Dirichlet's problem attached to Poisson's equation

$$\begin{aligned} &\Delta_\xi u(\xi) = f(\xi),\ \forall \xi \in \Omega, \\ &u(y) = \varphi(y),\ \forall y \in \partial\Omega, \end{aligned} \tag{3.5.29}$$

in which Ω is, as usual, a bounded domain from $\mathbb{R}^n$ whose boundary $\partial\Omega$ is a closed Lyapunov surface.

Theorem 3.5.4 *If the function f is locally Hőlder on Ω, then the Dirichlet inside problem, attached to the Poisson's equation, is reduced to the Dirichlet inside problem attached to the Laplace equation.*

Proof We attach to function f the Newtonian potential

$$U(\xi) = \frac{1}{(n-2)\omega_n} \int_\Omega \frac{f(x)}{r^{n-2}_{\xi x}} \mathrm{d}x.$$

From the study of the potential of volume, we know that if the density (in our case, the density is the function f) is a locally Hőlder function on Ω, then the respective potential satisfies the Poisson's equation, in which the right-hand side is just the density. Therefore, we get

$$\Delta_\xi U(\xi) = f(\xi).$$

If u is the solution of the problem (3.5.29), then we define the function v by

$$v(\xi) = u(\xi) - U(\xi),\ \forall \xi \in \Omega.$$

Then, v satisfies the following problem:

$$\Delta_\xi v(\xi) = \Delta_\xi u(\xi) - \Delta_\xi U(\xi) = f(\xi) - f(\xi) = 0,\ \forall \xi \in \Omega,$$
$$v(y) = u(y) - U(y) = \varphi(y) - \frac{1}{(n-2)\omega_n} \int_\Omega \frac{f(x)}{r_{yx}^{n-2}} \mathrm{d}x,\ \forall y \in \partial\Omega.$$

Thus, we reduced the problem (3.5.29) to the inside Dirichlet's problem, for the Laplace equation

$$\Delta_\xi v(\xi) = 0,\ \forall \xi \in \Omega,$$
$$u(y) = \varphi_1(y),\ \forall y \in \partial\Omega,$$

where

$$\varphi_1(y) = \varphi(y) - \frac{1}{(n-2)\omega_n} \int_\Omega \frac{f(x)}{r_{yx}^{n-2}} \mathrm{d}x,$$

and the proof of the theorem is complete. ■

We can proceed analogously also in the case of other boundary value problem attached to the Poisson's equation.

Chapter 4
Boundary Value Problems for Elliptic Operators

4.1 The Method of Green's Function

Consider the bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 3$ with boundary $\partial\Omega$ and the linear operator of second order

$$Lu = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u. \tag{4.1.1}$$

Now, we attach the operator M, which is the adjoint operator of L, defined by

$$Mv = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \left(a_{ij}(x)v\right)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial \left(b_i(x)v\right)}{\partial x_i} + c(x)v, \tag{4.1.2}$$

in which $a_{ij} = a_{ji} \in C^2(\Omega),\ b_i \in C^1(\Omega)$ and $c \in C^0(\Omega)$.

We say that the operator L is positive definite if the matrix of the coefficients $A = [a_{ij}]$ is positive definite, that is:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)\lambda_i \lambda_j \geq 0,\ \forall x \in \Omega,\ \forall (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n,$$

the equality taking place if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

We will consider the equation with partial derivatives of second order

$$Lu(x) = f(x),\ \forall x \in \Omega,$$

which will be accompanied by a boundary condition. As we have already seen, three types of boundary conditions are usually used:

- Dirichlet's condition

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_4

$$u(y) = \varphi_1(y),\ \forall y \in \partial\Omega, \tag{4.1.3}$$

where the function φ_1 is given and continuous on $\partial\Omega$;

- Neumann's condition

$$\frac{\partial u}{\partial \gamma}(y) = \varphi_2(y),\ \forall y \in \partial\Omega, \tag{4.1.4}$$

where γ is a direction from the space $\mathbb{R}^n$ and the function φ_2 is given and continuous on $\partial\Omega$;

- mixed condition

$$\alpha\frac{\partial u}{\partial \gamma}(y) + \beta u(y) = \varphi_3(y),\ \forall y \in \partial\Omega, \tag{4.1.5}$$

where α and β are given constants and the function φ_3 is given and continuous on $\partial\Omega$.

If we take into account expressions (4.1.1) and (4.1.2) of the operators L and M, we obtain the equality

$$\int_\Omega (vLu - uMv)\mathrm{d}x = \int_{\partial\Omega}\left[\gamma\left(v\frac{\partial u}{\partial \gamma} - u\frac{\partial v}{\partial \gamma}\right) + buv\right]\mathrm{d}\sigma_x, \tag{4.1.6}$$

where

$$b = \sum_{i=1}^n\left(b_i - \sum_{j=1}^n\frac{\partial a_{ij}}{\partial x_j}\right)\cos\alpha_i,$$
$$\gamma = \sqrt{\sum_{j=1}^n\left(\sum_{i=1}^n a_{ij}\cos\alpha_i\right)^2},$$

with $\cos\alpha_i$, being the cosine directors of the outside unit normal to $\partial\Omega$. If the integrand from the right-hand side of equality (4.1.6) becomes null, we obtain

$$\int_\Omega v(x)Lu(x)\mathrm{d}x = \int_\Omega u(x)Mv(x)\mathrm{d}x,$$

that is,

$$(Lu, v) = (u, Mv),$$

and therefore, the operators L and M are adjoint to each other.

Thus, the idea appears to consider the boundary conditions (4.1.3), (4.1.4), or (4.1.5) in homogeneous form, which will ensure that the right-hand side from (4.1.6) becomes null. Moreover, considering the homogeneous conditions for u, we will obtain homogeneous conditions for v. Thus, we will attach the following conditions:

$$\begin{aligned}
&u(y) = 0, \ \forall y \in \partial\Omega, \\
&\frac{\partial u}{\partial \gamma}(y) = 0, \ \forall y \in \partial\Omega, \\
&\alpha \frac{\partial u}{\partial \gamma}(y) + \beta u(y) = 0, \ \forall y \in \partial\Omega.
\end{aligned} \tag{4.1.7}$$

Let us analyze the boundary conditions obtained for the function v. We will call the boundary conditions for v *adjoint conditions*. Denote by $I(u, v)$ the integrand from the right-hand side of the relation (4.1.6), that is

$$I(u, v) = \gamma \left(v \frac{\partial u}{\partial \gamma} - u \frac{\partial v}{\partial \gamma} \right) + buv.$$

If the boundary condition $(4.1.7)_1$ holds true, that is, $u(y) = 0, \ \forall y \in \partial\Omega$, then

$$I(u, v) = \gamma v \frac{\partial u}{\partial \gamma},$$

and then $I(u, v) = 0 \Leftrightarrow v(y) = 0, \ \forall y \in \partial\Omega$.

Suppose that the boundary condition $(4.1.7)_2$ is satisfied,

$$\frac{\partial u}{\partial \gamma}(y) = 0, \ \forall y \in \partial\Omega.$$

Then, we have

$$I(u, v) = -\gamma u \frac{\partial v}{\partial \gamma} + buv,$$

and therefore, $I(u, v) = 0$ if and only if

$$-\gamma \frac{\partial v}{\partial \gamma}(y) + bv(y) = 0, \ \forall y \in \partial\Omega.$$

Finally, suppose that the mixed boundary condition $(4.1.7)_3$ holds true. The integrand $I(u, v)$ becomes

$$\begin{aligned}
I(u, v) &= \frac{\gamma v}{\alpha} \left[\alpha \frac{\partial u}{\partial \gamma} + \beta u \right] - \frac{u}{\alpha} \left[\alpha\gamma \frac{\partial v}{\partial \gamma} + (\gamma\beta - \alpha b) v \right] \\
&= -\frac{u}{\alpha} \left[\alpha\gamma \frac{\partial v}{\partial \gamma} + (\gamma\beta - \alpha b) v \right].
\end{aligned}$$

Therefore, $I(u, v) = 0$ if and only if

$$\alpha\gamma\frac{\partial v}{\partial \gamma}(y) + (\gamma\beta - \alpha b)v(y) = 0,\ \forall y \in \partial\Omega.$$

The results proven above are summarized in the proposition below.

Proposition 4.1.1 *If the homogeneous boundary conditions (4.1.7) are satisfied, then the adjoint boundary conditions are*

$$\begin{aligned} &v(y) = 0,\ \forall y \in \partial\Omega,\\ &-\gamma\frac{\partial v}{\partial \gamma}(y) + bv(y) = 0,\ \forall y \in \partial\Omega,\\ &\alpha\gamma\frac{\partial v}{\partial \gamma}(y) + (\gamma\beta - \alpha b)v(y) = 0,\ \forall y \in \partial\Omega. \end{aligned} \tag{4.1.8}$$

It is natural to call the condition $(4.1.8)_1$ as being the adjoint Dirichlet's condition, the condition $(4.1.8)_2$ as the adjoint Neumann's condition and the condition $(4.1.8)_3$ as the mixed adjoint boundary condition.

Observation 4.1.1 *If the operator L is self-adjoint ($L = M$), as we have already proven, we have $b = 0$ and the conditions (4.1.8) become*

$$\begin{aligned} &v(y) = 0,\ \forall y \in \partial\Omega,\\ &\frac{\partial v}{\partial \gamma}(y) = 0,\ \forall y \in \partial\Omega,\\ &\alpha\frac{\partial v}{\partial \gamma}(y) + \beta v(y) = 0,\ \forall y \in \partial\Omega, \end{aligned}$$

which are just the homogeneous boundary conditions (4.1.7). In this situation, we say that the boundary conditions (4.1.7) are self-adjoint conditions.

The method of Green's function, that we will further present, consists in determining the function of Green for a domain Ω, an operator L defined on Ω and a condition on the boundary $\partial\Omega$ which must be satisfied by the operator L. Then, the solution of the equation $Lu(x) = f(x)$, which satisfies the mentioned boundary condition, will be expressed with the help of the Green's function.

A part of the notions which will be seen further (for instance, Levi's function) was defined in Chap. 2. In the following, when we say the condition (c), we refer to one of the following conditions: the Dirichlet's boundary condition, or the Neumann's boundary condition or the mixed boundary condition.

Definition 4.1.1 The function $G(x, \xi)$ given by

$$G(x, \xi) = \Lambda(x, \xi) + g(x, \xi),$$

is called the function of Green attached to the domain Ω, to the operator L and to the condition (c), if it satisfies the properties:

(i) the function $\Lambda(x, \xi)$ is a fundamental solution for the operator M, that is, it is Levi's function;

(ii) for $\forall \xi \in \Omega,\ \xi =$fixed, the function $G(x, \xi)$ satisfies, with respect to x, the adjoint condition of condition (c), denoted by (c^*);

(iii) for $\forall \xi \in \Omega,\ \xi =$fixed in Ω, the function $g(x, \xi)$ satisfies the equation

$$M_x g(x, \xi) = 0,\ \forall x \in \Omega;$$

(iv) for $\forall \xi \in \Omega,\ \xi =$fixed in Ω, $g(x, \xi)$ is of class $C^2(\Omega)$ with respect to x and for $\forall x \in \Omega,\ x =$fixed in Ω, $g(x, \xi)$ is of class $C^2(\Omega)$ with respect to ξ.

Observation 4.1.2 *If the operator L is self-adjoint, then the properties of the function of Green become*

(i') the function $\Lambda(x, \xi)$ is a fundamental solution for the operator L;

(ii') the function $G(x, \xi)$ satisfies, with respect to x, the condition (c), for any ξ;

(iii') the function $g(x, \xi)$ satisfies the equation

$$L_x g(x, \xi) = 0,\ \forall x \in \Omega;$$

(iv')≡(iv).

An important property of the function of Green is proven in the following theorem. Denote by (co^*) the adjoint homogeneous condition of the boundary condition above denoted by (c).

Theorem 4.1.1 *Let $G(x, \xi)$ be the function of Green which corresponds to the triplet (Ω, L, (c)) and $G^*(x, \eta)$ the function of Green which corresponds to the triplet (Ω, L, (co^*)). Then, the following equality holds true:*

$$G(\eta, \xi) = G^*(\xi, \eta),\ \forall \xi, \eta \in \Omega,\ \xi \neq \eta.$$

Proof We will isolate the points ξ and η with ellipsoids

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(\xi)(x_i - \xi_i)(x_j - \xi_j) \leq \varrho^2, \qquad (e_1)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(\eta)(x_i - \eta_i)(x_j - \eta_j) \leq \varrho^2. \qquad (e_2)$$

We choose ϱ sufficiently small so that the two ellipsoids be fully in Ω and, in addition, have no point in common. We will write Green's formula for the corona which remains if we eliminate the two ellipsoids and for the pair of functions $v = G(x, \eta)$ and $v = G^*(x, \xi)$:

$$\begin{aligned}
0 &= \int_{\Omega\setminus((e_1)\cup(e_2))} \left[G(x,\eta)L_x G^*(x,\xi) - G^*(x,\xi)M_x G(x,\eta)\right] \mathrm{d}x \\
&= \int_{\partial\Omega} \left\{\gamma\left[G(x,\eta)\frac{\partial G^*(x,\xi)}{\partial\gamma} - G^*(x,\xi)\frac{\partial G(x,\eta)}{\partial\gamma}\right] + bG(x,\eta)G^*(x,\xi)\right\}\mathrm{d}\sigma_x \\
&+ \int_{\partial e_1} \left\{\gamma\left[G(x,\eta)\frac{\partial G^*(x,\xi)}{\partial\gamma} - G^*(x,\xi)\frac{\partial G(x,\eta)}{\partial\gamma}\right] + bG(x,\eta)G^*(x,\xi)\right\}\mathrm{d}\sigma_x \\
&+ \int_{\partial e_2} \left\{\gamma\left[G(x,\eta)\frac{\partial G^*(x,\xi)}{\partial\gamma} - G^*(x,\xi)\frac{\partial G(x,\eta)}{\partial\gamma}\right] + bG(x,\eta)G^*(x,\xi)\right\}\mathrm{d}\sigma_x .
\end{aligned}$$

The conditions (c) and (co^*) have been imposed so that the integrand of the first integral from the right-hand side of this relation be null. If in addition, we take into account

$$L_x G^*(x,\xi) = 0, \;\; M_x G(x,\eta) = 0,$$

we obtain

$$\begin{aligned}
&\int_{\partial e_1} \left\{\gamma\left[G(x,\eta)\frac{\partial G^*(x,\xi)}{\partial\gamma} - G^*(x,\xi)\frac{\partial G(x,\eta)}{\partial\gamma}\right] + bG(x,\eta)G^*(x,\xi)\right\}\mathrm{d}\sigma_x \\
&= -\int_{\partial e_2} \left\{\gamma\left[G(x,\eta)\frac{\partial G^*(x,\xi)}{\partial\gamma} - G^*(x,\xi)\frac{\partial G(x,\eta)}{\partial\gamma}\right] + bG(x,\eta)G^*(x,\xi)\right\}\mathrm{d}\sigma_x .
\end{aligned} \tag{4.1.9}$$

Denote by I_1 the integral

$$I_1 = \int_{\partial e_1} G^*(x,\xi)\left[-\gamma\frac{\partial G(x,\eta)}{\partial\gamma} + bG(x,\eta)\right]\mathrm{d}\sigma_x .$$

If we take into account that $G^*(x,\xi) = \Lambda^*(x,\xi) + g^*(x,\xi)$, I_1 becomes

$$\begin{aligned}
I_1 &= \int_{\partial e_1} \Lambda^*(x,\xi)\left[-\gamma\frac{\partial G(x,\eta)}{\partial\gamma} + bG(x,\eta)\right]\mathrm{d}\sigma_x \\
&+ \int_{\partial e_1} g^*(x,\xi)\left[-\gamma\frac{\partial G(x,\eta)}{\partial\gamma} + bG(x,\eta)\right]\mathrm{d}\sigma_x .
\end{aligned}$$

Both of the integrands above are continuous functions. Therefore, if the ellipsoid (e_1) is deformed homothetically to its center ξ, we obtain that $I_1 \to 0$ if $(e_1) \to \xi$ (homothetically).

Consider now the integral I_2 given by

$$I_2 = \int_{\partial e_1} \gamma G(x,\eta)\frac{\partial G^*(x,\xi)}{\partial\gamma}\mathrm{d}\sigma_x .$$

If we take into account that $G^*(x,\xi) = \Lambda^*(x,\xi) + g^*(x,\xi)$, I_2 becomes

$$I_2 = \int_{\partial e_1} \gamma G(x,\eta) \frac{\partial \Lambda^*(x,\xi)}{\partial \gamma} \mathrm{d}\sigma_x + \int_{\partial e_1} \gamma G(x,\eta) \frac{\partial g^*(x,\xi)}{\partial \gamma} \mathrm{d}\sigma_x. \tag{4.1.10}$$

From the properties of the Levi function, we know that

$$\lim_{(e_1)\to\xi} \int_{\partial e_1} \gamma u(x) \frac{\partial \Lambda^*(x,\xi)}{\partial \gamma} \mathrm{d}\sigma_x = u(\xi),$$

and then

$$\lim_{(e_1)\to\xi} \int_{\partial e_1} \gamma G(x,\xi) \frac{\partial \Lambda^*(x,\xi)}{\partial \gamma} \mathrm{d}\sigma_x = G(\xi,\eta). \tag{4.1.11}$$

The integrand of the last integral from (4.1.10) is a continuous function, and therefore if (e_1) is deformed homothetic to ξ, this integral tends to zero. Taking into account this observation and the relation (4.1.11), we deduce that if we pass to the limit in (4.1.10) with $(e_1) \to \xi$ (homothetically), we obtain

$$\lim_{(e_1)\to\xi} I_2 = G(\xi,\eta).$$

In conclusion, taking into account that $I_1 \to 0$, the left-hand side from (4.1.9) tends to $G(\xi,\eta)$, when $(e_1) \to \xi$, homothetically.

Consider now the integral I_3 given by

$$I_3 = -\int_{\partial e_2} G(x,\eta) \left[\gamma \frac{\partial G^*(x,\xi)}{\partial \gamma} + bG^*(x,\xi) \right] \mathrm{d}\sigma_x.$$

Because $G(x,\eta) = \Lambda(x,\eta) + g(x,\eta)$, we deduce that the integral I_3 becomes

$$\begin{aligned} I_3 = &-\int_{\partial e_2} \Lambda(x,\eta) \left[\gamma \frac{\partial G^*(x,\xi)}{\partial \gamma} + bG^*(x,\xi) \right] \mathrm{d}\sigma_x \\ &-\int_{\partial e_2} g(x,\eta) \left[\gamma \frac{\partial G^*(x,\xi)}{\partial \gamma} + bG^*(x,\xi) \right] \mathrm{d}\sigma_x. \end{aligned}$$

Both integrands of I_3 are continuous functions, and then when the ellipsoid (e_2) is deformed homothetically to its center η, we obtain

$$\lim_{(e_2)\to\eta} I_3 = 0.$$

The last integral from (4.1.9) is

$$I_4 = \int_{\partial e_2} \gamma G^*(x,\xi) \frac{\partial G(x,\eta)}{\partial \gamma} \mathrm{d}\sigma_x.$$

If we take into account that $G(x, \eta) = \Lambda(x, \eta) + g(x, \eta)$, we deduce that I_4 becomes

$$I_4 = \int_{\partial e_2} \gamma G^*(x, \xi) \frac{\partial \Lambda(x, \eta)}{\partial \gamma} \mathrm{d}\sigma_x + \int_{\partial e_2} \gamma G^*(x, \xi) \frac{\partial g(x, \eta)}{\partial \gamma} \mathrm{d}\sigma_x. \quad (4.1.12)$$

From the properties of the Levi function, we know that

$$\lim_{(e_2) \to \eta} \int_{\partial e_2} \gamma u(x) \frac{\partial \Lambda(x, \eta)}{\partial \gamma} \mathrm{d}\sigma_x = u(\eta),$$

and then

$$\lim_{(e_2) \to \eta} \int_{\partial e_1} \gamma G^*(x, \xi) \frac{\partial \Lambda(x, \eta)}{\partial \gamma} \mathrm{d}\sigma_x = G^*(\eta, \xi).$$

The integrand of the last integral from (4.1.12) is a continuous function and then when (e_2) is deformed homothetically to its center η, this integral tends to zero. Therefore, $I_4 \to G^*(\eta, \xi)$. As we have already shown that $I_3 \to 0$, we deduce that the right-hand side of equality (4.1.9) tends to $G^*(\eta, \xi)$.

Finally, if we pass to the limit in the equality (4.1.9), (e_1) being deformed homothetically to the center or ξ and (e_2) being deformed homothetically to the center or η, then we obtain $G(\xi, \eta) = G^*(\eta, \xi)$. ∎

Corollary 4.1.1 *If the operator L is self-adjoint, then the attached function of Green is symmetrical with respect to its two arguments.*

Proof Because L is self-adjoint we have $L = M$ and because from Theorem 4.1.1 we have $G(\xi, \eta) = G^*(\eta, \xi)$, we obtain immediately $G(\xi, \eta) = G(\eta, \xi)$, that is, we demonstrated the symmetry of the function G. ∎

We will clarify further why the function of Green is used for the representation of the solution of a boundary value problem.

Let us consider the problem

$$\begin{aligned} &Lu(x) = f(x), \ \forall x \in \Omega, \\ &\alpha \frac{\partial u}{\partial \gamma}(y) + \beta u(y) = \varphi(y), \ \forall y \in \partial\Omega. \end{aligned} \quad (4.1.13)$$

Theorem 4.1.2 *If the problem (4.1.13) has a solution, then it is represented in the form*

$$u(\xi) = -\int_\Omega G(x, \xi) f(x) \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma}{\alpha} G(x, \xi) \varphi(x) \mathrm{d}\sigma_x,$$

where $G(x, \xi)$ is the function of Green which corresponds to the domain Ω, to the operator L and to the mixed boundary condition $(4.1.13)_2$.

Proof We will write Green's formula for the pair of functions $v = g(x, \xi)$ and u, where u is the solution of the problem (4.1.13)

$$0 = -\int_\Omega [g(x,\xi)L_x u(x) - u(x)M_x g(x,\xi)]\,\mathrm{d}x$$
$$+\int_{\partial\Omega}\left\{\gamma\left[g(x,\xi)\frac{\partial u(x)}{\partial\gamma} - u(x)\frac{\partial g(x,\xi)}{\partial\gamma}\right] + bu(x)g(x,\xi)\right\}\mathrm{d}\sigma_x.$$

Taking into account that $Lu(x) = f(x)$ and $M_x g(x,\xi) = 0$, the equality above becomes

$$0 = -\int_\Omega g(x,\xi)f(x)\mathrm{d}x$$
$$+\int_{\partial\Omega}\left\{\gamma\left[g(x,\xi)\frac{\partial u(x)}{\partial\gamma} - u(x)\frac{\partial g(x,\xi)}{\partial\gamma}\right] + bu(x)g(x,\xi)\right\}\mathrm{d}\sigma_x. \tag{4.1.14}$$

We now write the Riemann–Green's formula for the pair of functions $v = \Lambda(x, \xi)$ and u, where u is the solution of the problem (4.1.13)

$$u(\xi) = -\int_\Omega [\Lambda(x,\xi)L_x u(x) - u(x)M_x \Lambda(x,\xi)]\,\mathrm{d}x$$
$$+\int_{\partial\Omega}\left\{\gamma\left[\Lambda(x,\xi)\frac{\partial u(x)}{\partial\gamma} - u(x)\frac{\partial \Lambda(x,\xi)}{\partial\gamma}\right] + bu(x)\Lambda(x,\xi)\right\}\mathrm{d}\sigma_x.$$

Because $Lu(x) = f(x)$ and $M_x\Lambda(x,\xi) = 0$, the equality above becomes

$$u(\xi) = -\int_\Omega \Lambda(x,\xi)f(x)\mathrm{d}x$$
$$+\int_{\partial\Omega}\left\{\gamma\left[\Lambda(x,\xi)\frac{\partial u(x)}{\partial\gamma} - u(x)\frac{\partial \Lambda(x,\xi)}{\partial\gamma}\right] + bu(x)\Lambda(x,\xi)\right\}\mathrm{d}\sigma_x. \tag{4.1.15}$$

By summing member by member the relations (4.1.14) and (4.1.15), we deduce

$$u(\xi) = -\int_\Omega G(x,\xi)f(x)\mathrm{d}x + \int_{\partial\Omega}\frac{\gamma}{\alpha}G(x,\xi)\left[\alpha\frac{\partial u(x)}{\partial\gamma} + \beta u(x)\right]\mathrm{d}\sigma_x$$
$$-\int_{\partial\Omega}\frac{1}{\alpha}u(x)\left[\gamma\alpha\frac{\partial G(x,\xi)}{\partial\gamma} + (\gamma\beta - b\alpha)G(x,\xi)\right]\mathrm{d}\sigma_x$$

so that if we take into account the boundary condition $(4.1.13)_1$ and the properties of the function of Green, we obtain

$$u(\xi) = -\int_\Omega G(x,\xi)f(x)\mathrm{d}x + \int_{\partial\Omega}\frac{\gamma}{\alpha}G(x,\xi)\varphi(x)\mathrm{d}\sigma_x, \tag{4.1.16}$$

that is, it is just the formula from the statement of the theorem. ■

After determining the function of Green, we approach the problem of determining the conditions that must be imposed on the functions f and φ so that the function u defined in (4.1.16) is effectively a solution of the mixed boundary value problem (4.1.13).

Observation 4.1.3 *1º. If in problem (4.1.13), in fact on the right-hand side of the equation* $(4.1.13)_1$*, instead of the function* f *we take the nonlinear function* $F(x, u(x))$*, then the considerations from Theorem 4.1.2 and those that follow can be transposed, correspondingly, for the problem*

$$\begin{aligned} &Lu(x) = F(x, u(x)), \ \forall x \in \Omega, \\ &\alpha \frac{\partial u}{\partial \gamma}(y) + \beta u(y) = \varphi(y), \ \forall y \in \partial\Omega. \end{aligned} \tag{4.1.17}$$

We will proceed analogously as in Theorem 4.1.2 so that we deduce that if the problem (4.1.17) has a solution, then it is represented in the form

$$u(\xi) = -\int_{\Omega} G(x,\xi) F(x,u(x)) \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma}{\alpha} G(x,\xi)\varphi(x) \mathrm{d}\sigma_x. \tag{4.1.18}$$

2º. Through particularization of the boundary condition in (4.1.13) or (4.1.17), the boundary value problem of Dirichlet type and of Neumann type are obtained. Then, in the linear case, the form of the solution will be analogous to formula (4.1.16), and in the nonlinear case, the solution will have a similar representation to formula (4.1.18).

4.2 The Dirichlet's Problem

Consider the operator L and its adjoint M, defined in Sect. 4.1, with the standard assumptions established therein. Recall the inside Dirichlet's problem

$$\begin{aligned} &Lu(x) = f(x), \ \forall x \in \Omega, \\ &u(y) = \varphi(y), \ \forall y \in \partial\Omega, \end{aligned} \tag{4.2.1}$$

and we attach the homogeneous problem

$$\begin{aligned} &Lv(x) = 0, \ \forall x \in \Omega, \\ &v(y) = 0, \ \forall y \in \partial\Omega. \end{aligned} \tag{4.2.2}$$

We will assume, in the following, that the operator L is self-adjoint, and therefore $L = M$. In Sect. 4.1, we have seen that in this situation the function of Green is symmetrical in its arguments, that is, we have the equality $G(\eta, \xi) = G(\xi, \eta), \ \forall \eta, \xi \in \Omega$.

We are searching for a function of Green of the form

$$G(x, \xi) = \Lambda(x, \xi) + g(x, \xi),$$

in which the Levi function $\Lambda(x, \xi)$ is the fundamental solution for the operator L, and $g(x, \xi)$ is a solution of the equation $Lv(x) = 0$. We will impose on the function of Green $G(x, \xi)$ to satisfy the homogeneous condition $(4.2.2)_2$. To this end, we write Green's formula for the pair of functions $v = g(x, \xi)$ and u, where u is a solution of the problem (4.2.1):

$$0 = -\int_{\Omega} g(x, \xi) Lu(x) \mathrm{d}x - \int_{\partial\Omega} \gamma \left[g(x, \xi) \frac{\partial u(x)}{\partial \gamma} - u(x) \frac{\partial g(x, \xi)}{\partial \gamma} \right] \mathrm{d}\sigma_x,$$

in which we take into account that $b = 0$, from the hypothesis that the operator L is self-adjoint.

If we take into account that $Lu(x) = f(x), \ \forall x \in \Omega$ and $u(y) = \varphi(y), \ \forall y \in \partial\Omega$, the above formula becomes

$$0 = -\int_{\Omega} g(x, \xi) f(x) \mathrm{d}x - \int_{\partial\Omega} \gamma \left[g(x, \xi) \frac{\partial u(x)}{\partial \gamma} - \varphi(x) \frac{\partial g(x, \xi)}{\partial \gamma} \right] \mathrm{d}\sigma_x. \qquad (4.2.3)$$

We now write Riemann–Green's formula for the pair of functions $v = \Lambda(x, \xi)$ and u, where u is a solution of the problem (4.2.1)

$$u(\xi) = -\int_{\Omega} \Lambda(x, \xi) Lu(x) \mathrm{d}x + \int_{\partial\Omega} \gamma \left[\Lambda(x, \xi) \frac{\partial u(x)}{\partial \gamma} - u(x) \frac{\partial \Lambda(x, \xi)}{\partial \gamma} \right] \mathrm{d}\sigma_x,$$

in which we take into account that $b = 0$, from the hypothesis that the operator L is self-adjoint.

If we take into account that $Lu(x) = f(x), \ \forall x \in \Omega$ and $u(y) = \varphi(y), \ \forall y \in \partial\Omega$, the above formula becomes

$$u(\xi) = -\int_{\Omega} \Lambda(x, \xi) f(x) \mathrm{d}x + \int_{\partial\Omega} \gamma \left[\Lambda(x, \xi) \frac{\partial u(x)}{\partial \gamma} - \varphi(x) \frac{\partial \Lambda(x, \xi)}{\partial \gamma} \right] \mathrm{d}\sigma_x. \qquad (4.2.4)$$

We add, member by member, the relations (4.2.3) and (4.2.4) and we deduce

$$u(\xi) = -\int_{\Omega} G(x, \xi) f(x) \mathrm{d}x + \int_{\partial\Omega} \gamma \left[G(x, \xi) \frac{\partial u(x)}{\partial \gamma} - \varphi(x) \frac{\partial G(x, \xi)}{\partial \gamma} \right] \mathrm{d}\sigma_x \qquad (4.2.5)$$

because $\Lambda(x, \xi) + g(x, \xi) = G(x, \xi)$.

According to the properties of the function of Green, we have $G(x, \xi) = 0$ on $\partial\Omega$ and then (4.2.5) becomes

$$u(\xi) = -\int_{\Omega} G(x, \xi) f(x) \mathrm{d}x - \int_{\partial\Omega} \gamma \frac{\partial G(x, \xi)}{\partial \gamma} \varphi(x) \mathrm{d}\sigma_x. \tag{4.2.6}$$

Thus, we proved the following result.

Proposition 4.2.1 *If $G(x, \xi)$ is the function of Green attached to the domain Ω, to the operator L and to the (nonhomogeneous) Dirichlet boundary condition, then if a solution of the Dirichlet inside problem (4.2.1) exists then this is given by formula (4.2.6).*

In the particular case when L is the Laplace operator, $L = \Delta$, we have $\gamma = 1$ and the derivative in the direction γ becomes the derivative in the normal direction. The inside Dirichlet's problem for the equation of Poisson has the form

$$\begin{aligned} \Delta u(x) &= f(x), \ \forall x \in \Omega, \\ u(y) &= \varphi(y), \ \forall y \in \partial\Omega, \end{aligned} \tag{4.2.7}$$

where f is given and continuous on Ω and φ is given and continuous on $\partial\Omega$. Suppose that we know the function of Green attached to the domain Ω, to the Laplace operator and to the nonhomogeneous Dirichlet boundary condition. Then, if the problem (4.2.7) admits a classical solution, then this has the representation

$$u(\xi) = -\int_{\Omega} G(x, \xi) f(x) \mathrm{d}x - \int_{\partial\Omega} \frac{\partial G(x, \xi)}{\partial \nu_x} \varphi(x) \mathrm{d}\sigma_x.$$

Proposition 4.2.2 *If $G(x, \xi)$ is the function of Green attached to the domain Ω, to the operator L and to the nonhomogeneous Dirichlet boundary condition, then $G(x, \xi)$ is nonnegative, $G(x, \xi) \geq 0, \ \forall x, \xi \in \overline{\Omega}$.*

Proof If we take into account the expression of the Levi function attached to the operator Δ,

$$\Lambda(x, \xi) = \frac{1}{(n-2)\omega_n} \frac{1}{r_{\xi x}^{n-2}},$$

and the fact that, by definition, $G(x, \xi) = \Lambda(x, \xi) + g(x, \xi)$, we can write

$$G(x, \xi) = \frac{1}{(n-2)\omega_n} \frac{1}{r_{\xi x}^{n-2}} + g(x, \xi).$$

Because $g(x, \xi)$ is a continuous function, we deduce that $g(x, \xi)$ is a bounded function on $\overline{\Omega}$. Then

$$\lim_{x \to \xi} G(x, \xi) = \infty.$$

For $\forall \delta > 0$, we take the ball $B(\xi, \delta)$ which is fully included in Ω, on which $G(x, \xi) > 0$. Consider the corona $K = \Omega \setminus B(\xi, \delta)$. In K, by definition, we have that $G(x, \xi)$ is of class C^2 and satisfies the equation

$$\Delta_x G(x, \xi) = 0, \ \forall x \in K,$$

that is, G is a harmonic function in the open corona.

On the other hand, we have

$$G(y, \xi) > 0, \ \forall y \in \partial B(\xi, \delta)$$

and

$$G(y, \xi) = 0, \ \forall y \in \partial \Omega,$$

because, from the definition of the function of Green, we have that G verifies the adjoint homogeneous boundary condition. Being harmonic on the open corona, the function G lends itself to the min-max principle for harmonic functions. Then, G reaches its infimum on the boundary of the corona, that is,

$$\inf_{x \in \overline{K}} G(x, \xi) \geq 0,$$

and therefore $G(x, \xi) \geq 0, \forall x \in K$. This ends the proof of the proposition. ■

Consider the following problem of eigenvalues and eigenfunctions:

$$\begin{aligned} &\Delta u(x) - \lambda u(x) = 0, \ \forall x \in \Omega, \\ &u(y) = \varphi(y), \ \forall y \in \partial \Omega. \end{aligned} \tag{4.2.8}$$

We notice that if we pass $\lambda u(x)$ to the right-hand side and consider it as a function $f(x)$, then we can do analogous considerations to those from above. If the problem (4.2.8) admits a solution, then its form is

$$u(\xi) = -\lambda \int_{\Omega} G(x, \xi) u(x) \mathrm{d}x - \int_{\partial \Omega} \frac{\partial G(x, \xi)}{\partial \nu_x} \varphi(x) \mathrm{d}\sigma_x. \tag{4.2.9}$$

Thus, the solution u of the problem (4.2.8) satisfies an integral equation of Fredholm type. Equation (4.2.9) is self-adjoint because its kernel is symmetric, taking into account that the function $G(x, \xi)$ is symmetric.

Consider now a more general problem

$$\Delta u(x) = F\left(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right), \ \forall x \in \Omega,$$
$$u(y) = \varphi(y), \ \forall y \in \partial\Omega. \tag{4.2.10}$$

We will build the function of Green attached to the domain Ω, to the operator L and to boundary condition $(4.2.10)_2$. Using the previous procedure, if the problem (4.2.10) admits a solution, then this is represented by

$$u(\xi) = -\lambda \int_{\Omega} G(x, \xi) F\left(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) \mathrm{d}x - \int_{\partial\Omega} \frac{\partial G(x, \xi)}{\partial \nu_x} \varphi(x) \mathrm{d}\sigma_x, \tag{4.2.11}$$

that is, u satisfies an integral equation of Fredholm type. Therefore, we reduce the problem of finding a solution of the problem (4.2.10) to the problem of finding a solution for the integral equation (4.2.11).

Without going into details, we suggest a method for determining a solution of the nonlinear integral equation (4.2.11).

Consider the operator T defined by

$$Tu = -\lambda \int_{\Omega} G(x, \xi) F\left(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) \mathrm{d}x - \int_{\partial\Omega} \frac{\partial G(x, \xi)}{\partial \nu_x} \varphi(x) \mathrm{d}\sigma_x,$$

We can show that the operator T has a fixed point in the space of the solutions, having an appropriate topology and this fixed point will be the solution of the equation (4.2.11).

From the above considerations, we deduce clearly that in determining the solution of a boundary value problem, the construction of the function of Green, associated to a domain, to an operator, defined on this domain and to some boundary conditions, is essential.

In general, it is difficult to build a function of Green. The problem becomes easier in the case of some particular operators and, especially, in the case of some particular domains.

We illustrate, in the following, the construction of the function of Green in the case when the domain is a sphere from the n-dimensional space and the operator is the Laplacian.

Consider the ball $B(0, R) \subset \mathbb{R}^n$. Let the points x, x', ξ, ξ' so that

$$\overline{0\xi}.\overline{0\xi'} = \overline{0x}.\overline{0x'} = R^2$$

or, otherwise written

$$\frac{\overline{0\xi}}{\overline{0x`}} = \frac{\overline{0x}}{\overline{0\xi`}},$$

from where we deduce the similarity of the triangles $\triangle 0\xi x \sim \triangle 0\xi`x`$.

The complete quotients of similarity are

$$\frac{\overline{0\xi}}{\overline{0x`}} = \frac{\overline{0x}}{\overline{0\xi`}} = \frac{\overline{\xi x}}{\overline{\xi`x`}},$$

from where we deduce

$$\frac{1}{r_{\xi x}} = \frac{1}{\overline{\xi x}} = \frac{\overline{0x`}}{\overline{\xi`x`}.\overline{0\xi`}}.$$

If $x \in \partial B(0, R)$, then $x` \equiv x$ and

$$\frac{1}{r_{\xi x}} = \frac{R}{\overline{\xi`x}.\overline{0\xi}}.$$

We define the Green function $G(x, \xi)$ by

$$G(x, \xi) = \frac{1}{(n-2)\omega_n}\left[\frac{1}{r_{\xi x}^{n-2}} - \frac{R^{n-2}}{\overline{\xi`x}^{n-2}\overline{0\xi}^{n-2}}\right], \tag{4.2.12}$$

so that

$$\Lambda(x, \xi) = \frac{1}{(n-2)\omega_n}\frac{1}{r_{\xi x}^{n-2}},\ \ g(x, \xi) = -\frac{1}{(n-2)\omega_n}\frac{R^{n-2}}{\overline{\xi`x}^{n-2}\overline{0\xi}^{n-2}}.$$

It is clear that

$$\frac{1}{\overline{\xi`x}^{n-2}} = \frac{1}{r_{\xi`x}^{n-2}}$$

is a regular function. Indeed, it is impossible to achieve the singularity $\xi` = x$, because $x$ is on the sphere and $\xi`$ is outside the sphere. Also, inside the sphere we have

$$\Delta_x\left(\frac{1}{r_{\xi`x}^{n-2}}\right) = 0,$$

and on the boundary

$$G(y, \xi) = 0,\ \forall y \in \partial B(0, R),$$

that is, the function $G(x, \xi)$ defined in (4.2.12) satisfies the conditions of a Green function. From the formula of representation of the solution of a boundary value problem with the help of the function of Green, it is certified that we need a derivative of the function $G(x, \xi)$ in the normal direction ν, the derivative computed on the boundary. If we denote by β the angle between 0ξ‘ and $0x$‘ and write the generalized Theorem of Pythagoras, we obtain

$$G(x,\xi) = \frac{1}{(n-2)\omega_n}\left[\frac{1}{[\overline{0x}^2 + \overline{0\xi}^2 - 2\overline{0x}.\overline{0\xi}\cos\beta]^{(n-2)/2}}\right.$$
$$\left. - \frac{R^{n-2}}{\overline{0\xi}^{n-2}[\overline{0x}^2 + \overline{0\xi‘}^2 - 2\overline{0x}.\overline{0\xi‘}\cos\beta]^{(n-2)/2}}\right].$$

Because the direction of the outside normal is just the direction $0x$, we have

$$\frac{\partial G(x,\xi)}{\partial \nu_x} = \frac{1}{\omega_n}\left[\frac{-\overline{0x} + \overline{0\xi}\cos\beta}{[\overline{0x}^2 + \overline{0\xi}^2 - 2\overline{0x}\overline{0\xi}\cos\beta]^{n/2}}\right.$$
$$\left. - \frac{R^{n-2}[\overline{0\xi}^2\overline{0x} - R^2\overline{0\xi}\cos\beta]}{[R^4 + \overline{0x}^2.\overline{0\xi}^2 - 2R^2\overline{0x}.\overline{0\xi}\cos\beta]^{n/2}}\right].$$

But we only need the derivative of the function of Green in the normal direction, only on the boundary, so we deduce that

$$\left.\frac{\partial G(x,\xi)}{\partial \nu_x}\right|_{x\in\partial B(0,R)} = -\frac{R^2 - \varrho^2}{R\omega_n\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}},$$

where

$$\varrho^2 = \overline{0\xi} = \sqrt{\sum_{i=1}^{n}\xi_i^2}.$$

If we replace these calculations in the formula of representation of the solution of the Dirichlet problem with the help of the function of Green, we obtain

$$u(\xi) = -\int_{B(0,R)} G(x,\xi) f(x)\mathrm{d}x$$
$$+ \frac{R^2 - \varrho^2}{R\omega_n}\int_{\partial B(0,R)} \frac{\varphi(x)}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}}\mathrm{d}\sigma_x, \tag{4.2.13}$$

where $G(x, \xi)$ has the form (4.2.12).

In conclusion, for the particular problem

$$\Delta u(x) = f(x),\ \forall x \in B(0, R),$$
$$u(y) = \varphi(y),\ \forall y \in \partial B(0, R), \tag{4.2.14}$$

the solution, if it exists, has the form (4.2.13).

It remains to be found in what conditions the function u from (4.2.13) is an effective solution for the problem (4.2.14).

Theorem 4.2.1 *If the function f is Hőlder on compact sets from the ball $B(0, R)$ and has a null value outside it, and the function φ is continuous on $\partial B(0, R)$, then the function u from (4.2.13) is an effective solution for the problem (4.2.14).*

Proof Write the right-hand side from (4.2.13) in the form

$$-\frac{1}{(n-2)\omega_n}\int_{B(0,R)}\frac{1}{r_{\xi x}^{n-2}}f(x)\mathrm{d}x-\int_{B(0,R)}g(x,\xi)f(x)\mathrm{d}x$$

$$-\frac{1}{(n-2)\omega_n}\int_{\partial B(0,R)}\frac{\partial}{\partial\nu}\left(\frac{1}{r_{\xi x}^{n-2}}\right)\varphi(x)\mathrm{d}\sigma_x-\int_{\partial B(0,R)}\frac{\partial g(x,\xi)}{\partial\nu}\varphi(x)\mathrm{d}\sigma_x$$

$$=I_1+I_2+I_3+I_4.$$

The significance of the notations for the integrals I_1, I_2, I_3 and I_4 in the above equality is clear.
Then

$$\Delta_\xi u(\xi)=\Delta_\xi I_1+\Delta_\xi I_2+\Delta_\xi I_3+\Delta_\xi I_4.$$

But $\Delta_\xi I_2=\Delta_\xi I_3=\Delta_\xi I_4=0$ due to the fact that the functions which appear under the three integrals are harmonic functions. Then, we have $\Delta_\xi I_1=f$, because the potential of volume satisfies the equation of Poisson in which the right-hand side is just the density, if this is a Hőlder function (in our case, the density is f, and this is a Hőlder function, according to the assumptions). From the above relations, we deduce

$$\Delta_\xi u(\xi)=f(\xi),\ \forall\xi\in\partial B(0,R),$$

that is, the function u from (4.2.13) verifies Poisson's equation $(4.2.14)_1$. We want now to verify the boundary condition $(4.2.14)_2$. As we defined the function of Green in (4.2.12), it is easy to see that it is null outside the inner ball and therefore $G(y,\xi)=0,\ \forall y\in\partial B(0,R)$. Then for $y\in\partial B(0,R)$, the first integral from (4.2.13) is null. Denote by $\tilde{u}$ the value of the second integral from (4.2.13) computed on $\partial B(0,R)$, that is,

$$\tilde{u}=\frac{R^2-\varrho^2}{R\omega_n}\int_{\partial B(0,R)}\frac{\varphi(x)}{\left[R^2+\varrho^2-2R\varrho\cos\beta\right]^{n/2}}\mathrm{d}\sigma_x.$$

We will prove that if ξ tends by inside points from the ball $B(0,R)$ to the point $y\in\partial B(0,R)$, then $\tilde{u}$ tends to $\varphi(y)$.

We arbitrarily fix $y \in \partial B(0, R)$ and we take a ball $B(y, \eta)$ with center in y and the radius η sufficient small. Because φ was supposed to be continuous on $\partial B(0, R)$, we deduce that φ is uniform continuous in the ball $B(y, \eta)$.

We use the notations

$$\sigma = \partial B(0, R) \cap B(y, \eta),$$
$$\Sigma = \partial B(0, R) \setminus \sigma.$$

A classical result of mathematical analysis states that

$$\frac{R^2 - \varrho^2}{R\omega_n} \int_{\partial B(0,R)} \frac{1}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}} \mathrm{d}\sigma_x \equiv 1.$$

We have the evaluations

$$\begin{aligned} |\tilde{u}(\xi) - \varphi(y)| &\le \frac{R^2 - \varrho^2}{R\omega_n} \int_{\partial B(0,R)} \frac{|\varphi(x) - \varphi(y)|}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}} \mathrm{d}\sigma_x \\ &= \frac{R^2 - \varrho^2}{R\omega_n} \int_{\sigma} \frac{|\varphi(x) - \varphi(y)|}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}} \mathrm{d}\sigma_x \\ &+ \frac{R^2 - \varrho^2}{R\omega_n} \int_{\Sigma} \frac{|\varphi(x) - \varphi(y)|}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}} \mathrm{d}\sigma_x. \end{aligned} \tag{4.2.15}$$

On σ the difference $[\varphi(x) - \varphi(y)]$ can be made no matter how small, due to uniform continuity. Then, the first integral from the right-hand side of the relation (4.2.15) can be made no matter how small. If we take ξ inside the ball $B(y, \eta/2)$, then the difference $R - \varrho$ can be made less than $\eta/2$. On the other hand, the denominator of the last integral from (4.2.15) is continuous on Σ and strictly positive. Then if $\xi \to y$, the last integral from (4.2.15) is no matter how small. In conclusion, if we pass to the limit in (4.2.15) with $\xi \to y$, then $\tilde{u}(\xi) \to \varphi(y)$ and this ends the proof of Theorem 4.2.1. ■

4.3 Properties of Harmonic Functions

In Sect. 4.2, we represented the solution of Dirichlet's problem for the sphere $B(0, R)$ in the form

$$\begin{aligned} u(\xi) &= -\int_{B(0,R)} G(x, \xi) f(x) \mathrm{d}x \\ &+ \frac{R^2 - \varrho^2}{R\omega_n} \int_{\partial B(0,R)} \frac{\varphi(x)}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}} \mathrm{d}\sigma_x. \end{aligned} \tag{4.3.1}$$

The last term from this formula leads to

$$\frac{R^2-\varrho^2}{R\omega_n}\int_{\partial B(0,R)}\frac{\varphi(x)}{\left[R^2+\varrho^2-2R\varrho\cos\beta\right]^{n/2}}\mathrm{d}\sigma_x=\varphi(y),\ \forall y\in\partial B(0,R). \tag{4.3.2}$$

Usually, the formula (4.3.1) is called *the generalized Poisson formula*. In the present section, many considerations will be based on the formula of Poisson, or the kernel of Poisson, defined by

$$H(x,\xi)=\frac{R^2-\varrho^2}{R\omega_n}\frac{1}{\left[R^2+\varrho^2-2R\varrho\cos\beta\right]^{n/2}}=\frac{R^2-\varrho^2}{R\omega_n}\frac{1}{r_{\xi x}^n}, \tag{4.3.3}$$

where β is the angle between $0x$ and 0ξ.

In the following theorem, we prove that if a function u is a harmonic function in the domain Ω, then u is an analytic function in Ω.

Theorem 4.3.1 *Let Ω be a domain and the function $u \in C^2(\Omega)$. If u is harmonic on the domain Ω, then u is an analytic function on Ω.*

Proof Let $0 \in \Omega$ be the origin of a system of coordinates, so that for arbitrary points x and ξ, we have

$$\overline{0\xi}=\sqrt{\sum_{i=1}^{n}\xi_i^2},\ \overline{0x}=\sqrt{\sum_{i=1}^{n}x_i^2}.$$

Because Ω is a domain and, therefore, is an open set, and 0 is considered to be inside of Ω, we deduce that there is a closed ball $B(0, R) \subset \Omega$.

Denote $\varrho = \overline{0\xi}$ and define the function $w(x, \xi)$ by

$$w(x,\xi)=\frac{\varrho^2-2\varrho\overline{0x}\cos\beta}{R^2}. \tag{4.3.4}$$

We now use the known function k, defined by

$$k(w)=\frac{1}{(1+w)^{n/2}},$$

which is an elementary function and, also, an analytic function on the open disk $|w| < 1$.

With the help of the function k, we will establish a relation with the kernel of Poisson

$$\left(1-\frac{\varrho^2}{R^2}\right)k(w(x,\xi))\Big|_{x\in\partial B(0,R)}=\frac{R^n\left(1-\frac{\varrho^2}{R^2}\right)}{\left(R^2+\varrho^2-2R\varrho\cos\beta\right)^{n/2}}. \tag{4.3.5}$$

Obviously, we can write

$$\left(1-\frac{\varrho^2}{R^2}\right)k(w(x,\xi)) = R^{n-3}\omega_n \frac{R^2-\varrho^2}{R\omega_n}\frac{1}{\left(R^2+\varrho^2-2R\varrho\cos\beta\right)^{n/2}}$$

and then

$$\left(1-\frac{\varrho^2}{R^2}\right)k(w(x,\xi)) = R^{n-3}\omega_n H(x,\xi),$$

where $H(x,\xi)$ is the kernel of Poisson defined in (4.3.3).

If we take into account that

$$\varrho = |\overline{0\xi}|,\ \overline{0\xi}.\overline{0x} = \sum_{i=1}^{n} x_i\xi_i,\ \cos\beta = \frac{\sum\limits_{i=1}^{n} x_i\xi_i}{\varrho.|\overline{0x}|},$$

we deduce that w from (4.3.4) is a polynomial, and, namely, a homogeneous polynomial with regard to the variables $x_1, x_2, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n$, indifferently if $x \in \partial B(0,R)$ or $x \notin \partial B(0,R)$.

Let us consider $\varepsilon > 0$, arbitrarily small and according to ε we make the choice

$$\begin{aligned} &x \in B(0, R(1+\varepsilon)),\\ &\xi \in B\left(0, \frac{R}{3+2\varepsilon}\right). \end{aligned} \tag{4.3.6}$$

We want to show that for x and ξ having the form as in (4.3.6), $w(x,\xi)$ from (4.3.4) satisfies the condition $|w(x,\xi)| < 1$, if ε is chosen sufficiently small, in a sense to be specified later.

Indeed, with the help of (4.3.6) and taking into account that

$$\overline{0x} < R(1+\varepsilon),\ \varrho < \frac{R}{3+2\varepsilon},\ \cos\beta < 1,$$

from (4.3.4) we obtain

$$|w(x,\xi)| \le \frac{R^2}{R^2(3+2\varepsilon)^2} + \frac{2R(1+\varepsilon)R}{R^2(3+2\varepsilon)^2} = \frac{7+10\varepsilon+4\varepsilon^2}{9+12\varepsilon+4\varepsilon^2},$$

that is,

$$|w(x,\xi)| \le \frac{7}{9} + \eta(\varepsilon), \tag{4.3.7}$$

where

$$\eta(\varepsilon) = \frac{7 + 10\varepsilon + 4\varepsilon^2}{9 + 12\varepsilon + 4\varepsilon^2} - \frac{7}{9} = \frac{6\varepsilon + 8\varepsilon^2}{9(9 + 12\varepsilon + 4\varepsilon^2)}.$$

It is clear that if $\varepsilon \to 0$, then $\eta(\varepsilon) \to 0$.

Thus, if ε is sufficiently small so that $\eta(\varepsilon)$ is sufficiently small, then for x and ξ chosen as in (4.3.6), we have $w(x, \xi) < 1$.

Because in choosing (4.3.6) there is also the possibility that $0x = R$, we deduce that the kernel of Poisson, up to a constant factor, is equal to the left-hand member from (4.3.5), which is an analytic function in w, because $|w| < 1$.

Because $k(w)$ is an analytic function, as a function of w, and w is a homogeneous polynomial with regard to the variables $x_1, x_2, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n$, in agreement with the theorem of Weierstrass, we deduce that the whole expression from (4.3.5) is an analytic function.

Then, the kernel of Poisson is an analytic function of variables $x_1, x_2, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n$, with the condition that x and ξ must be chosen as in (4.3.6), for ε sufficiently small, in the sense stated above.

On the other hand, taking into account that the function u is harmonic on Ω, the expression of u from (4.3.1) is reduced to

$$u(\xi) = \frac{R^2 - \varrho^2}{R\omega_n} \int_{\partial B(0,R)} \frac{u(x)}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}} \mathrm{d}\sigma_x.$$

Considering that the integrand is an analytic function with regard to the variables x and ξ, we deduce that it can be integrated and the result is an analytic function and then u is an analytic function. ■

Corollary 4.3.1 *The following formula holds true:*

$$u(0) = \frac{1}{R^{n-1}\omega_n} \int_{\partial B(0,R)} u(x)\mathrm{d}\sigma_x. \tag{4.3.8}$$

Proof The result is obtained immediately if in the formula of Poisson, we consider the particular case $\xi \equiv 0$ and therefore $0\xi = 0$. ■

We want to mention that formula (4.3.8) is known as *the formula of Gauss*.

Another important property of harmonic functions is proven in the following Theorem, which is due to Weierstrass.

Theorem 4.3.2 *Let Ω be a bounded domain with boundary $\partial\Omega$ and $\{u_n\}_{n\geq 1}$. Also, we consider a sequence of the functions defined on Ω, having the properties*

1^o. $u_n \in C(\overline{\Omega}) \cap C^2(\Omega)$, $\forall n = 1, 2, \ldots$;

2^o. $\Delta_x u_n(x) = 0$, $\forall x \in \Omega$, $\forall n = 1, 2, \ldots$;

3^o. If $u_n(y) = \varphi_n(y)$, $\forall y \in \partial\Omega$, $\forall n = 1, 2, ..$, then $\lim\limits_{n\to\infty} \varphi_n(y) = \varphi(y)$, uniformly with respect to $y \in \partial\Omega$.

Then, we have

(i). There is

$$\lim_{n\to\infty} u_n(x) = u(x), \ \forall x \in \overline{\Omega},$$

the limit taking place uniformly with respect to $x \in \overline{\Omega}$*;*

(ii). The limit function $u(x)$ *is harmonic on* Ω*.*

Proof (i). Consider two arbitrary terms of the sequences u_q and u_p. Then, we have, obviously

$$u_p - u_q \in C(\overline{\Omega}) \cap C^2(\Omega),$$
$$\Delta_x \left[u_p(x) - u_q(x)\right] = 0, \ \forall x \in \Omega.$$

Then for the difference $u_p - u_q$, we can use the min-max principle for harmonic functions. According to this principle, the difference function $u_p - u_q$ reaches its supremum and infimum on the boundary

$$\left|u_p(x) - u_q(x)\right| \le \sup_{y\in\partial\Omega} \left|u_p(y) - u_q(y)\right|.$$

If we take into account the property 3^o, we deduce that

$$\sup_{y\in\partial\Omega} \left|u_p(y) - u_q(y)\right| = \sup_{y\in\partial\Omega} \left|\varphi_p(y) - \varphi_q(y)\right|.$$

Thus, $\forall\varepsilon > 0, \ \exists N_0(\varepsilon)$ so that $\forall p, q > N_0(\varepsilon)$, we have

$$\left|u_p(x) - u_q(x)\right| \le \sup_{y\in\partial\Omega} \left|\varphi_p(y) - \varphi_q(y)\right| < \varepsilon, \ \forall x \in \overline{\Omega},$$

in which we take into account the properties 1^o and 3^o of the functions u_n.

According to the criterion of Cauchy, we deduce that the sequence u_n is uniformly convergent on $\overline{\Omega}$, that is, there is the uniform limit

$$u(x) = \lim_{n\to\infty} u_n(x).$$

(ii). We write the formula of Poisson for the function u_n which is a harmonic function

$$u_n(\xi) = \frac{R^2 - \varrho^2}{R\omega_n} \int_{\partial B(0,R)} \frac{u_n(x)}{\left[R^2 + \varrho^2 - 2R\varrho\cos\beta\right]^{n/2}} \mathrm{d}\sigma_x. \tag{4.3.9}$$

The right-hand side from (4.3.9) is the general term of a uniform convergent sequence, because u_n is uniformly convergent. We can then pass to the limit in (4.3.9) and interchanging on the right-hand side, the limit with the integral, we obtain

$$u(\xi) = \lim_{n\to\infty} u_n(\xi) = \frac{R^2-\varrho^2}{R\omega_n}\int_{\partial B(0,R)} \frac{u(x)}{\left[R^2+\varrho^2-2R\varrho\cos\beta\right]^{n/2}}\mathrm{d}\sigma_x.$$

Therefore, the function u satisfies the formula of Poisson and, consequently, u is a harmonic function. ■

The inequalities which will be proven in the following proposition and which are called *the inequalities of Harnack* are very helpful in the study the harmonic functions.

Proposition 4.3.1 *Let* Ω *be a bounded domain and* u *be a function defined on* Ω *which satisfies the properties*

1°. $u \in C^2(\Omega)$*;*

2°. $\Delta_x u(x) = 0,\ \forall x \in \Omega$*;*

3°. $u(x) \geq 0,\ \forall x \in \Omega$.

Then, if $\overline{B(0,R)} \subset \Omega$*, the following inequalities are satisfied:*

$$\left(\frac{R}{R+\varrho}\right)^{n-2}\frac{R-\varrho}{R+\varrho}u(\eta) \leq u(\xi) \leq u(\eta)\frac{R+\varrho}{R-\varrho}\left(\frac{R}{R-\varrho}\right)^{n-2}, \tag{4.3.10}$$

for $\forall \xi \in B(\eta, R)$*, where* $\varrho = |\overline{\eta\xi}|$.

Proof We choose $\eta \equiv 0$, that is, the origin of a system of coordinates and we use the kernel of Poisson

$$H(x,\xi) = \frac{R^2-\varrho^2}{R\omega_n}\frac{1}{r^n_{\xi x}}.$$

Based on the triangle inequality, we have

$$\frac{1}{(R+\varrho)^n} \leq \frac{1}{r^n_{\xi x}} \leq \frac{1}{(R-\varrho)^n}, \tag{4.3.11}$$

so that if we multiply in both members by

$$\frac{R^2-\varrho^2}{R\omega_n}u(x),$$

which, based on the assumption, is a positive quantity, we obtain

$$\frac{R^2-\varrho^2}{R\omega_n}\frac{1}{(R+\varrho)^n}u(x) \leq \frac{R^2-\varrho^2}{R\omega_n}\frac{1}{r^n_{\xi x}}u(x) \leq \frac{R^2-\varrho^2}{R\omega_n}\frac{1}{(R-\varrho)^n}u(x). \tag{4.3.12}$$

In (4.3.12), we had the right to write $u(x)$ because $\eta \equiv 0$.

We integrate in (4.3.12), term by term, on the surface $\partial B(0, R)$ so that we are led to the estimates

$$\begin{aligned}
&\frac{R-\varrho}{R\omega_n}\frac{1}{(R+\varrho)^{n-1}}\int_{\partial B(0,R)} u(x)\mathrm{d}\sigma_x \\
&\quad\le \frac{R^2-\varrho^2}{R\omega_n}\int_{\partial B(0,R)} \frac{u(x)}{\left[R^2+\varrho^2-2R\varrho\cos\beta\right]^{n/2}}\mathrm{d}\sigma_x \\
&\qquad\qquad \le \frac{R+\varrho}{R\omega_n}\frac{1}{(R-\varrho)^{n-1}}\int_{\partial B(0,R)} u(x)\mathrm{d}\sigma_x .
\end{aligned} \tag{4.3.13}$$

But using the formula (4.3.8) of Gauss, for the last integral from (4.3.13), we have

$$\int_{\partial B(0,R)} u(x)\mathrm{d}\sigma_x = R^{n-1}\omega_n u(0)$$

and then (4.3.13) becomes

$$\left(\frac{R}{R+\varrho}\right)^{n-2}\frac{R-\varrho}{R+\varrho}u(0) \le u(\xi) \le u(0)\frac{R+\varrho}{R-\varrho}\left(\frac{R}{R-\varrho}\right)^{n-2},$$

which are just the inequalities (4.3.10) of Harnack, taking into account the fact that we have chosen $\eta \equiv 0$. ■

With the help of the inequalities of Harnack, we will prove in the following theorem, which is due to Harnack, another important property of harmonic functions.

Theorem 4.3.3 *Let Ω be a bounded domain with boundary $\partial\Omega$ and $\{u_n\}_{n\ge 1}$ be a sequence of the functions defined on Ω, having the properties*

1^o. $u_n \in C^2(\Omega)$, $\forall n = 1, 2, \dots$;

2^o. $\Delta_x u_n(x) = 0$, $\forall x \in \Omega$, $\forall n = 1, 2, \dots$;

3^o. The sequence $\{u_n(x)\}_{n\ge 1}$ is ascending, $\forall x \in \Omega$.

If the sequence $\{u_n(x)\}_{n\ge 1}$ is convergent in a point $x_0 \in \Omega$, then $\{u_n(x)\}_{n\ge 1}$ is convergent in any point $x \in \Omega$ and, namely, it is uniformly convergent on compact sets from Ω to a harmonic function.

Proof Consider two arbitrary terms of the sequence, u_p and u_q, with $p > q$. Then, the difference function $u_p(x) - u_q(x)$ has the properties

- $u_p - u_q \in C^2(\Omega)$;
- $\Delta_x\left[u_p(x) - u_q(x)\right] = 0, \ \forall x \in \Omega$;
- $u_p(x) - u_q(x) \ge 0$.

Thus, the difference $u_p(x) - u_q(x)$ satisfies the conditions of Proposition 4.3.1, and therefore, we can use the inequalities of Harnack so that we have the estimate

$$\left|u_p(\xi) - u_q(\xi)\right| \le \left|u_p(x_0) - u_q(x_0)\right| \frac{R+\varrho}{R-\varrho}\left(\frac{R}{R-\varrho}\right)^{n-2}, \tag{4.3.14}$$

for $\forall \xi \in B(x_0, R)$.

On the other hand, due to convergence of the sequence $\{u_n\}_{n\geq 1}$ in the arbitrarily fixed point $x_0 \in \Omega$, we deduce that for $\forall \varepsilon > 0$, $\exists N(\varepsilon)$ so that if $p \geq q \geq N(\varepsilon)$, we have

$$\left|u_p(x_0) - u_q(x_0)\right| < \varepsilon. \tag{4.3.15}$$

If we use (4.3.15) in (4.3.14), we deduce that the sequence $\{u_n(\xi)\}_{n\geq 1}$ is a Cauchy sequence (or fundamental sequence), so we can deduce its uniform convergence with respect to $\forall \xi \in B(0, R)$.

The fact that the uniform limit of the sequence $\{u_n\}_{n\geq 1}$ is a harmonic function can be obtained from Theorem 4.3.1, because the assumptions of this theorem are, obviously, satisfied. ■

As a direct consequence of inequalities of Harnack, in the form

$$\left(\frac{R}{R+\varrho}\right)^{n-2}\frac{R-\varrho}{R+\varrho}u(0) \leq u(\xi) \leq u(0)\frac{R+\varrho}{R-\varrho}\left(\frac{R}{R-\varrho}\right)^{n-2}, \tag{4.3.16}$$

which have been proved in Proposition 4.3.1, we will obtain the famous results due to Liouville.

Theorem 4.3.4 (Liouville). *If $u(x)$ is a harmonic function on the whole space $\mathbb{R}^n$ and $u(x)$ is a nonnegative (or nonpositive) function everywhere on $\mathbb{R}^n$, then $u(x)$ is identically equal to a constant.*

Proof Without loss of generality, we can suppose that $u(x) \geq 0$, $\forall x \in \mathbb{R}^n$. For the case in which $u(x) \leq 0$, $\forall x \in \mathbb{R}^n$, the proof is analogous.

Then, because $\varrho = |0x| < R$ and $|0\eta| = R$, we have

$$R - |0x| \leq |0\eta - 0x| \leq R + |0x|,$$

and then we obtain the inequalities of Harnack in the form (4.3.16). If we arbitrarily fix $x \in \mathbb{R}^n$ and we pass to the limit in (4.3.16) with $R \to \infty$, we obtain immediately that $u(x) = u(0)$ and because x is arbitrary, we deduce that the function u is a constant. ■

Theorem 4.3.5 (Liouville). *If $u(x)$ is a harmonic function on the whole space $\mathbb{R}^n$ and $u(x)$ is bounded from below (or from above) everywhere on $\mathbb{R}^n$, then $u(x)$ is identically equal to a constant.*

Proof To fix the ideas, suppose that $u(x) \leq M$, for any $x \in \mathbb{R}^n$, where M is a constant. Analogous is the proof in the situation in which $u(x) \geq M$, $\forall x \in \mathbb{R}^n$.

Because the function $M - u(x)$ is harmonic on the whole space $\mathbb{R}^n$ and is a nonnegative function, according to Theorem 4.3.4, we deduce that

$$M - u(x) = M - u(0), \ \forall x \in \mathbb{R}^n,$$

from where we deduce immediately that $u(x) = u(0),\ \forall x \in \mathbb{R}^n$, that is, the function $u(x)$ is identically equal to a constant. ■

With the help of theorems of Liouville, we can prove a result of uniqueness, for the problem of Dirichlet.

Proposition 4.3.2 *The problem of Dirichlet for the half-space $x_n > 0$ has only one solution in the class of bounded functions.*

Proof Suppose by absurd that the problem of Dirichlet, considered for the half-space $x_n > 0$, has two solutions, $u_1(x)$ and $u_2(x)$. Then, their difference, $v(x) = u_1(x) - u_2(x)$, satisfies the boundary condition in its homogeneous form

$$v(x_n) = 0, \text{ for } x_n = 0.$$

We will build the function $w(x)$ as follows:

$$w(x) = \begin{cases} v(x_1, x_2, \ldots, x_n) & , \text{ for } x_n \geq 0, \\ -v(x_1, x_2, \ldots, -x_n) & , \text{ for } x_n < 0. \end{cases}$$

The function $w(x)$ is harmonic in the half-space $x_n > 0$ and also in the half-space $x_n < 0$. Moreover, $w(x)$ is harmonic on the whole space $\mathbb{R}^n$, because, for any $R > 0$, $w(x)$ coincides, inside the ball $\varrho < R$, with the harmonic function $w^*(x)$, which satisfies the boundary condition $w^*(x) = w(x)$, for $\varrho = R$. By hypothesis, the function $w(x)$ is bounded and then Theorem 4.3.5 of Liouville leads to the conclusion that $w(x)$ is identically equal to a constant. Finally, we have that $w(x) = 0$ for $x_n = 0$ and then $w(x) = 0$ on the whole space $\mathbb{R}^n$, and this, obviously, involves the fact that $v(x) = 0$ on the whole half-space $x_n \geq 0$, that is, $u_1(x) = u_2(x)$. ■

Chapter 5
Operational Calculus

5.1 The Laplace Transform

A useful tool in approaching ordinary differential equations and also partial differential equations is the Laplace transform, which will be studied in this paragraph.

Definition 5.1.1 A function $f : R \to R$ is called an original function for the Laplace transform if it fulfills the following conditions:

(i) $f(t)$ and $f'(t)$ exist and are either continuous on the whole real axis or continuous, except for a sequence of points $\{t_n\}_{n\geq 1}$ in which they can have discontinuities of first order;

(ii) $f(t) = 0, \ \forall t < 0$;

(iii) There are the constants $M > 0, \ s_0 \geq 0$, so that

$$|f(t)| \leq Me^{s_0 t}, \ \forall t \in \mathbb{R}.$$

Usually, s_0 is called *the growth index* of the original. A classical example of an original function is the step function of Heaviside, denoted by θ and given by

$$\theta(t) = \begin{cases} 0, \text{ if } t < 0, \\ 1, \text{ if } t \geq 0. \end{cases}$$

If a function f satisfies the conditions (i) and (iii), but does not satisfy the condition (ii), from the definition of the original function, we make the convention that the function f is implicitly multiplied by the step function of Heaviside, θ, that is,

$$f(t) = f(t)\theta(t) = \begin{cases} 0, & \text{if } t < 0, \\ f(t), & \text{if } t \geq 0. \end{cases}$$

This convention is made to enrich the set of original functions.

Let us denote by $\mathcal{O}$ the set of original functions.

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_5

In the following theorem, we prove what is the structure of the set $\mathcal{O}$ of originals. More precisely, we prove that $\mathcal{O}$ has the structure of a linear space and also the structure of an algebra.

Theorem 5.1.1 *Let $\mathcal{O}$ be the set of originals for the Laplace transform. Then, $\mathcal{O}$ has the structure of a vector space and, that of an algebra:*
1^o. $f+g \in \mathcal{O}, \ \forall f, g \in \mathcal{O}$;
2^o. $\lambda f \in \mathcal{O}, \ \forall f \in \mathcal{O}, \ \forall \lambda \in \mathbb{C}$;
3^o. $f.g \in \mathcal{O}, \ \forall f, g \in \mathcal{O}$.

Proof 1^o. Because $f, \ g \in \mathcal{O}$, we deduce that $f+g$ obviously satisfies the properties (i) and (ii) of the originals. We now want to verify condition (iii). If

$$|f(t)| \leq M_1 e^{s_1 t}, \ |g(t)| \leq M_2 e^{s_2 t}, \ \forall t \in \mathbb{R},$$

then

$$|f(t)+g(t)| \leq |f(t)|+|g(t)| \leq M_1 e^{s_1 t}+M_2 e^{s_2 t} \leq M_3 e^{s_3 t}, \ \forall t \in \mathbb{R},$$

where $s_3 = \max\{s_1, s_2\}$ and $m_3 = \max\{M_1, M_2\}$.

2^o. λf obviously satisfies the properties (i) and (ii) of the originals. We now want to verify the condition (iii). Because

$$|f(t)| \leq M_1 e^{s_1 t}, \ \forall t \in \mathbb{R}$$

we deduce that

$$|\lambda f(t)| = |\lambda||f(t)| \leq |\lambda| M_1 e^{s_1 t}, \ \forall t \in \mathbb{R},$$

that is, λf has the same growth index as f.

3^o. As far as condition (iii) for the product $f \cdot g$, we have

$$|f(t) \cdot g(t)| = |f(t)| \cdot |g(t)| \leq M_1 M_2 e^{(s_1+s_2)t}, \ \forall t \in \mathbb{R},$$

and the proof is complete, because the properties (i) and (ii) are obvious. ■

Observation 5.1.1 *1^o. From the proof of Theorem 5.1.1, it is verified that*

$$ind(f+g) = \max\{ind(f), ind(g)\},$$
$$ind(f.g) = ind(f)+ind(g).$$

2^o. If $f_i \in \mathcal{O}, \ i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n} \lambda_i f_i \in \mathcal{O}, \ \forall \lambda_i \in \mathbb{R}, \ sau \ \lambda_i \in \mathbb{C}, i = 1, 2, \ldots, n.$$

This statement can be deduced from the first two points of Theorem 5.1.1

3°. If $f_i \in \mathcal{O},\ i = 1, 2, \ldots, n$, then

$$\prod_{i=1}^{n} f_i \in \mathcal{O}.$$

This statement can be easily obtained, by repeatedly applying the point 3° from Theorem 5.1.1.

In particular, if $f \in \mathcal{O}$ then $f^n \in \mathcal{O},\ \forall n \in \mathbb{N}^*$.

4°. The function $f(t) = e^{\lambda t}$ is an original function, $\forall \lambda \in \mathbb{C},\ \lambda\alpha + i\beta$, having growth index

$$s_0 = \begin{cases} 0, \ \text{if } \alpha < 0, \\ \alpha, \ \text{if } \alpha \geq 0. \end{cases}$$

Consequently, we obtain that the functions

$$\sin \lambda t,\ \cos \lambda t,\ \sinh \lambda t,\ \cosh \lambda t$$

are also originals functions.

If we represent the function $e^{\lambda t}$ as a power series

$$e^{\lambda t} = 1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \cdots + \frac{\lambda^n t^n}{n!} + \cdots,\ t \geq 0$$

and we take into account that

$$\frac{\lambda^n t^n}{n!} < e^{\lambda t},\ \forall t \geq 0,$$

we deduce immediately that

$$t^n < \frac{n!}{\lambda^n t^n} e^{\lambda t},\ \forall t \geq 0,$$

and then we obtain that the function $f(t) = t^n,\ t \geq 0$ is an original function.

Based on the above observations, we obtain that the function

$$f(t) = e^{\lambda t} \left[P(t) \cos \alpha t + Q(t) \sin \alpha t \right]$$

is an original function, for any two polynomials P and Q.

Definition 5.1.2 If $f(t)$ is an original function, with growth index s_0, then we call the Laplace transform of this function, or its image by the Laplace transform, the function F defined by

$$F(p) = \int_0^\infty f(t) e^{-pt} \mathrm{d}t,\ \forall p \in \mathbb{C},\ Re(p) \geq s_0. \tag{5.1.1}$$

We will prove that the image F from (5.1.1) is defined on the whole semi-plane $[s_0, \infty)$ and, moreover, that F is an analytic function in this semi-plane.

Theorem 5.1.2 *If f is an original function with growth index s_0, then the function $F : [s_0, \infty) \to \mathbb{C}$ makes sense for any complex number p for which $Re(p) \geq s_0$ and F is an analytic function in this semi-plane.*

Proof Starting from (5.1.1), we obtain

$$\begin{aligned} |F(p)| &\leq \int_0^\infty \left| f(t)e^{-pt} \right| \mathrm{d}t \\ &\leq M \int_0^\infty e^{s_0 t} e^{-pt} \mathrm{d}t = \frac{M}{s - s_0} e^{(s_0 - p)t} \Big|_0^\infty = \frac{M}{s - s_0}, \end{aligned}$$

and this inequality proves that the function F is well defined.

If $Re(p) \geq s_1 \geq s_0$, then we can differentiate under the integral sign in (5.1.1) and obtain

$$F'(p) = \int_0^\infty -tf(t)e^{-pt} \mathrm{d}t,$$

and then we have the bounds

$$\begin{aligned} |F'(p)| &\leq \int_0^\infty |tf(t)| e^{-pt} \mathrm{d}t \\ &\leq M \int_0^\infty t e^{(s_0 - p)t} \mathrm{d}t \leq M \int_0^\infty t e^{(s_0 - s_1)t} \mathrm{d}t \\ &= Mt \frac{e^{(s_0 - s_1)t}}{s_0 - s_1} \Big|_0^\infty + \frac{M}{s_0 - s_1} \int_0^\infty e^{(s_0 - s_1)t} \mathrm{d}t = \frac{M}{(s_0 - s_1)^2}, \end{aligned}$$

where we have integrated by parts. The derivative being bounded, we deduce that F is an analytic function in an open semi-plane (s_0, ∞). ■

As a consequence of Theorem 5.1.2, it can be seen that

$$\lim_{|p| \to \infty} |F(p)| = 0.$$

It is natural to ask what the original function is whose Laplace transform is just F if we know a transform F. The answer is given in the following theorem.

Theorem 5.1.3 *If we know the Laplace transform F, then the original function can be expressed, in each point t of continuity, by the following inversion formula:*

$$f(t) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} F(p) e^{pt} \mathrm{d}p, \tag{5.1.2}$$

where $a \in \mathbb{R},\ a \geq s_0$.

We do not write the proof, this being very laborious and very technical.

Theorem 5.1.3 asserts that if we are given a transform of an original, then this is the transform of a single original, in other words, the Laplace transform performs a biunivocal correspondence on the set of the originals.

The integral from the right-hand side of the formula (5.1.2) is an improper integral in the Cauchy sense.

$$\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\mathrm{d}p = \lim_{b\to\infty}\int_{a-ib}^{a+ib} F(p)e^{pt}\mathrm{d}p.$$

The fact that the original function is uniquely determined by its Laplace transform is reinforced in the following theorem.

Theorem 5.1.4 *Consider the transformed function F with the following properties:*

1^o. $F(p)$ is an analytic function in the semi-plane $Re(p) \geq a > s_0$;

2^o. $\lim\limits_{|p|\to\infty} |F(p)| = 0$, for $Re(p) \geq a > s_0$, the limit taking place uniformly with respect to p;

3^o. The integral $\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\mathrm{d}p$ is absolutely convergent.

Then, the function $f(t)$, defined by

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\mathrm{d}p, \tag{5.1.3}$$

is the original whose Laplace transform is the function $F(p)$.

Proof We first observe that the Laplace transform $F(p)$ is also denoted by $\mathcal{L}(f(t))$ or, simpler, by $\mathcal{L}(f)$. It is being understood that the argument of the original function is denoted by t and the argument of the Laplace transform is denoted by p.

If we apply the Laplace transform in (5.1.3), we deduce that

$$\mathcal{L}(f) = \int_0^\infty \left\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}\mathrm{d}p\right\} e^{-p_0 t}\mathrm{d}t. \tag{5.1.4}$$

We intend to prove that $\mathcal{L}(f) = F(p_0)$, where $p_0 = a + i\sigma$ is fixed arbitrarily in the semi-plane $[s_0, \infty)$.

Because we have

$$\left|\int_{a-i\infty}^{a+i\infty} F(p)e^{p_0 t}\mathrm{d}p\right| \leq \int_{a-i\infty}^{a+i\infty} |F(p)|\left|e^{p_0 t}\right|\mathrm{d}p$$
$$= \int_{a-i\infty}^{a+i\infty} |F(p)|\left|e^{at}\right|\left|\left|e^{i\sigma t}\right|\mathrm{d}p = \int_{a-i\infty}^{a+i\infty} |F(p)|\, e^{at}\mathrm{d}p,$$

and the last integral was assumed to be convergent (by the hypothesis 3^o), we deduce that in (5.1.4) we can invert the order of integration and obtain

$$\mathcal{L}(f) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(p) \left\{ \int_0^{\infty} e^{(p-p_0)t} \mathrm{d}t \right\} \mathrm{d}p.$$

Because $Re(p - p_0) = a - s < 0$ and $e^{(p-p_0)t}\Big|_0^{\infty} = -1$, we obtain

$$\mathcal{L}(f) = \frac{1}{2\pi i} \int_{a+i\infty}^{a-i\infty} \frac{F(p)}{p - p_0} \mathrm{d}p.$$

We consider a circle with its center in the origin and a radius R. We consider the vertical segment included between $a - ib$ and $a + ib$ and the area C_R from the quadrants I and IV intersected by this segment on the considered circle. We now apply the formula of Cauchy (from the theory of complex functions), taking into account that $p = p_0$ is a polar singularity, so that we are lead to

$$F(p_0) = \frac{1}{2\pi i} \int_{a+ib}^{a-ib} \frac{F(p)}{p - p_0} \mathrm{d}p + \frac{1}{2\pi i} \int_{C_R} \frac{F(p)}{p - p_0} \mathrm{d}p. \tag{5.1.5}$$

For the last integral from (5.1.5), we have the bound

$$\left| \frac{1}{2\pi i} \int_{C_R} \frac{F(p)}{p - p_0} \mathrm{d}p \right| \leq \frac{1}{2\pi} 2\pi R \frac{M_R}{|R| - |p_0|},$$

where

$$M_R = \sup_{p \in C_R} |F(p)|.$$

Based on the assumption 2^o, we deduce that $M_R \to 0$, as $R \to \infty$. We get that the last integral from (5.1.5) is convergent to zero, as $R \to \infty$.

Therefore, if we pass to the limit in (5.1.5) for $R \to \infty$, we obtain

$$F(p_0) = \frac{1}{2\pi i} \int_{a+i\infty}^{a-i\infty} \frac{F(p)}{p - p_0} \mathrm{d}p = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{F(p)}{p - p_0} \mathrm{d}p,$$

that is, $F(p_0) = \mathcal{L}(f)$. ■

In the propositions which follow, we will prove the main properties of the Laplace transform.

Proposition 5.1.1 (The property of linearity) *If f and g are the original functions, having the images F and G, respectively, and $\alpha,\ \beta \in \mathbb{R}$, then*

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha F(p) + \beta G(p).$$

Proof The result is immediately obtained based on the linearity of the Riemann integral. ■

Proposition 5.1.2 (The property of similarity) *If f is an original function, having the image F, and $\alpha \in \mathbb{C}^*$, then*

$$\mathcal{L}(f(\alpha t)) = \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right).$$

Proof With change of variables $\alpha t = \tau$, we obtain

$$\mathcal{L}(f(\alpha t)) = \int_0^\infty f(\alpha t) e^{-pt} \mathrm{d}t =$$
$$= \int_0^\infty f(\tau) e^{-\frac{p}{\alpha}\tau} \frac{1}{\alpha} \mathrm{d}\tau = \frac{1}{\alpha} \int_0^\infty f(\tau) e^{-\frac{p}{\alpha}\tau} \mathrm{d}\tau,$$

and the result is proven. ■

Proposition 5.1.3 (The transform of the derivative) *If f is an original function, having the image F, then in a point t in which f is differentiable, the following formula holds true:*

$$\mathcal{L}(f'(t)) = pF(p) - f(0).$$

Proof We start from the definition of the Laplace transform. By direct calculations, we obtain

$$\mathcal{L}(f'(t)) = \int_0^\infty f'(t) e^{-pt} \mathrm{d}t = e^{-pt} f(t)\Big|_0^\infty$$
$$- \int_0^\infty (-p) f(t) e^{-pt} \mathrm{d}t = -f(0) + p \int_0^\infty f(t) e^{-pt} \mathrm{d}t = pF(p) - f(0)$$

and the result is proven. ■

Corollary 5.1.1 *As far as the transform of a derivative of the original functions is concerned, we will prove a more general result*

$$\mathcal{L}(f^{(n)}(t)) = p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

Proof Using Proposition 5.1.3, we successively have

$$\mathcal{L}(f'(t)) = p\mathcal{L}(f(t)) - f(0),$$
$$\mathcal{L}(f''(t)) = p\mathcal{L}(f'(t)) - f'(0),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathcal{L}(f^{(n)}(t)) = p\mathcal{L}(f^{(n-1)}(t)) - f^{(n-1)}(0).$$

We multiply the first relation by p^{n-1}, the second by $p^{n-2}, \ldots,$ and the last by p^0. Then, the obtained relations are summed up and we deduce the desired formula. ■

Proposition 5.1.4 (The derivative of the transform) *If f is an original function, having the image F, then we have*

$$F'(p) = \mathcal{L}(-tf(t)).$$

Proof We proved that the integral by which the Laplace transform can be defined is convergent. Then, we can differentiate under the integral sign with respect to p:

$$F(p) = \int_0^\infty f(t)e^{-pt}\mathrm{d}t \Rightarrow F'(p) = \int_0^\infty f(t)(-t)e^{-pt}\mathrm{d}t,$$

from where the desired result can be deduced. ■

Corollary 5.1.2 *As far as the derivative of a transform is concerned, a more general result holds true*

$$F^{(n)}(p) = \mathcal{L}((-t)^n f(t)).$$

Proof The desired formula is obtained by deriving successively under the integral and then by using mathematical induction. ■

As a consequence of these properties, we can immediately get

$$\mathcal{L}(t^n) = \frac{n!}{p^{n+1}}.$$

Proposition 5.1.5 (The transform of the integral) *Let f be an original function whose transform is F. Then, the integral $\int_0^t f(\tau)\mathrm{d}\tau$ is also an original function, with the same growth index as f. Moreover, the following formula holds true:*

$$\mathcal{L}\left(\int_0^t f(\tau)\mathrm{d}\tau\right) = \frac{1}{p}F(p).$$

Proof The conditions (i) and (ii) from the definition of an original are immediately verified by the integral $\int_0^t f(\tau)\mathrm{d}\tau$, taking into account that f satisfies these conditions. Denote by g this integral

$$g(t) = \int_0^t f(\tau)\mathrm{d}\tau.$$

Let us show that g satisfies the condition (iii) from the definition of an original

$$|g(t)| \leq \int_0^t |f(\tau)|\mathrm{d}\tau \leq M\int_0^t e^{s_0\tau}\mathrm{d}\tau$$
$$= \frac{M}{s_0}\left(e^{s_0t} - 1\right) \leq \frac{M}{s_0}e^{s_0t}.$$

Note that g has the same growth index as f.

On the other hand, because $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$, we deduce immediately that $g(0) = 0$ and $g'(t) = f(t)$. Then

$$\mathcal{L}(f(t)) = \mathcal{L}(g'(t)) = pG(p) - g(0) = pG(p),$$

in which we used the transform of the derivative and we denoted by G the Laplace transform of g, that is,

$$G(p) = \mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(\tau)\mathrm{d}\tau\right),$$

so that the proof is complete. ■

Proposition 5.1.6 (The integral of a transform) *Let f be an original function whose transform is F. If we assume that the integral $\int_p^\infty F(q)\mathrm{d}q$ is convergent, then*

$$\int_p^\infty F(q)\mathrm{d}q = \mathcal{L}\left(\frac{f(t)}{t}\right).$$

Proof If we take into account the expression of F, we obtain

$$\begin{aligned}\int_p^\infty F(q)\mathrm{d}q &= \int_p^\infty \left\{\int_0^\infty f(t)e^{-qt}\mathrm{d}t\right\}\mathrm{d}q\\ &= \int_0^\infty \left\{\int_p^\infty e^{-qt}\mathrm{d}q\right\} f(t)\mathrm{d}t = \int_0^\infty \left(\left.\frac{e^{-qt}}{t}\right|_p^\infty\right) f(t)\mathrm{d}t\\ &= \int_0^\infty \frac{f(t)}{t}e^{-pt}\mathrm{d}t = \mathcal{L}\left(\frac{f(t)}{t}\right).\end{aligned}$$

that is, we obtained the desired result. ■

Proposition 5.1.7 (The property of the delay) *If the argument of the original function f is "delayed", then the following formula holds true:*

$$\mathcal{L}(f(t-\tau)) = e^{-p\tau}F(p),\ \forall \tau > 0,$$

where, as usual, F is the Laplace transform of the function f.

Proof By starting from the definition of the Laplace transform, we obtain

$$\mathcal{L}(f(t-\tau)) = \int_0^\infty f(t-\tau)e^{-pt}\mathrm{d}t,$$

so that if we use the change of variables $t - \tau = u$, we get

$$\mathcal{L}(f(t-\tau)) = \int_{-\tau}^{\infty} f(u)e^{-pu}e^{-p\tau}\mathrm{d}u$$
$$= \int_{-\tau}^{0} f(u)e^{-pu}e^{-p\tau}\mathrm{d}u + e^{-p\tau}\int_{0}^{\infty} f(u)e^{-pu}\mathrm{d}u$$
$$= e^{-p\tau}\int_{0}^{\infty} f(u)e^{-pu}\mathrm{d}u = e^{-p\tau}F(p),$$

because the function f is an original and therefore $f(u) = 0,\ \forall u < 0$. ■

Even though the product of two original functions is an original function, the Laplace transform of the product cannot be computed. But if the usual product is replaced by the product of convolution, then the Laplace transform can be computed. We know that the product of convolution for two functions can be calculated in a more general framework. In the case of the original functions, the product of convolution can be defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau. \tag{5.1.6}$$

Observation 5.1.2 *We can verify, without difficulty, the following properties of the product of convolution:*

- *commutativity:* $f * g = g * f$;
- *associativity:* $f * (g * h) = (f * g) * h = f * g * h$;
- *distributivity in relation to summation:* $f * (g + h) = f * g + f * h$;
- *if* $f * g = 0$ *then* $f \equiv 0$ *or* $g \equiv 0$.

Proposition 5.1.8 *If f and g are the original functions, then their product of convolution (5.1.6) is an original function.*

Proof The conditions (i) and (ii) from the definition of an original are immediately satisfied, taking into account that f and g satisfy these conditions. Because f and g satisfy condition (iii), we have

$$|f(t)| \le M_1 e^{s_1 t},\ |g(t)| \le M_2 e^{s_2 t},$$

and then

$$|(f * g)(t)| \le \int_0^t |f(\tau)||g(t-\tau)|\mathrm{d}\tau \le M_1 M_2 \int_0^t e^{s_1\tau}e^{s_2(t-\tau)}\mathrm{d}\tau.$$

If $s_2 \le s_1$, then

$$|(f * g)(t)| \le M_1 M_2 \int_0^t e^{s_1\tau}e^{s_1(t-\tau)}\mathrm{d}\tau = M_1 M_2 \int_0^t e^{s_1 t}\mathrm{d}\tau = M_1 M_2 t e^{s_1 t}.$$

It is elementary that $t + 1 \leq e^t \Rightarrow t \leq e^t - 1 \leq e^t$. Then

$$|(f * g)(t)| \leq M_1 M_2 e^{(s_1+1)t}.$$

If $s_1 < s_2$, then we can change the roles of the functions f and g. Using the commutativity of the product of convolution, we get the desired result. ■

Proposition 5.1.9 *If f and g are the original functions, then the Laplace transform of their product of convolution is equal to the usual product of the transforms*

$$\mathcal{L}(f * g) = F(p) \cdot G(p).$$

Proof Taking into account (5.1.6), we obtain

$$\begin{aligned}
\mathcal{L}\left((f * g)(t)\right) &= \mathcal{L}\left(\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\right) \\
&= \int_0^\infty \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau e^{-pt}\mathrm{d}t = \int_0^\infty \int_\tau^\infty g(t-\tau)e^{-pt}\mathrm{d}t f(\tau)\mathrm{d}\tau \\
&= \int_0^\infty \int_0^\infty g(u)e^{-p(\tau+u)}\mathrm{d}u f(\tau)\mathrm{d}\tau = \int_0^\infty f(\tau)e^{-p\tau}\int_0^\infty g(u)e^{-pu}\mathrm{d}u\mathrm{d}\tau \\
&= \int_0^\infty f(\tau)e^{-p\tau}G(p)\mathrm{d}\tau = G(p)\int_0^\infty f(\tau)e^{-p\tau}\mathrm{d}\tau = G(p).F(p),
\end{aligned}$$

in which we used the change of variables $t - \tau = u$. We thus immediately obtained the desired result. ■

Corollary 5.1.3 *In some technical fields, especially electrical fields, the following formula is helpful, called the formula of Duhamel:*

$$pF(p)G(p) = \mathcal{L}\left(f(t)g(0) + \int_0^t f(\tau)g'(t-\tau)\mathrm{d}\tau\right).$$

Proof Denote by h the product of convolution of the functions f and g, that is,

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau.$$

then $h(0) = 0$ and

$$h'(t) = f(t)g(0) + \int_0^t f(\tau)g'(t-\tau)\mathrm{d}\tau.$$

We apply the transform of the product of convolution, the transform of the derivative and the fact that $h(0) = 0$ and we obtain, without difficulty formula of Duhamel. ■

It was proven that the Laplace transform is a useful tool to transform some specific operations from mathematical analysis, in more affordable operations. For instance, by using the Laplace transform, solving some differential equations (or some integral equations) is reduced to solving some algebraic equations. Therefore, by applying the Laplace transform, the problem becomes easier to solve, but the solution will be obtained in the set of the transformed functions, even though the initial problem was formulated in the set of the original functions. It is natural to pose the problem of the transposition of the solution of the respective problem, from the set of the transforms in the set of the originals. This is the object of the so-called formulas of development. To this end, we will give two results, of course, the most used.

Theorem 5.1.5 *If the series*

$$\sum_{k=1}^{\infty} \frac{c_k}{p^k} \tag{5.1.7}$$

is convergent for $|p| \geq R$*, then the function*

$$\theta(t) \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1} \tag{5.1.8}$$

is an original, and its Laplace transform is the series (5.1.7). We denoted by θ *the step function of Heaviside.*

Proof According to the criterion of Cauchy of convergence, we have

$$c_k \leq MR^k \Rightarrow \left| \theta(t) \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1} \right| \leq M \sum_{k=1}^{\infty} \frac{R^k |t|^{k-1}}{(k-1)!} \leq MRe^{R|t|},$$

from where we deduce that the function (5.1.8) is an original function.

For the second statement of the theorem, we use the formula which gives the Laplace transform of the function $f(t) = t^k$

$$\mathcal{L}\left(\theta(t) t^{k-1}\right) = \frac{(k-1)!}{p^k}.$$

Then, based on the linearity of the Laplace transform, we have

$$\mathcal{L}\left(\theta(t)\sum_{k=1}^{\infty}\frac{c_k}{(k-1)!}t^{k-1}\right) = \int_0^{\infty}\theta(t)\sum_{k=1}^{\infty}\frac{c_k}{(k-1)!}t^{k-1}e^{-pt}\mathrm{d}t$$
$$= \sum_{k=1}^{\infty}\int_0^{\infty}e^{-pt}\theta(t)\frac{t^{k-1}}{(k-1)!}c_k\mathrm{d}t = \sum_{k=1}^{\infty}\frac{c_k}{p^k},$$

that is, we get just the desired result. ■

Theorem 5.1.6 *Let us consider P and Q to be two polynomials so that $gr\,P < gr\,Q$, and Q has only simple roots $p_0, p_1, \ldots, p_n$. Then, the function*

$$F(p) = \frac{P(p)}{Q(p)}$$

is the Laplace transform of the function f given by

$$f(t) = \sum_{k=0}^{n}\frac{P(p_k)}{Q'(p_k)}e^{p_k t}.$$

Proof Taking into account the hypothesis on polynomial Q, we can write $Q(p) = c(p-p_0)(p-p_1)\ldots(p-p_n)$ and then decompose the function F in simple fractions

$$F(p) = \frac{a_0}{p-p_0} + \frac{a_1}{p-p_1} + \cdots + \frac{a_n}{p-p_n}. \tag{5.1.9}$$

Note that the function F has simple poles $p_0, p_1, \ldots, p_n$. We take the circles $c_j(p_j, r_j)$ with centers in points p_j and a radius r_j sufficiently small so that in each closed disk, there is no other pole except for the center of the respective circle. The coefficient a_j can be determined by integrating the equality (5.1.9) on the circle c_j

$$\int_{c_j}F(p)\mathrm{d}p = \sum_{k=0}^{n}a_k\int_{c_j}\frac{1}{p-p_k}\mathrm{d}p. \tag{5.1.10}$$

According to the known theorem of Cauchy, the integrals from the right-hand side of the relation (5.1.10) are null, except for the integral which corresponds to $k = j$, for which we have

$$\int_{c_j}\frac{1}{p-p_j}\mathrm{d}p = 2\pi i.$$

Then, relation (5.1.10) becomes

$$\int_{c_j} F(p)\mathrm{d}p = 2\pi i a_j. \tag{5.1.11}$$

On the other hand, the integral from the right-hand side of the relation (5.1.10) can be calculated with the help of the theorem of residues

$$\int_{c_j} F(p)\mathrm{d}p = 2\pi i \; rez(F, p_j) = 2\pi i \frac{P(p_j)}{Q'(p_j)},$$

so that by replacing it in (5.1.11), we obtain

$$a_j = \frac{P(p_j)}{Q'(p_j)}.$$

Then, formula (5.1.9) becomes

$$F(p) = \sum_{k=0}^{n} \frac{P(p_k)}{Q'(p_k)} \frac{1}{p - p_k} = \sum_{k=0}^{n} \frac{P(p_k)}{Q'(p_k)} \mathcal{L}\left(e^{p_k t}\right).$$

Finally, using the linearity of the Laplace transform, we deduce that

$$F(p) = \mathcal{L}\left(\sum_{k=0}^{n} \frac{P(p_k)}{Q'(p_k)} e^{p_k t}\right),$$

from where the desired result is certified. ■

Corollary 5.1.4 *If one of the roots of the polynomial Q is null, then the original function becomes*

$$f(t) = \frac{P(0)}{Q(0)} + \sum_{k=1}^{n} \frac{P(p_k)}{R'(p_k)} e^{p_k t}, \tag{5.1.12}$$

where R is the polynomial defined so that $Q(p) = pR(p)$.

Proof Suppose that the null root is $p_0 = 0$ and then write $Q(p) = pR(p)$. Then, $Q'(p) = R(p) + R'(p)$. For the other roots of Q, we have that $Q(p_k) = 0 \Leftrightarrow R(p_k) = 0$. Then, $Q'(p_k) = R(p_k) + p_k Q'(p_k) = p_k Q'(p_k)$. Then, the desired result is obtained with the help of Theorem 5.1.6. ■

Formula (5.1.12) is known as the formula of Heaviside.

At the end of the paragraph, we want to find the image of two functions by means of the Laplace transform that are frequently found in applications. Let us first consider the function $f(t) = t^{\alpha}$, where α is a complex constant so that $Re(\alpha) > -1$. If $Re(\alpha) \geq 0$, then f is an original function, and then its Laplace transform is

$$\mathcal{L}(t^\alpha) = \int_0^\infty t^\alpha e^{-pt} \mathrm{d}t. \tag{5.1.13}$$

If $Re(\alpha) \in (-1, 0)$, then $\lim_{t \searrow 0} f(t) = \infty$ and f is not an original function, but the integral (5.1.13) is convergent and in this case, we can study the integral (5.1.13) for $Re(\alpha) > -1$. Taking into account the definition of the function Γ of Euler, from (5.1.13) we obtain that

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}. \tag{5.1.14}$$

It is interesting to outline that formula (5.1.14) allows an elegant proof of the known connection between the two functions of Euler, Γ and β

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \;\; Re(x) > -1, \;\; Re(y) > -1.$$

Indeed, if we start by using the equalities

$$\mathcal{L}(t^{x-1}) = \frac{\Gamma(x)}{p^x}, \;\; \mathcal{L}(t^{y-1}) = \frac{\Gamma(y)}{p^y},$$

and we take into account Proposition 5.1.9, regarding the Laplace transform for the product of convolution, we have

$$\frac{\Gamma(x)\Gamma(y)}{p^{x+y}} = \mathcal{L}\left(t^{x+y+1}\int_0^\infty \theta^{x-1}(1-\theta)^{y-1}\mathrm{d}\theta\right),$$

in which we used the change of variables $\tau = t\theta$. The last integral is equal to $\beta(x, y)$ and

$$\mathcal{L}(t^{x+y+1}) = \frac{\Gamma(x+y)}{p^{x+y}},$$

and then $\Gamma(x)\Gamma(y) = \beta(x, y)\Gamma(x+y)$.

Let us now consider the function of Bessel of first order and of order $n \in \mathbb{N}$, J_n. It is known that the function J_n admits the integral representation

$$J_n(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(t\sin\theta - n\theta)}\mathrm{d}\theta.$$

Clearly, J_n is a function of class C^1 on $\mathbb{R}$ and, in addition, it satisfies $|J_n(t)| \le 1, \; \forall t \in \mathbb{R}, \; \forall n \in \mathbb{N}$. We deduce that J_n is an original function with growth index $s_0 = 0$. The image by the Laplace transform of the function $J_n(t)$ is given by

$$\mathcal{L}(J_n(t)) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\mathrm{d}\theta \int_0^{\infty} e^{(i\sin\theta - p)t}\mathrm{d}t.$$

If $Re(p) > s_0 = 0$, then

$$\int_0^{\infty} e^{(i\sin\theta - p)t}\mathrm{d}t = \frac{1}{p - i\sin\theta}, \Rightarrow \mathcal{L}(J_n(t)) = \frac{1}{2\pi}\int_0^{\infty} \frac{e^{-in\theta}}{p - i\sin\theta}\mathrm{d}\theta.$$

By using the substitution $e^{-i\theta} = z$, the integral from the right-hand side becomes a complex integral, computable with the help of the theorem of residues, so that, finally, we obtain

$$\mathcal{L}(J_n(t)) = \frac{1}{\sqrt{p^2+1}(p+\sqrt{p^2+1})^n}.$$

In a particular case, for $n = 0$, we have a result frequently met in applications, namely,

$$\mathcal{L}\left(J_0\left(2\sqrt{t}\right)\right) = \frac{1}{p}e^{-\frac{1}{p}}.$$

5.2 The Fourier Transform for Functions from L^1

Let us recall, first the fact that a function $f : \mathbb{R} \to \mathbb{R}$ is from $L^1(\mathbb{R})$ and we can write it shorter $f \in L^1$, if

$$\int_{-\infty}^{+\infty} |f(t)|\mathrm{d}t < +\infty.$$

Definition 5.2.1 If the function $f : \mathbb{R} \to \mathbb{R}$, $f \in L^1$, then its Fourier transform can be defined by

$$\mathcal{F}(f(t))(x) = \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t, \tag{5.2.1}$$

where i is the complex unit, $i^2 = -1$.

For a more convenient writing, we will use the notation $\mathcal{F}(f(t))(x) = \widehat{f}(x)$.

Theorem 5.2.1 *If $f \in L^1$, then its Fourier transform $\widehat{f}$ is bounded and continuous on $\mathbb{R}$. In addition, we have*

$$\left|\widehat{f}(x)\right| \le \|\widehat{f}\|_{B(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}, \tag{5.2.2}$$

where with $B(\mathbb{R})$ we denote the set of bounded functions on $\mathbb{R}$.

Proof We start from the definition (5.2.1), we obtain

$$\left|\widehat{f}(x)\right| \leq \int_{-\infty}^{+\infty} |f(t)| \left|e^{ixt}\right| \mathrm{d}t$$
$$= \int_{-\infty}^{+\infty} |f(t)| \mathrm{d}t = \|f\|_{L^1(\mathbb{R})}.$$

If in this inequality we pass to the supremum, we deduce that

$$\|\widehat{f}\|_{B(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})},$$

which proves that the Fourier transform is a bounded function. In addition, we also prove the double inequality (5.2.2). We want to prove now that $\widehat{f}$ is a continuous function. We use the following evaluations:

$$\left|\widehat{f}(x+h) - \widehat{f}(x)\right| = \left|\int_{-\infty}^{+\infty} f(t)\left[e^{i(x+h)t} - e^{ixt}\right] \mathrm{d}t\right|$$
$$\leq \int_{-\infty}^{+\infty} |f(t)|\left|e^{ixt}\right|\left|e^{iht}-1\right| \mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\left|e^{iht}-1\right| \mathrm{d}t \leq 2\int_{-\infty}^{+\infty} |f(t)| \mathrm{d}t, \quad (5.2.3)$$

from where we deduce that the difference from the left-hand side of the inequality (5.2.3) is bounded by a summable function.

On the other hand, we have

$$\left|\widehat{f}(x+h) - \widehat{f}(x)\right| \leq \int_{-\infty}^{+\infty} \left|e^{iht} - 1\right| |f(t)| \mathrm{d}t$$

and

$$\lim_{h\to 0} \left|e^{iht} - 1\right| |f(t)| = 0.$$

This means that we fulfill the conditions of the theorem of Lebesgue of passing to the limit under the integral sign, and therefore, we get

$$\lim_{h\to 0} \left|\widehat{f}(x+h) - \widehat{f}(x)\right| = \int_{-\infty}^{+\infty} \lim_{h\to 0} \left|e^{iht} - 1\right| |f(t)| \mathrm{d}t = 0,$$

that is, $\widehat{f}(x)$ is continuous in any point $x \in \mathbb{R}$. ■

Corollary 5.2.1 *If we have a sequence $\{f_n\}_{n\geq 1}$ of the functions from $L^1(\mathbb{R})$ which is convergent so that*

$$\lim_{n\to\infty} f_n = f, \ \textit{in } L^1(\mathbb{R}),$$

then

$$\lim_{n\to\infty} \widehat{f_n}(x) = \widehat{f}(x), \text{ uniformly with respect to } x \in \mathbb{R}.$$

Proof The result is immediately obtained based on inequality (5.2.2)

$$\left|\widehat{f_n}(x) - \widehat{f}(x)\right| \le \|f_n - f\|_{L^1(\mathbb{R})},$$

from where we get the result from the conclusion of the corollary. ■

The properties of the Fourier transform proved in the following theorem are important in computations.

Theorem 5.2.2 *If $f \in L^1$, then its Fourier transform $\widehat{f}$ satisfies the following rules of calculation:*

$$\mathcal{F}(f(t+a)) = e^{-iax}\mathcal{F}(f(t)) = e^{-iax}\widehat{f}(x),$$
$$\widehat{f}(x+b) = \mathcal{F}\left(e^{ibt}f(t)\right) = \widehat{e^{ibt}f(t)}. \tag{5.2.4}$$

Proof We start from the definition of the Fourier transform

$$\mathcal{F}(f(t+a)) = \int_{-\infty}^{+\infty} f(t+a)e^{ixt}\mathrm{d}t$$
$$= e^{-iax}\int_{-\infty}^{+\infty} f(\tau)e^{ix\tau}\mathrm{d}\tau = e^{-iax}\widehat{f}(x),$$

where we used the change of variables $t + a = \tau$. We proved thus formula $(5.2.4)_1$. For formula $(5.2.4)_2$ we also start from the definition of the Fourier transform

$$\mathcal{F}(e^{ibt}f(t)) = \widehat{e^{ibt}f(t)} = \int_{-\infty}^{+\infty} e^{ibt}f(t)e^{ixt}\mathrm{d}t$$
$$= \int_{-\infty}^{+\infty} f(t)e^{i(x+b)t}\mathrm{d}t = \widehat{f}(x+b),$$

so that we are led to $(5.2.4)_2$. ■

In the following theorem, due to the great mathematicians Riemann and Lebesgue, the behavior of the Fourier transform at infinity is given.

Theorem 5.2.3 *If $f \in L^1$, then for its Fourier transform $\widehat{f}$ we have*

$$\lim_{x\to\pm\infty} \widehat{f}(x) = \lim_{x\to\pm\infty} \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t = 0.$$

Proof We can write

$$-\widehat{f}(x) = e^{i\pi}\widehat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{i\pi}e^{ixt}\mathrm{d}t$$
$$= \int_{-\infty}^{+\infty} f(t)e^{ix(t+\pi/x)}\mathrm{d}t = \int_{-\infty}^{+\infty} f(\tau - \pi/x)e^{ix\tau}\mathrm{d}\tau.$$

In the last formula, we did the change of variables $t + \pi/x = \tau$.

Then

$$2\widehat{f}(x) = \widehat{f}(x) - (-\widehat{f}(x)) = \int_{-\infty}^{+\infty} \left[f(t) - f\left(t - \frac{\pi}{x}\right)\right] e^{ixt}\mathrm{d}t. \tag{5.2.5}$$

For the last integrand from (5.2.5), we have the bound

$$\left|\left[f(t) - f\left(t - \frac{\pi}{x}\right)\right] e^{ixt}\right| \leq |f(t)| + \left|f\left(t - \frac{\pi}{x}\right)\right|,$$

that is, the last integrand from (5.2.5) is bounded from above by a summable function (by the hypothesis that $f \in L^1$). We can then use the theorem of Lebesgue of passing to the limit under the integral sign in (5.2.5). Taking into account that

$$\lim_{x\to\pm\infty} \left|f(t) - f\left(t - \frac{\pi}{x}\right)\right| = 0,$$

so that the result formulated in the statement is immediately obtained. ■

Corollary 5.2.2 *If $f \in L^1(\mathbb{R})$, then*

$$\lim_{x\to\pm\infty} \int_{-\infty}^{+\infty} f(t)\cos xt\mathrm{d}t = 0, \quad \lim_{x\to\pm\infty} \int_{-\infty}^{+\infty} f(t)\sin xt\mathrm{d}t = 0.$$

Proof The result is immediately obtained from Theorem 5.1.2, using the formula of Euler $e^{ixt} = \cos xt + i\sin xt$. ■

According to Theorems 5.1.2 and 5.1.3, the Fourier transform is a continuous function on $\mathbb{R}$ and has null limits to $-\infty$ and $+\infty$. Now, we want to pose the converse problem. If we have a function g which is continuous on $\mathbb{R}$ and has null limits to $-\infty$ and $+\infty$, then is g the Fourier transform of a function from $L^1(\mathbb{R})$? The answer is negative and we will prove this statement by a counterexample

Lemma 5.2.1 *If a function g has the properties of a Fourier transform, then g is not necessarily the image of a function from $L^1(\mathbb{R})$.*

Proof We define the function g by

$$g(x) = \begin{cases} -g(-x) \;, \text{ if } x < 0, \\ \quad x/e \quad \;, \text{ if } 0 \le x \le e, \\ \; 1/\ln x \;\;, \text{ if } x > e. \end{cases}$$

From the definition, g is a symmetrical function in relation to the origin. Then

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1}{\ln x} = 0.$$

On the other hand, g is a continuous function, because for $x = e$, we have

$$g(e-0) = g(e+0) = 1.$$

Therefore, the function g has the properties of a Fourier transform. However, g is not the image of a function from $L^1(\mathbb{R})$. Suppose, by absurd, that there is a function $f \in L^1(\mathbb{R})$ so that

$$g(x) = \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t. \tag{5.2.6}$$

Let us compute the following limit:

$$\lim_{n\to\infty} \int_e^n \frac{g(x)}{x}\mathrm{d}x = \lim_{n\to\infty} \int_e^n \frac{1}{x\ln x}\mathrm{d}x$$

$$= \lim_{n\to\infty} \left[\ln(\ln x)\right]_e^n = \lim_{n\to\infty} \ln(\ln n) = \infty. \tag{5.2.7}$$

Therefore, if we start from the definition of g, we obtain that the limit from (5.2.7) is infinite. We want to prove that if we use the form (5.2.6) of the function g, the limit from (5.2.7) is finite. Indeed, taking into account the form (5.2.6) of the function g, we obtain

$$g(x) = -g(-x) = \int_{-\infty}^{+\infty} f(t)e^{-ixt}\mathrm{d}t$$

and if we add this relation member by member to the relation (5.2.6) we are led to

$$2g(x) = \int_{-\infty}^{+\infty} f(t)\left[e^{ixt} + e^{-ixt}\right]\mathrm{d}t = 2i\int_{-\infty}^{+\infty} f(t)\sin xt\mathrm{d}t.$$

Therefore, we can write

$$g(x) = i\int_{-\infty}^{0} f(t)\sin xt\mathrm{d}t + i\int_{0}^{\infty} f(t)\sin xt\mathrm{d}t$$
$$= i\int_{0}^{\infty} [f(t) - f(-t)]\sin xt\mathrm{d}t.$$

Then, the integral under the limit from (5.2.7) becomes

$$\int_{e}^{n} \frac{g(x)}{x}\mathrm{d}x = i\int_{e}^{n}\left\{\int_{0}^{\infty} [f(t) - f(-t)]\frac{\sin xt}{x}\mathrm{d}t\right\}\mathrm{d}x. \tag{5.2.8}$$

In the last integral, we can change the order of integration, because $f(t) - f(-t)$ is summable (according to the hypothesis that $f \in L^1(\mathbb{R})$). Thus,

$$\int_{e}^{n} \frac{g(x)}{x}\mathrm{d}x = i\int_{0}^{\infty} [f(t) - f(-t)]\left\{\int_{e}^{n}\frac{\sin xt}{x}\mathrm{d}x\right\}\mathrm{d}t$$
$$= i\int_{0}^{\infty} [f(t) - f(-t)]\left\{\int_{et}^{nt}\frac{\sin\xi}{\xi}\mathrm{d}\xi\right\}\mathrm{d}t < \infty,$$

because the integral $\int_{et}^{nt}\frac{\sin\xi}{\xi}\mathrm{d}\xi$ is convergent, and the function $f(t) - f(-t)$ is summable.

We arrived in this way to a contradiction which proves that the function g cannot be the Fourier transform of a function from $L^1(\mathbb{R})$. ■

Another natural question in relation with the Fourier transform is the following: if $f \in L^1(\mathbb{R})$, then is $\widehat{f} \in L^1(\mathbb{R})$? The answer is again negative and we will prove this also by means of a counterexample.

Lemma 5.2.2 *If a function is from $L^1(\mathbb{R})$, then its Fourier transform is not necessarily a function from $L^1(\mathbb{R})$.*

Proof We define the function f by

$$f(t) = \begin{cases} 0 & , \text{ if } t < 0, \\ e^{-t} & , \text{ if } t \leq 0. \end{cases}$$

Taking into account that

$$\int_{-\infty}^{+\infty} f(t)\mathrm{d}t = \int_{0}^{+\infty} e^{-t}\mathrm{d}t = 1,$$

we deduce that $f \in L^1(\mathbb{R})$.

But the Fourier transform of the function f is

$$\widehat{f}(x) = \int_{0}^{+\infty} e^{-t}e^{ixt}\mathrm{d}t = \int_{0}^{+\infty} e^{(ix-1)t}\mathrm{d}t = \frac{1}{1-ix} = \frac{1+ix}{1+x^2},$$

from where we clearly deduce that $\widehat{f} \notin L^1(\mathbb{R})$. ■

We give now, without proof, two theorems, due to Jordan, which establishes the relation between the Fourier transform and the original function.

Theorem 5.2.4 *Assume that $f \in L^1(\mathbb{R})$ and, in addition, f is a function with bounded variation ($f \in BV(\mathbb{R})$). Then, in a neighborhood of a fixed point u, the following formula of inversion holds true:*

$$\lim_{a \to \infty} \frac{1}{2\pi} \int_{-a}^{a} \widehat{f}(x) e^{-ixu} \mathrm{d}x = \frac{1}{2}[f(u+0) - f(u+0)].$$

If u is a point of continuity for the function f, then the formula of inversion becomes

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(x) e^{-ixu} \mathrm{d}x,$$

where $\widehat{f}$ is the Fourier transform of the function f.

Theorem 5.2.5 *If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then in a point u of continuity of the function f, the formula of inversion holds true*

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(x) e^{-ixu} \mathrm{d}x.$$

We finish this paragraph with some considerations on the product of convolution for functions from $L^1(\mathbb{R})$.

By definition, if $f, g \in L^1(\mathbb{R})$, then their product of convolution is

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-\tau) g(\tau) \mathrm{d}\tau. \tag{5.2.9}$$

Theorem 5.2.6 *If $f, g \in L^1(\mathbb{R})$, then their product of convolution is defined almost everywhere on $\mathbb{R}$ and is a function from $L^1(\mathbb{R})$.*

Proof With the change of variables $t - \tau = u$, we have

$$\int_{-\infty}^{+\infty} |f(t-\tau)| \mathrm{d}\tau = \int_{-\infty}^{+\infty} |f(u)| \mathrm{d}u,$$

so that, taking into account that $f \in L^1(\mathbb{R})$, we deduce that the integrand from the right-hand side of the relation of definition (5.2.9) is a function defined almost everywhere and summable. We can therefore invert the order of integration and get

$$\begin{aligned}\int_{-\infty}^{+\infty} |f * g|(t)\mathrm{d}t &= \int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} |f(t-\tau)||g(\tau)|\mathrm{d}\tau\right\} \mathrm{d}t \\ &= \int_{-\infty}^{+\infty} |g(\tau)| \left\{\int_{-\infty}^{+\infty} |f(t-\tau)|\mathrm{d}t\right\} \mathrm{d}\tau \\ &= \|f\|_{L^1} \int_{-\infty}^{+\infty} |g(\tau)|\mathrm{d}\tau = \|f\|_{L^1}\|g\|_{L^1},\end{aligned}$$

which proves that $f * g \in L^1(\mathbb{R})$. ∎

Proposition 5.2.1 *If* $f, g \in L^1(\mathbb{R})$, *then*

$$\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}.$$

Proof Because $f, g \in L^1(\mathbb{R})$, then according to Theorem 1.10, we have $f * g \in L^1(\mathbb{R})$. Using the definition (5.2.9) of the product of convolution and the definition of the norm in $L^1(\mathbb{R})$, we obtain

$$\begin{aligned}\|f * g\|_{L^1} &= \int_{-\infty}^{+\infty} \left|\int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\mathrm{d}\tau\right| \mathrm{d}t \\ &\le \int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} |f(t-\tau)g(\tau)|\mathrm{d}\tau\right| \mathrm{d}t \\ &= \int_{-\infty}^{+\infty} |g(\tau)| \left\{\int_{-\infty}^{+\infty} |f(t-\tau)|\mathrm{d}t\right\} \mathrm{d}\tau \\ &= \int_{-\infty}^{+\infty} |g(\tau)| \left\{\int_{-\infty}^{+\infty} |f(u)|\mathrm{d}u\right\} \mathrm{d}\tau \\ &= \|f\|_{L^1} \int_{-\infty}^{+\infty} |g(\tau)|\mathrm{d}\tau = \|f\|_{L^1}\|g\|_{L^1},\end{aligned}$$

and the proof is complete. ∎

Because we proved that the product of convolution $f * g$ is a function from $L^1(\mathbb{R})$, we can compute its Fourier transform.

Theorem 5.2.7 *If* $f, g \in L^1(\mathbb{R})$, *then the Fourier transform of their product of convolution is equal to usual product of the Fourier transforms, namely,*

$$\mathcal{F}((f * g)(t)) = \mathcal{F}(f(t)) \cdot \mathcal{F}(g(t)).$$

Proof We take into account the definition of the Fourier transform for functions from $L^1(\mathbb{R})$ and the definition of the product of convolution so that we obtain

$$
\begin{aligned}
\mathcal{F}((f*g)(t)) &= \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\mathrm{d}\tau\right\}e^{ixt}\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} g(\tau)\left\{\int_{-\infty}^{+\infty} f(t-\tau)e^{ixt}\mathrm{d}t\right\}\mathrm{d}\tau \\
&= \int_{-\infty}^{+\infty} g(\tau)\left\{\int_{-\infty}^{+\infty} f(u)e^{ixu}\mathrm{d}u\right\}e^{ix\tau}\mathrm{d}\tau \\
&= \widehat{f}(x)\int_{-\infty}^{+\infty} g(\tau)e^{ix\tau}\mathrm{d}\tau = \widehat{f}(x).\widehat{g}(x).
\end{aligned}
$$

In the third formula, we performed the change of variables $t-\tau=u$. ■

5.3 The Fourier Transform for Functions from L^2

In the calculations that follow, the result from the following lemma is useful.

Lemma 5.3.1 *For* $\forall\varepsilon>0$ *and* $\forall\alpha\in\mathbb{R}$*, the following equality holds true:*

$$
\int_{-\infty}^{+\infty} e^{i\alpha t}e^{-\varepsilon t^2}\mathrm{d}t = \left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{\frac{-\alpha^2}{4\varepsilon}}.
$$

Proof With the change of variables $t=x/\sqrt{\varepsilon}$, we obtain

$$
\int_{-\infty}^{+\infty} e^{i\alpha t}e^{-\varepsilon t^2}\mathrm{d}t = \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{+\infty} e^{i\alpha\frac{x}{\sqrt{\varepsilon}}}e^{-x^2}\mathrm{d}x. \tag{5.3.1}
$$

The value of the integral of Gauss is well known

$$
\int_{-\infty}^{+\infty} e^{-(x+i\beta)^2}\mathrm{d}x = \sqrt{\pi}, \tag{5.3.2}
$$

and this value can be obtained with the help of the Laplace transform, or by using techniques from the theory of complex integrals.

We can write the integral from (5.3.2) in the form

$$
\int_{-\infty}^{+\infty} e^{-x^2}e^{-2\beta xi}e^{\beta^2}\mathrm{d}x = e^{\beta^2}\int_{-\infty}^{+\infty} e^{-x^2}e^{-2\beta xi}\mathrm{d}x
$$

and then

$$
\int_{-\infty}^{+\infty} e^{-x^2}e^{-2\beta xi}\mathrm{d}x = \sqrt{\pi}e^{-\beta^2}.
$$

We come back to the result from (5.3.1), and taking $\beta = -\alpha/(2\sqrt{\varepsilon})$ we get

$$\int_{-\infty}^{+\infty} e^{i\alpha t} e^{-\varepsilon t^2} \mathrm{d}t = \frac{1}{\sqrt{\varepsilon}} \sqrt{\pi} e^{\frac{-\alpha^2}{4\varepsilon}},$$

and this ends the proof. ∎

In the following theorem, we prove a fundamental result, which predicts the Fourier transform for functions from $L^2(\mathbb{R})$.

Theorem 5.3.1 *Let us consider the function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, $\widehat{f}$, considered to be the Fourier transform of a function from $L^1(\mathbb{R})$, is a function from $L^2(\mathbb{R})$. In addition, we have*

$$\|\widehat{f}\|_{L^2(\mathbb{R})} = \sqrt{2\pi}\|f\|_{L^2(\mathbb{R})}.$$

Proof Because $f \in L^1(\mathbb{R})$, we know that there is its Fourier transform, $\widehat{f}$, and it is given by

$$\widehat{f}(x) = \int_{-\infty}^{+\infty} f(t) e^{ixt} \mathrm{d}t.$$

Then, we obtain

$$\left|\widehat{f}(x)\right|^2 = \widehat{f}(x)\overline{\widehat{f}(x)} = \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t \int_{-\infty}^{+\infty} \overline{f(u)}e^{-ixu}\mathrm{d}u.$$

We multiply this equality with $e^{-x^2/n}$ and the obtained equality is integrated on $\mathbb{R}$. Thus, we are led to

$$I \equiv \int_{-\infty}^{+\infty} \left|\widehat{f}(x)\right|^2 e^{-\frac{x^2}{n}} \mathrm{d}x = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{n}} \left\{ \int_{-\infty}^{+\infty} f(t)e^{ixt}\mathrm{d}t \int_{-\infty}^{+\infty} \overline{f(u)}e^{-ixu}\mathrm{d}u \right\} \mathrm{d}x.$$

Because f and $\overline{f}$ are absolutely integrable functions, we can invert the order of integration

$$I = \int_{-\infty}^{+\infty} \overline{f(u)} \left\{ \int_{-\infty}^{+\infty} f(t) \left[\int_{-\infty}^{+\infty} e^{-\frac{x^2}{n}} e^{ix(t-u)}\mathrm{d}x \right] \mathrm{d}t \right\} \mathrm{d}u$$
$$= \sqrt{\pi n} \int_{-\infty}^{+\infty} \overline{f(u)} \left\{ \int_{-\infty}^{+\infty} f(t) e^{-\frac{n(t-u)^2}{4}} \mathrm{d}t \right\} \mathrm{d}u,$$

in which we used the result from Lemma 5.3.1 with $\varepsilon = 1/n$ and $\alpha = t - u$.

In the last integral, we make the change of variables $t - u = s$ and change the notation of s by t and so we are led to

$$I = \sqrt{\pi n}\int_{-\infty}^{+\infty}\overline{f(u)}\left\{\int_{-\infty}^{+\infty} f(t+u)e^{-\frac{nt^2}{4}}\mathrm{d}t\right\}\mathrm{d}u$$
$$= \sqrt{\pi n}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\overline{f(u)}f(t+u)\mathrm{d}u\right\}e^{-\frac{nt^2}{4}}\mathrm{d}t. \qquad (5.3.3)$$

We introduce the notation

$$g(t) = \int_{-\infty}^{+\infty}\overline{f(u)}f(t+u)\mathrm{d}u. \qquad (5.3.4)$$

We now want to prove that the function g is continuous in $t = 0$. Indeed,

$$|g(t)-g(0)|^2 = \left|\int_{-\infty}^{+\infty}\overline{f(u)}[f(t+u)-f(u)]\mathrm{d}u\right|^2$$
$$\le \int_{-\infty}^{+\infty}\left|\overline{f(u)}\right|^2\cdot\int_{-\infty}^{+\infty}|f(t+u)-f(u)|^2\,\mathrm{d}u$$
$$= \|f\|_{L^2}^2\cdot\int_{-\infty}^{+\infty}|f(t+u)-f(u)|^2\,\mathrm{d}u,$$

in which we used the inequality of Hőlder.

It is known that any function from $L^p,\ p>1$ (in our case $f\in L^2$) is continuous in average and then

$$\int_{-\infty}^{+\infty}|f(t+u)-f(u)|^2\,\mathrm{d}u \to 0,\ \text{as}\ t\to 0,$$

which proves that

$$|g(t)-g(0)|^2 \to 0,\ \text{as}\ t\to 0,$$

that is, the function g is continuous in the origin.

We will come back to the relation (5.3.3) and write it in the form

$$\int_{-\infty}^{+\infty}e^{-\frac{x^2}{n}}\left|\widehat{f}(x)\right|^2\mathrm{d}x = \sqrt{\pi n}\int_{-\infty}^{+\infty}e^{-\frac{nt^2}{4}}g(t)\mathrm{d}t$$
$$= 2\sqrt{\pi}\int_{-\infty}^{+\infty}e^{-\tau^2}g\left(\frac{2}{\sqrt{n}}\tau\right)\mathrm{d}\tau. \qquad (5.3.5)$$

From the definition (5.3.4) of the function g, we deduce that

$$|g(t)| \le \left\{\int_{-\infty}^{+\infty}\left|\overline{f}(u)\right|^2\mathrm{d}u\right\}^{1/2}\cdot\left\{\int_{-\infty}^{+\infty}|f(t+u)|^2\,\mathrm{d}u\right\}^{1/2}$$
$$= \left(\|f\|_{L^2}^2\right)^{1/2}\left(\|f\|_{L^2}^2\right)^{1/2} = \|f\|_{L^2}^2.$$

Also, from (5.3.4) we obtain

$$g(0) = \int_{-\infty}^{+\infty} \overline{f}(u) f(u) \mathrm{d}u = \int_{-\infty}^{+\infty} |f(u)|^2 \,\mathrm{d}u = \|f\|_{L^2}^2. \qquad (5.3.6)$$

Because the function

$$e^{-\tau^2} g\left(\frac{2}{\sqrt{n}}\tau\right)$$

is bounded by a summable function, namely, $e^{-\tau^2}\|f\|_{L^2}^2$, we deduce that in (5.3.5) we can use the theorem of Lebesgue of passing to the limit under the integral sign. Thus, as $n \to \infty$, from (5.3.5) we deduce that

$$\int_{-\infty}^{+\infty} \left|\widehat{f}(x)\right|^2 \mathrm{d}x = 2\sqrt{\pi} \int_{-\infty}^{+\infty} e^{-t^2} g(0) \mathrm{d}t$$
$$= 2\sqrt{\pi}\|f\|_{L^2}^2 \int_{-\infty}^{+\infty} e^{-t^2} \mathrm{d}t = 2\sqrt{\pi}\|f\|_{L^2}^2 \sqrt{\pi} = 2\pi\|f\|_{L^2}^2,$$

in which we take into account (5.3.6).

Thus,

$$\|\widehat{f}\|_{L^2}^2 = 2\pi\|f\|_{L^2}^2 \Rightarrow \|\widehat{f}\|_{L^2} = \sqrt{2\pi}\|f\|_{L^2},$$

and this ends the proof the theorem. ■

We make yet another step in our intention to introduce the Fourier transform for functions from $L^2(\mathbb{R})$. To this end, we recall the definition of the truncated of a function. Thus, if $f \in L^2(\mathbb{R})$, then its truncated f_a can be defined by

$$f_a(t) = \begin{cases} f(t), & \text{if } |t| \le a, \\ 0, & \text{if } |t| > a. \end{cases} \qquad (5.3.7)$$

Theorem 5.3.2 *If the function $f \in L^2(\mathbb{R})$, then its truncated f_a is a function from $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and therefore, it admits the Fourier transform $\widehat{f_a}$ and we have $\widehat{f_a} \in L^2(\mathbb{R})$. In addition, as $a \to 0$ we have*

$$\widehat{f_a}(t) \to \widehat{f}(t), \text{ in the norm from } L^2.$$

Proof Let us observe, first, that from (5.3.7) we deduce that

$$|f_a(t)| \le |f(t)|, \ \forall t \in \mathbb{R} \Rightarrow |f_a(t)|^2 \le |f(t)|^2, \ \forall t \in \mathbb{R},$$

so that by integrating the last inequality, we obtain

$$\int_{-\infty}^{+\infty} |f_a(t)|^2 \, \mathrm{d}t \leq \int_{-\infty}^{+\infty} |f(t)|^2 \mathrm{d}t \Rightarrow \|f_a(t)\|_{L^2} \leq \|f(t)\|_{L^2},$$

which proves that $f_a \in L^2(\mathbb{R})$.

On the other hand, from the definition of the truncated function, we obtain

$$\int_{-\infty}^{+\infty} |f_a(t)| \, \mathrm{d}t = \int_{-a}^{+a} |f(t)| \mathrm{d}t \leq \sqrt{2a} \left(\int_{-a}^{+a} |f(t)|^2 \mathrm{d}t \right)^{1/2},$$

in which we take into account the inequality of Hőlder.

But

$$\int_{-a}^{+a} |f(t)|^2 \mathrm{d}t \leq \int_{-\infty}^{+\infty} |f(t)|^2 \mathrm{d}t = \|f\|_{L^2}^2,$$

and then

$$\int_{-\infty}^{+\infty} |f_a(t)| \, \mathrm{d}t \leq \sqrt{2a} \|f\|_{L^2}^2,$$

which proves that $f_a \in L^1(\mathbb{R})$. Thus, the truncated function f_a is a function which belongs to the set $L^1(\mathbb{R})$ and also belongs to the set $L^2(\mathbb{R})$, that is, it satisfies the hypotheses of Theorem 5.3.1. Then, its Fourier transform exists in the sense of the transform of a function from $L^1(\mathbb{R})$:

$$\widehat{f_a}(x) = \int_{-\infty}^{+\infty} f_a(t) e^{ixt} \mathrm{d}t = \int_{-a}^{+a} f(t) e^{ixt} \mathrm{d}t.$$

By using, also, Theorem 5.3.1, we deduce that $\widehat{f_a} \in L^1(\mathbb{R})$. It remains only to prove that $\widehat{f_a}$ is convergent in $L^2(\mathbb{R})$. For this, we use the criterion of Cauchy for fundamental sequences (this is possible because $L^2(\mathbb{R})$ is complete space). For $b > 0$, we have

$$\left\| \widehat{f_a} - \widehat{f_{a+b}} \right\|_{L^2}^2 \leq \left| \int_{-a+b}^{+a} |f(t)|^2 \mathrm{d}t + \int_{a}^{a+b} |f(t)|^2 \mathrm{d}t \right|.$$

Thus, $\forall \varepsilon > 0, \ \exists n_0(\varepsilon)$ so that if $a > n_0(\varepsilon)$ and $b > 0$, we have

$$\left\| \widehat{f_a} - \widehat{f_{a+b}} \right\|_{L^2}^2 < \varepsilon,$$

which proves that the sequence $\{\widehat{f_a}\}$ is convergent in the norm of $L^2(\mathbb{R})$. ∎

We can now define the Fourier transform for a function from $L^2(\mathbb{R})$.

Definition 5.3.1 If the function $f \in L^2(\mathbb{R})$, then we can attach its truncated function f_a and to this, as a function from $L^1(\mathbb{R})$, we can attach the Fourier transform

$$\widehat{f_a}(x) = \int_{-\infty}^{+\infty} f_a(t) e^{ixt} \mathrm{d}t = \int_{-a}^{+a} f(t) e^{ixt} \mathrm{d}t.$$

By definition

$$\widehat{f}(x) = \lim_{a\to 0} \widehat{f_a}(x), \text{ in } L^2.$$

The result from the following theorem is due to Parseval.

Theorem 5.3.3 *If the function* $f \in L^2(\mathbb{R})$*, then* $\widehat{f} \in L^2(\mathbb{R})$ *and*

$$\left\|\widehat{f}\right\|_{L^2} = \sqrt{2\pi}\, \|f\|_{L^2}.$$

Proof The fact that $\widehat{f} \in L^2(\mathbb{R})$ is certified from Theorem 5.3.1. Then, we have the equality

$$\left| \left\|\widehat{f_n}\right\|_{L^2} - \left\|\widehat{f_m}\right\|_{L^2} \right| \le \left\|\widehat{f_n} - \widehat{f_m}\right\|_{L^2}. \tag{5.3.8}$$

In Theorem 5.3.2, we proved that the sequence $\{\widehat{f_n}\}$ is convergent and then

$$\lim_{n\to\infty} \left\|\widehat{f_n}\right\|_{L^2} = \left\|\widehat{f}\right\|_{L^2}.$$

On the other hand, since $f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we can write for the truncated function

$$\left\|\widehat{f_n}\right\|_{L^2} = \sqrt{2\pi}\, \|f_n\|_{L^2},$$

so that if we pass to the limit we obtain the desired result. ■

In the following theorem, we prove a formula of inversion, due to Plancherel.

Theorem 5.3.4 *If the functions* $f, g \in L^2(\mathbb{R})$*, then the following formula of inversion holds true:*

$$\int_{-\infty}^{+\infty} \widehat{f}(x)\overline{\widehat{g}(x)}\mathrm{d}x = 2\pi \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\mathrm{d}x.$$

Proof According to the formula of Parseval, we can write

$$\left\|\widehat{f} + \widehat{g}\right\|_{L^2}^2 = 2\pi\, \|f + g\|_{L^2}^2,$$

that is,

$$\int_{-\infty}^{+\infty} \left(\widehat{f}(x) + \widehat{g}(x)\right)\left(\overline{\widehat{f}(x)} + \overline{\widehat{g}(x)}\right)\mathrm{d}x$$
$$= 2\pi \int_{-\infty}^{+\infty} \left(f(x) + g(x)\right)\left(\overline{f(x)} + \overline{g(x)}\right)\mathrm{d}x.$$

After simple calculations, taking into account the formula of Parseval, we deduce

$$\int_{-\infty}^{+\infty} \widehat{f}(x)\overline{\widehat{g}(x)}\mathrm{d}x + \int_{-\infty}^{+\infty} \widehat{g}(x)\overline{\widehat{f}(x)}\mathrm{d}x$$
$$= 2\pi \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\mathrm{d}x + 2\pi \int_{-\infty}^{+\infty} \overline{f(x)}g(x)\mathrm{d}x. \tag{5.3.9}$$

If we consider again the calculations above by taking ig instead of g, we deduce

$$-i \int_{-\infty}^{+\infty} \widehat{f}(x)\overline{\widehat{g}(x)}\mathrm{d}x + i \int_{-\infty}^{+\infty} \overline{\widehat{f}(x)}\widehat{g}(x)\mathrm{d}x$$
$$= -2\pi i \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\mathrm{d}x + 2\pi i \int_{-\infty}^{+\infty} \overline{f(x)}g(x)\mathrm{d}x.$$

We can simplify here with $(-i)$ and then add the obtained equality member by member to equality (5.3.9) so that we obtain the result of Plancherel. ■

A formula of inversion is also the result of the following theorem.

Theorem 5.3.5 *If the functions $f, g \in L^2(\mathbb{R})$, then the following formula of inversion holds true:*

$$\int_{-\infty}^{+\infty} \widehat{f}(x)g(x)\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\mathrm{d}x. \tag{5.3.10}$$

Proof Since the functions $f, g \in L^2(\mathbb{R})$, we deduce that we can attach to them the truncated functions f_n and g_k, respectively.

As we already proved, $f_n, g_k \in L^1(\mathbb{R})$. Then, to these truncated functions we can attach the transforms, in the sense of the functions from $L^1(\mathbb{R})$. From the formula of the Fourier transform for the truncated function, we deduce

$$\int_{-\infty}^{+\infty} \widehat{f_n}(x)g_k(x)\mathrm{d}x = \int_{-\infty}^{+\infty} g_k(x)\left\{\int_{-\infty}^{+\infty} f_n(t)e^{ixt}\mathrm{d}t\right\}\mathrm{d}x$$
$$= \int_{-\infty}^{+\infty} f_n(x)\left\{\int_{-\infty}^{+\infty} g_k(t)e^{ixt}\mathrm{d}x\right\}\mathrm{d}t = \int_{-\infty}^{+\infty} f_n(t)\widehat{f_k}(t)\mathrm{d}t. \tag{5.3.11}$$

In these calculations, it was possible to reverse the order of integration because in fact, the integrals can be calculated on finite intervals, taking into account the definition of the truncated function.

The equality (5.3.11) proves that the formula of inversion (5.3.10) is true for the truncated function.

We keep f_n fixed and we use the result from Theorem 5.3.2. According to this, the sequence $\{\widehat{g_k}\}$ is convergent, almost everywhere, in the sense of L^2, to a function from L^2. Analogously is obtained the fact that the sequence $\{\widehat{f_n}\}$ is convergent, almost everywhere, in the sense of L^2, to a function from L^2.

We use then the fact that the two limits are g and f, respectively. We deduce (5.3.10) from (5.3.11) by using the theorem of Lebesgue of passing to the limit under the integral sign. So the theorem is proven. ∎

We finish this paragraph with a last inversion formula of the Fourier transform for functions from the space $L^2(\mathbb{R})$.

Theorem 5.3.6 *Let us consider the function $f \in L^2(\mathbb{R})$ and we define the function g by*

$$g(x) = \overline{\widehat{f}(x)}, \;\; \forall x \in \mathbb{R}.$$

Then, we have

$$f(x) = \frac{1}{2\pi}\overline{\widehat{g}(x)}, \;\; \forall x \in \mathbb{R}.$$

Proof According to the definition of the norm in L^2, we have

$$\begin{aligned}\left\| f - \frac{1}{2\pi}\overline{\widehat{g}}\right\|^2_{L^2} &= \int_{-\infty}^{+\infty}\left(f(x) - \frac{1}{2\pi}\overline{\widehat{g}(x)}\right)\left(\overline{f(x)} - \frac{1}{2\pi}\widehat{g}(x)\right)\mathrm{d}x \\ &= \|f\|^2_{L^2} + \frac{1}{4\pi^2}\|\widehat{g}\|^2_{L^2} - \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\mathrm{d}x \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{f(x)\widehat{g}(x)}\mathrm{d}x.\end{aligned} \tag{5.3.12}$$

Using twice the formula of Parseval, we obtain

$$\begin{aligned}\frac{1}{4\pi^2}\|\widehat{g}\|^2_{L^2} &= \frac{2\pi}{4\pi^2}\|g\|^2_{L^2} \\ &= \frac{1}{2\pi}\left\|\overline{\widehat{f}}\right\|^2_{L^2} = \frac{1}{2\pi}\left\|\widehat{f}\right\|^2_{L^2} = \frac{1}{2\pi}\|f\|^2_{L^2}.\end{aligned} \tag{5.3.13}$$

On the other hand, with formula (5.3.10) and then with the formula of Parseval, we deduce that

$$\begin{aligned}-\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\mathrm{d}x &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{f}(x)g(x)\mathrm{d}x \\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{f}(x)\overline{\widehat{f}(x)}\mathrm{d}x = -\frac{1}{2\pi}\left\|\widehat{f}\right\|^2_{L^2} \\ &= -\frac{2\pi}{2\pi}\|f\|^2_{L^2} = -\|f\|^2_{L^2}.\end{aligned} \tag{5.3.14}$$

analogously,

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{\widehat{g}(x)}\,\widehat{f}(x)\mathrm{d}x=\overline{-\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(x)\widehat{g}(x)\mathrm{d}x}= \tag{5.3.15}$$
$$=\overline{-\,\|f\|_{L^2}^2}=-\,\|f\|_{L^2}^2\,,$$

in which we used the result from (5.3.14) and the fact that the conjugate of a real number is the number itself.

If we take into account the formulas (5.3.13), (5.3.14), and (5.3.15) in (5.3.12), we obtain the formula from the statement of the theorem. ■

Chapter 6
Parabolic Equations

6.1 Initial-Boundary Value Problems

The prototype of a parabolic equation is given by the equation of propagation of heat in a body. Let Ω be a bounded domain from $\mathbb{R}^n$ having boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. For a constant of time $T > 0$, arbitrarily fixed, consider the interval of time $\mathcal{T}_T$ given by

$$\mathcal{T}_T = \{t \ : 0 < t \leq T\}, \overline{\mathcal{T}_T} = \{t \ : 0 \leq t \leq T\}.$$

Then the equation of propagation of heat (shorter, the equation of heat) is

$$u_t(t, x) - a^2 \Delta u(t, x) = f(t, x), \ \forall (t, x) \in \mathcal{T}_T \times \Omega, \tag{6.1.1}$$

in which we use the notation $u_t = \partial u / \partial t$, a is a given positive constant, and Δ is the Laplace operator.

Commonly, Eq. (6.1.1) is accompanied by an initial condition of the form:

$$u(0, x) = \varphi(x), \ \forall x \in \overline{\Omega}. \tag{6.1.2}$$

As in the case of elliptic equations, the boundary condition is one of the following types:

- Dirichlet boundary condition

$$u(t, y) = \alpha(t, y), \ \forall (t, y) \in \overline{\mathcal{T}_T} \times \partial\Omega; \tag{6.1.3}$$

- Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t, y) = \beta(t, y), \ \forall (t, y) \in \overline{\mathcal{T}_T} \times \partial\Omega; \tag{6.1.4}$$

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- Mixed boundary condition

$$\lambda_1 \frac{\partial u}{\partial \nu}(t, y) + \lambda_2 u(t, y) = \gamma(t, y),\ \forall (t, y) \in \overline{\mathcal{T}_T} \times \partial\Omega; \qquad (6.1.5)$$

If we consider, for instance, the problem (6.1.1)–(6.1.3), we have the following physical interpretation:

- $u(t, x)$, which is the unknown function of the problem and represents the temperature of the body Ω, at any moment t;
- $\varphi(x)$ represents the temperature (which is known) at the initial moment in all points of the body (including on the boundary);
- $\alpha(t, y)$ represents the temperature (which is known) at any moment on the surface $\partial\Omega$ which borders the body.

Thus, the problem (6.1.1)–(6.1.3) consists in determining the temperature in all points of the body Ω, at any moment, by knowing the temperature of the body at the initial moment and by knowing at any moment the temperature on the surface of the body, $\partial\Omega$.

In the following, we will consider, in particular, the problem (6.1.1), (6.1.2), (6.1.3). In the study of this problem, we will consider, at present, the following standard hypotheses:

(i) the function $f : \mathcal{T}_T \times \partial\Omega \to \mathbb{R}$ is known (given) and $f \in C(\mathcal{T}_T \times \partial\Omega)$;
(ii) the function $\varphi : \overline{\Omega} \to \mathbb{R}$ is given and $\varphi \in C(\overline{\Omega})$;
(iii) the function $\alpha : \overline{\mathcal{T}_T} \times \partial\Omega \to \mathbb{R}$ is given and $\alpha \in C(\overline{\mathcal{T}_T} \times \partial\Omega)$.

A function $u = u(t, x), u : \overline{\mathcal{T}_T} \times \overline{\Omega} \to \mathbb{R}$, is called a classical solution of the problem (6.1.1), (6.1.2), (6.1.3), which satisfies the following properties:

- $u \in C(\overline{\mathcal{T}_T} \times \overline{\Omega})$;
- $u_t, u_{x_i x_i} \in C\,(\mathcal{T}_T \times \Omega)$;
- u satisfies Eq. (6.1.1), the initial condition (6.1.2), and the Dirichlet boundary condition (6.1.3).

In the formulation of the problem (6.1.1), (6.1.2), (6.1.3), the initial and the boundary conditions are given on the set $\overline{\mathcal{T}_T} \times \partial\Omega$ or on the set $\{0\} \times \overline{\Omega}$.

We define the set Γ by

$$\Gamma = \left\{(t, x)\ :\ (t, x) \in \left(\overline{\mathcal{T}_T} \times \partial\Omega\right) \cup \left(\{0\} \times \overline{\Omega}\right)\right\}, \qquad (6.1.6)$$

and we call it the *parabolic border*, which is different from the topological border. Practically, to obtain the parabolic border, "the lid" for $t = T$ is removed from the topological border.

We now prove a theorem of extreme values for the case of a homogeneous parabolic equation

$$u_t(t, x) - \Delta u(t, x) = 0,\ \forall (t, x) \in \mathcal{T}_T \times \Omega. \qquad (6.1.7)$$

Theorem 6.1.1 *Let us consider the domain* Ω *and* $\mathcal{T}_T$ *defined as above and consider the function* u *so that* $u \in C(\overline{\mathcal{T}}_T \times \overline{\Omega})$, $u_t, u_{x_i x_i} \in C(\mathcal{T}_T \times \Omega)$. *If* u *satisfies the homogeneous equation (6.1.7), then the extreme values*

$$\sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} u(t,x), \qquad \inf_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} u(t,x)$$

are reached necessarily on Γ.

Proof If we perform the proof for the supremum value, the result for the infimum value is immediately obtained by passing from u to $-u$.

We must outline that in the conditions of the theorem, u reaches its effective extreme values, according to a classical theorem due to Weierstrass.

Suppose, by contradiction, that u reaches its supremum value inside the domain, not on the boundary Γ. This means that we suppose that there is a point $(t_0, x^0) \in \overline{\mathcal{T}}_T \times \overline{\Omega} \setminus \Gamma$ so that

$$M = \sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} u(t,x) = u(t_0, x^0).$$

Denote by m the supremum value of the function u reached on Γ, that is,

$$m = \sup_{(t,x)\in\Gamma} u(t,x).$$

According to the assumption that we made, we have

$$M > m. \tag{6.1.8}$$

In the following, we will prove that (6.1.8) leads to a contradiction. We define the function $v(t,x)$ by

$$v(t,x) = u(t,x) + \frac{M-m}{2d^2}\sum_{i=1}^{n}\left(x_i - x_i^0\right)^2, \tag{6.1.9}$$

where d is the diameter of the set $\overline{\Omega}$.

By evaluating the function v on Γ, we obtain

$$v(t,x)|_\Gamma \le m + \frac{M-m}{2} = \frac{M+m}{2} < \frac{M+M}{2} = M. \tag{6.1.10}$$

On the other hand,

$$v(t_0,x^0) = u(t_0,x^0) + \frac{M-m}{2d^2}\sum_{i=1}^{n}\left(x_i^0 - x_i^0\right)^2 = M,$$

that is, v, which verifies the same conditions of regularity as u, reaches its biggest value also in the point (t_0, x^0) as the function u. Because values of v on Γ are strictly less than M, we deduce that there is a point (t_1, x^1) inside the parabolic border so that

$$\sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x) = v(t_1, x^1),$$

while v cannot reach its supremum value on Γ. We write the conditions of the extremum for $v(t,x)$ in the point (t_1, x^1)

$$\left.\frac{\partial v(t,x)}{\partial t}\right|_{(t_1,x^1)} \geq 0. \tag{6.1.11}$$

If $t_1 \in (0, T)$ then we have equality in (6.1.11) and we obtain the condition of Fermat. If $t_1 = T$, then the values on the right-hand side of T do not exist and then a point of extremum in t_1 means that on the left-hand side of T, the function v is positive and increasing. On the other hand, the function $v(t_1, x)$, considered only as a function of the n spatial variables $(x_1, x_2, \dots, x_n)$, reaches its supremum on $\overline{\Omega}$ in the point $(x_1^1, x_2^1, \dots, x_n^1)$, and then we have the necessary condition of the maximum:

$$\left.\frac{\partial^2 v(t,x)}{\partial x_i^2}\right|_{(t_1,x^1)} \leq 0,\ i = 1, 2, \dots, n,$$

from where we obtain

$$\Delta v(t_1, x^1) \leq 0. \tag{6.1.12}$$

From (6.1.11) and (6.1.12), we obtain

$$(-v_t(t,x) + \Delta v(t,x))_{(t_1,x^1)} \leq 0. \tag{6.1.13}$$

Starting from the form (6.1.9) of the function v, we obtain

$$\begin{aligned}(-v_t(t,x) + \Delta v(t,x))_{(t_1,x^1)} &= (-u_t(t,x) + \Delta u(t,x))_{(t_1,x^1)} \\ &+ \frac{(M-m)n}{d^2} = \frac{(M-m)n}{d^2} > 0,\end{aligned}$$

in which we take into account (6.1.8).

This inequality is in contradiction with the inequality (6.1.13), and this proves that the assumption (6.1.8) is false and the theorem is proven. ■

As an immediate consequence of the theorem of extreme values, we will prove the uniqueness of the classical solution for the initial-boundary value problem that consists of (6.1.1), (6.1.2), (6.1.3).

Theorem 6.1.2 *The problem consisting of Eq. (6.1.1), the initial condition (6.1.2), and the boundary condition (6.1.3) has at most a classical solution.*

Proof Suppose that the problem (6.1.1), (6.1.2), (6.1.3) admits two classical solutions $u_1(t,x)$ and $u_2(t,x)$. Then we have

$$\begin{aligned}&\Delta u_i(t,x) - \frac{\partial u_i}{\partial t}(t,x) = f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \Omega,\\&u_i(0,x) = \varphi(x),\ \forall x \in \overline{\Omega},\\&u_i(t,y) = \alpha(t,y),\ \forall (t,y) \in \overline{\mathcal{T}}_T \times \partial\Omega,\end{aligned} \tag{6.1.14}$$

where $i = 1, 2$ and the functions f, φ, and α are given and are continuous on their domain of definition.

On the other hand, u_1 and u_2 satisfy the conditions for classical solutions. We define the function $v(t,x)$ by

$$v(t,x) = u_1(t,x) - u_2(t,x),\ \forall (t,x) \in \overline{\mathcal{T}}_T \times \overline{\Omega}.$$

Taking into account the considerations above, we obtain that v satisfies the conditions of regularity of a classical solution and, in addition, verifies the problem

$$\begin{aligned}&\Delta v(t,x) - \frac{\partial v}{\partial t}(t,x) = 0,\ \forall (t,x) \in \mathcal{T}_T \times \Omega,\\&v(0,x) = 0,\ \forall x \in \overline{\Omega},\\&v(t,y) = 0,\ \forall (t,y) \in \overline{\mathcal{T}}_T \times \partial\Omega.\end{aligned} \tag{6.1.15}$$

The function v satisfies the conditions of Theorem 6.1.1. Then its extreme values

$$\sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x), \qquad \inf_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x)$$

are reached necessarily on Γ. According to $(6.1.15)_2$ and $(6.1.15)_3$, v becomes null on the parabolic boundary and then

$$\sup_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x) = \inf_{(t,x)\in\overline{\mathcal{T}}_T\times\overline{\Omega}} v(t,x) = 0,$$

that is, we have $v(t,x) = 0$, $\forall (t,x) \in \overline{\mathcal{T}}_T \times \overline{\Omega}$ and from here we deduce that $u_1(t,x) \equiv u_2(t,x)$, so that the proof for Theorem 6.1.2 is concluded. ∎

Also, as an application of the theorem of extreme values, we will prove in next theorem, a result of stability with respect to the initial conditions and the boundary conditions, for the problem (6.1.1), (6.1.2), (6.1.3).

Theorem 6.1.3 *Suppose that the function $f(t,x)$ is given and continuous on $\mathcal{T}_T \times \Omega$. Consider also the functions $\varphi_1(t,x)$ and $\varphi_2(t,x)$ which are given and continuous* *on* $\overline{\Omega}$ *and the functions* $\alpha_1(t, x)$ *and* $\alpha_2(t, x)$ *which are given and continuous on* $\overline{\mathcal{T}_T} \times \partial\Omega$. *We attach the problems*

$$\Delta u_i(t, x) - \frac{\partial u_i}{\partial t}(t, x) = f(t, x), \ \forall (t, x) \in \mathcal{T}_T \times \Omega,$$
$$u_i(0, x) = \varphi_i(x), \ \forall x \in \overline{\Omega},$$
$$u_i(t, y) = \alpha_i(t, y), \ \forall (t, y) \in \overline{\mathcal{T}}_T \times \partial\Omega,$$

where $i = 1, 2$.
If $\forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon)$ *such that*

$$|\varphi(x)| = |\varphi_1(x) - \varphi_2(x)| < \delta,$$
$$|\alpha(x)| = |\alpha_1(x) - \alpha_2(x)| < \delta,$$

then

$$|u(x)| = |u_1(x) - u_2(x)| < \varepsilon.$$

Proof The function $u(t, x)$ defined as in the statement by

$$u(t, x) = u_1(t, x) - u_2(t, x),$$

satisfies the conditions of a classical solution. Also, u satisfies the problem

$$\begin{aligned} &\Delta u(t, x) - \frac{\partial u}{\partial t}(t, x) = f(t, x) - f(t, x) = 0, \\ &u(0, x) = u_1(0, x) - u_2(0, x) = \varphi_1(x) - \varphi_2(x) = \varphi(x), \\ &u(t, y) = u_1(t, y) - u_2(t, y) = \alpha_1(t, y) - \alpha_2(t, y) = \alpha(t, y). \end{aligned} \tag{6.1.16}$$

Because the function u satisfies the mentioned conditions of regularity and homogeneous equation $(6.1.16)_1$, we deduce that we are in the conditions of the theorem of the extreme values. Then the extreme values of the function u are reached on the parabolic boundary Γ. But on Γ the function u is reduced to φ or to α and because also φ and α satisfy the conditions $|\varphi| < \delta$, $|\alpha| < \delta$, we obtain the result of the theorem by taking $\delta = \varepsilon$. ∎

A *particular solution* of the problem (6.1.1), (6.1.2), (6.1.3) is the solution obtained by fixing the right-hand side f, the initial data φ, and the boundary data α.

The family of all particular solutions obtained by varying the functions f, φ, and α in the class of continuous functions is *the general solution* of the problem (6.1.1), (6.1.2), (6.1.3).

We prove now that a particular solution for the homogeneous equation of heat is the function V defined by

$$V(t,\tau,x,\xi)=\frac{1}{\left(2\sqrt{\pi}\right)^n\left(\sqrt{t-\tau}\right)^n}\exp\left(-\frac{\sum\limits_{i=1}^n(x_i-\xi_i)^2}{4(t-\tau)}\right). \tag{6.1.17}$$

Proposition 6.1.1 *The function $V(t,\tau,x,\xi)$, for $0\le\tau<t\le T$, is of class C^∞ and satisfies the following equations:*

$$\Delta_x V(t,\tau,x,\xi)-\frac{\partial V(t,\tau,x,\xi)}{\partial t}=0,$$
$$\Delta_\xi V(t,\tau,x,\xi)+\frac{\partial V(t,\tau,x,\xi)}{\partial \tau}=0.$$

Proof Through a simple calculation of the derivatives, we obtain

$$\frac{\partial V(t,\tau,x,\xi)}{\partial x_i}=V(t,\tau,x,\xi)\left(-\frac{(x_i-\xi_i)}{2(t-\tau)}\right)=-\frac{\partial V(t,\tau,x,\xi)}{\partial \xi_i}.$$

Then

$$\frac{\partial^2 V(t,\tau,x,\xi)}{\partial x_i^2}=V(t,\tau,x,\xi)\left(\frac{(x_i-\xi_i)^2}{4(t-\tau)^2}-\frac{1}{2(t-\tau)}\right)=\frac{\partial^2 V(t,\tau,x,\xi)}{\partial \xi_i^2},$$

so that by summing up, for $i=1,2,\ldots,n$, we are led to

$$\Delta_x V(t,\tau,x,\xi)=V(t,\tau,x,\xi)\left(\frac{1}{4(t-\tau)^2}\sum_{i=1}^n(x_i-\xi_i)^2-\frac{n}{2(t-\tau)}\right)$$
$$=\Delta_\xi V(t,\tau,x,\xi).$$

On the other hand, by differentiating in (6.1.17) with respect to t and τ, we obtain

$$\frac{\partial V(t,\tau,x,\xi)}{\partial t}=V(t,\tau,x,\xi)\left(\frac{1}{4(t-\tau)^2}\sum_{i=1}^n(x_i-\xi_i)^2-\frac{n}{2(t-\tau)}\right)$$
$$=-\frac{\partial V(t,\tau,x,\xi)}{\partial \tau}.$$

Then, the results formulated in the statement of the proposition are immediately obtained. The fact that the function $V(t,\tau,x,\xi)$ is of class C^∞ is motivated by the fact that $t\neq\tau$ and, essentially, the function $V(t,\tau,x,\xi)$ is an exponential function. ■

Observation 6.1.1 *It is easy to verify the fact that if $x\neq\xi$, then the function $V(t,\tau,x,\xi)$ is dominated by an exponential function and*

$$\lim_{t-\tau\to 0^+} V(t,\tau,x,\xi)=0,$$

and if $x=\xi$, *the exponential function disappears and*

$$\lim_{t-\tau\to 0^+} V(t,\tau,x,\xi)=+\infty.$$

Another important property of the function $V(t,\tau,x,\xi)$ is proven in the following theorem.

Theorem 6.1.4 *The following equalities are true:*

$$\int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}x=1,\ \int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi=1.$$

Proof We write in extenso the integral of the volume as

$$\int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi=$$

$$=\frac{1}{\left(2\sqrt{\pi}\right)^n}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}\frac{1}{\left(\sqrt{t-\tau}\right)^n}\exp\left(-\frac{\sum\limits_{i=1}^{n}(x_i-\xi_i)^2}{4(t-\tau)}\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\dots\mathrm{d}\xi_n.$$

We make the change of variables $\xi_i-x_i=2\sqrt{t-\tau}\eta_i$ and by direct calculations, we obtain that the Jacobean of the change of variables has the value

$$\left|\frac{D\xi}{D\eta}\right|=2^n\left(\sqrt{t-\tau}\right)^n.$$

then

$$\begin{aligned}\int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi&=\frac{1}{\left(\sqrt{\pi}\right)^n}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}e^{-\sum\limits_{i=1}^{n}\eta_i^2}\mathrm{d}\eta_1\mathrm{d}\eta_2\dots\mathrm{d}\eta_n\\&=\frac{1}{\left(\sqrt{\pi}\right)^n}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}e^{-\eta_1^2}e^{-\eta_2^2}\dots e^{-\eta_n^2}\mathrm{d}\eta_1\mathrm{d}\eta_2\dots\mathrm{d}\eta_n\\&=\frac{1}{\left(\sqrt{\pi}\right)^n}\left(\int_{-\infty}^{+\infty}e^{-s^2}ds\right)^n=\frac{1}{\left(\sqrt{\pi}\right)^n}\left(\sqrt{\pi}\right)^n=1,\end{aligned}$$

in which we used the integral of Gauss

$$\int_{-\infty}^{+\infty}e^{-s^2}\mathrm{d}s=\sqrt{\pi}.$$

The other equality from the statement is proven analogously. ■

We now prove a result which generalizes the results from Theorem 6.1.4.

Theorem 6.1.5 *Let Ω be a bounded domain. If we denote by I_Ω the integral*

$$I_\Omega(t-\tau, x) = \int_\Omega V(t, \tau, x, \xi)\mathrm{d}\xi,$$

then for $x \in \Omega$, we have

$$\lim_{t-\tau \to 0^+} I_\Omega(t-\tau, x) = 1,$$

the limit taking place uniformly with respect to x, on compact sets from Ω, and for $x \in \mathbb{R}^n \setminus \overline{\Omega}$

$$\lim_{t-\tau \to 0^+} I_\Omega(t-\tau, x) = 0,$$

the limit taking place uniformly with respect to x, on compact sets from $\mathbb{R}^n \setminus \overline{\Omega}$.

Proof We prove, first, the case when $x \in \Omega$. We use the notations

$$d_0 = \mathrm{dist}(x, \Omega),\ \ d_1 = \mathrm{dist}(Q, \partial\Omega),$$

where Q is a compact set arbitrarily fixed in Ω so that $x \in Q$.

We recall that, by definition, that we have

$$\begin{aligned} d_0 &= \mathrm{dist}(x, \partial\Omega) = \sup_{y \in \partial\Omega} |x-y|, \\ d_1 &= \mathrm{dist}(Q, \partial\Omega) = \sup_{y \in \partial\Omega, x \in Q} |x-y|. \end{aligned}$$

Consider balls $B(x, d_0)$ and $B(x, d_1)$ and then

$$B(x, d_1) \subset B(x, d_0) \subset \Omega. \tag{6.1.18}$$

Now we use the monotony of the integral, and by taking into account the inclusion (6.1.8), we deduce that

$$\begin{aligned} I_\Omega(t-\tau, x) &= \int_\Omega V(t, \tau, x, \xi)\mathrm{d}\xi \\ &\geq \int_{B(x,d_0)} V(t, \tau, x, \xi)\mathrm{d}\xi \geq \int_{B(x,d_1)} V(t, \tau, x, \xi)\mathrm{d}\xi \end{aligned} \tag{6.1.19}$$

$$= \frac{1}{\left(2\sqrt{\pi}\right)^n \left(\sqrt{t-\tau}\right)^n} \int_{B(x,d_1)} \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\xi_i)^2}{4(t-\tau)} \right) \mathrm{d}\xi.$$

We make the change of variables $\xi_i - x_i = 2\sqrt{t-\tau}\eta_i$, with $i = 1, 2, \ldots, n$.

As in the proof of Theorem 6.1.4, the value of the Jacobean of this change of variables is $2^n \left(\sqrt{t-\tau}\right)^n$. With this change of variables, the last integral from (6.1.19) becomes

$$\frac{1}{\left(\sqrt{\pi}\right)^n} \int_{B(0,\frac{d}{2\sqrt{t-\tau}})} e^{-\sum_{i=1}^{n}\eta_i^2} \mathrm{d}\eta, \tag{6.1.20}$$

in which

$$\sqrt{\sum_{i=1}^{n}(\xi_i - x_i)^2} = 2\sqrt{t-\tau}\sqrt{\sum_{i=1}^{n}\eta_i^2}.$$

If we pass to the limit in (6.1.19) with $t-\tau \to 0^+$ and we take into account (6.1.20), we obtain

$$\begin{aligned} \lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \; &\geq \lim_{t-\tau\to 0^+} \frac{1}{\left(\sqrt{\pi}\right)^n} \int_{B(0,\frac{d}{2\sqrt{t-\tau}})} e^{-\sum_{i=1}^{n}\eta_i^2} \mathrm{d}\eta \\ &= \frac{1}{\left(\sqrt{\pi}\right)^n} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{n}\eta_i^2} \mathrm{d}\eta = 1, \end{aligned}$$

in which we used the integral of Gauss. We have taken into account also the fact that for $t-\tau \to 0^+$, we have

$$\frac{d}{2\sqrt{t-\tau}} \to \infty$$

and then the ball $B(0, \frac{d}{2\sqrt{t-\tau}})$ becomes the whole space $\mathbb{R}^n$.

Thus, we proved that

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \geq 1. \tag{6.1.21}$$

Since $\Omega \subset \mathbb{R}^n$, we have, obviously, that

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \leq \int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi = 1,$$

and then

$$\lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \le 1. \tag{6.1.22}$$

From (6.1.21) and (6.1.22), the first part of the proof is complete.

The limit holds true uniformly with respect to x, on compact sets from Ω that contain x, because d used in the considerations above depends only on the compact set that contains x, and does not depend on the choice of x in the respective compact.

We now approach the case when $x \in \mathbb{R}^n \setminus \overline{\Omega}$. Having in mind that Ω was assumed to be a (bounded) domain, with the help of the theorem of Jordan, we deduce that $\mathbb{R}^n \setminus \overline{\Omega}$ is also a domain. We take thus, a compact set $Q^* \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that $x \in Q^*$ and consider the distances $d_0^* = \text{dist}(x, \partial\Omega)$, $d_1^* = \text{dist}(Q^*, \partial\Omega)$ and the balls $B(x, d_0^*)$ and $B(x, d_0^*)$. Because $d_0^* > d_1^*$, we deduce that

$$\begin{aligned} &B(x, d_1^*) \subset B(x, d_0^*) \Rightarrow \\ &\Rightarrow \Omega \subset \mathbb{R}^n \setminus B(x, d_0^*) \subset \mathbb{R}^n \setminus B(x, d_1^*). \end{aligned}$$

Accordingly, for $I_\Omega(t-\tau, x)$ we have the evaluations

$$\begin{aligned} 0 \le I_\Omega(t-\tau, x) &= \int_\Omega V(t, \tau, x, \xi)\mathrm{d}\xi \\ \le \int_{\mathbb{R}^n \setminus B(x, d_0^*)} V(t, \tau, x, \xi)\mathrm{d}\xi &\le \int_{\mathbb{R}^n \setminus B(x, d_1^*)} V(t, \tau, x, \xi)\mathrm{d}\xi. \end{aligned} \tag{6.1.23}$$

We make the change of variables $\xi_i - x_i = 2\sqrt{t-\tau}\eta_i$, for $i = 1, 2, \ldots, n$. Based on the considerations from the first part of the proof, the last integral from (6.1.23) becomes

$$\frac{1}{\left(\sqrt{\pi}\right)^n} \int_{\mathcal{D}} e^{-\sum\limits_{i=1}^n \eta_i^2} \mathrm{d}\eta, \tag{6.1.24}$$

where the domain of the integral $\mathcal{D}$ is $\mathcal{D} = \mathbb{R}^n \setminus B(0, \frac{d^*}{2\sqrt{t-\tau}})$. If we pass to the limit with $t - \tau \to 0^+$, the radius $\frac{d^*}{2\sqrt{t-\tau}}$ becomes infinite and then the ball $B(0, \frac{d^*}{2\sqrt{t-\tau}})$ becomes the whole space $\mathbb{R}^n$. Then the integral from (6.1.24) tends to zero and, by returning in (6.1.23), we deduce that

$$0 \le \lim_{t-\tau\to 0^+} I_\Omega(t-\tau, x) \le \int_{\mathcal{D}} e^{-\sum\limits_{i=1}^n \eta_i^2} \mathrm{d}\eta = 0,$$

in which the domain of integration $\mathcal{D}$ is defined as above.

The limit holds true uniformly with respect to x, on compact sets from Ω that contain x, because d^* used in the considerations above depends only on the compact set that contains x, and does not depend on the choice of x in the respective compact set. ■

The usefulness of the results proven in Theorems 6.1.4 and 6.1.5 is certified in the following theorem.

Theorem 6.1.6 *Let Ω be a bounded domain from $\mathbb{R}^n$ and suppose that the function f is continuous and bounded on Ω.*

Then we have the following two alternatives:

• *(i). If $x \in \Omega$, then*

$$\lim_{t-\tau\to 0^+} \int_\Omega V(t,\tau,x,\xi) f(\xi) \mathrm{d}\xi = f(x),$$

the limit taking place uniformly with respect to x, on compact sets from Ω.

• *(ii). If $x \in \mathbb{R}^n \setminus \Omega$, then*

$$\lim_{t-\tau\to 0^+} \int_\Omega V(t,\tau,x,\xi) f(\xi) \mathrm{d}\xi = 0,$$

the limit taking place uniformly with respect to x, on compact sets from $\mathbb{R}^n \setminus \Omega$.

Proof (i). Let Q be a compact set arbitrarily fixed, $Q \subset \Omega$, such that $x \in Q$. We have the evaluations

$$\begin{aligned} \left| \int_\Omega V(t,\tau,x,\xi) f(\xi) \mathrm{d}\xi - f(x) \right| &\le \left| \int_\Omega V(t,\tau,x,\xi) f(\xi) \mathrm{d}\xi \right. \\ &\left. -f(x) \int_\Omega V(t,\tau,x,\xi) f(\xi) \mathrm{d}\xi \right| + \left| f(x) \int_\Omega V(t,\tau,x,\xi) \mathrm{d}\xi - f(x) \right| \\ \le \int_\Omega V(t,\tau,x,\xi) |f(x) - f(\xi)| \mathrm{d}\xi &+ |f(x)| \left| \int_\Omega V(t,\tau,x,\xi) \mathrm{d}\xi - 1 \right| \qquad (6.1.25) \\ \le \int_{B(x,\delta)} V(t,\tau,x,\xi) |f(x) - f(\xi)| \mathrm{d}\xi & \int_D V(t,\tau,x,\xi) |f(x) - f(\xi)| \mathrm{d}\xi \\ &+ c_0 \left| \int_\Omega V(t,\tau,x,\xi) \mathrm{d}\xi - 1 \right|, \end{aligned}$$

in which the domain D is $D = \mathbb{R}^n \setminus B(x,\delta)$ and we denote by c_0 the constant given by $c_0 = \sup_{x\in\Omega} |f(x)|$.

To use the continuity of the function f, we take an arbitrarily small ε and then there is $\eta(\varepsilon)$ so that if $|x - \xi| < \eta(\varepsilon) \Rightarrow |f(x) - f(\xi)| < \varepsilon$.

If in the evaluations from (6.1.25), we take $\delta < \eta(\varepsilon)$ we deduce that

$$\int_{B(x,\delta)} V(t,\tau,x,\xi)|f(x)-f(\xi)|\mathrm{d}\xi < \varepsilon \int_{B(x,\delta)} V(t,\tau,x,\xi)\mathrm{d}\xi$$
$$\leq \varepsilon \int_{\mathbb{R}^n} V(t,\tau,x,\xi)\mathrm{d}\xi = \varepsilon.$$

Then

$$\int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)|f(x)-f(\xi)|\mathrm{d}\xi < 2c_0 \int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)\mathrm{d}\xi,$$

and

$$\lim_{t-\tau\to 0^+} \int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi = 0,$$

because $x \notin \Omega \setminus B(x,\delta)$. Then we can use the second part from Theorem 6.1.5 to obtain that the limit is null.

Finally, for the last integral from (6.1.25), we have

$$\lim_{t-\tau\to 0^+} \left| \int_{\Omega\setminus B(x,\delta)} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi - 1 \right| = 0,$$

because $x \in \Omega$ and then we can use the first part from Theorem 6.1.5. If we take into account these evaluations in (6.1.25), we obtain that the point (i) is proven. We must mention that the limit from (i) holds true uniformly with respect to x because the last integrals from (6.1.25) are convergent to zero, uniformly on compact sets from Ω.

(ii). We take an arbitrary compact set Q^* so that $x \in Q^*$ and $Q^* \subset \mathbb{R}^n \setminus \overline{\Omega}$. Because our hypothesis is that f is a bounded function, we have

$$\left|\int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi\right| \leq \int_{\Omega} |V(t,\tau,x,\xi)|\,|f(\xi)|\,\mathrm{d}\xi$$
$$\leq c_0 \int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi,$$

and then

$$0 \leq \lim_{t-\tau\to 0^+} \left|\int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi\right| \leq c_0 \int_{\Omega} V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi.$$

Because $x \notin \Omega$, based on the second part from Theorem 6.1.5, these inequalities lead to the conclusion that

$$\lim_{t-\tau\to 0^+} \int_{\Omega} V(t,\tau,x,\xi)\mathrm{d}\xi = 0 \tag{6.1.26}$$

and then

$$\lim_{t-\tau\to 0^+}\left|\int_\Omega V(t,\tau,x,\xi)f(\xi)\mathrm{d}\xi\right| = 0,$$

the limit taking place uniformly with respect to x, on compact sets from $\mathbb{R}^n \setminus \overline{\Omega}$, because the limit from (6.1.26) was obtained in the same way. ■

In the following theorem, we generalize the results from Theorem 6.1.6.

Theorem 6.1.7 *Let us consider the function $g(\tau,\xi)$ supposed to be continuous and bounded on $\mathcal{T}_T \times \Omega$. If, in addition,*

$$\lim_{\tau\to t^+} g(\tau,\xi) = g(t,\xi),$$

and the limit takes place uniformly with respect to ξ, on compact sets from Ω, then

- *(i). If $x \in \Omega$, then*

$$\lim_{\tau\to t^-}\int_\Omega V(t,\tau,x,\xi)g(\tau,\xi)\mathrm{d}\xi = g(t,x),$$

  and the limit takes place uniformly with respect to x, on compact sets from Ω.
- *(ii). If $x \in \mathbb{R}^n \setminus \Omega$, then*

$$\lim_{\tau\to t^-}\int_\Omega V(t,\tau,x,\xi)g(\tau,\xi)\mathrm{d}\xi = 0,$$

and the limit takes place uniformly with respect to x, on compact sets from $\mathbb{R}^n \setminus \Omega$.

Proof (i). Let Q be an arbitrarily fixed compact, $Q \subset \Omega$, so that $x \in Q \subset \Omega$. Then

$$\begin{aligned}&\left|\int_\Omega V(t,\tau,x,\xi)g(\tau,\xi)\mathrm{d}\xi - g(t,x)\right|\\ &\le \left|\int_\Omega V(t,\tau,x,\xi)[g(\tau,\xi)-g(t,\xi)]\mathrm{d}\xi\right| + \left|\int_\Omega V(t,\tau,x,\xi)\mathrm{d}\xi - g(t,x)\right|.\end{aligned} \tag{6.1.27}$$

If in (6.1.27) we pass to the limit with $\tau \to t^-$, then the first integral from the right-hand side tends to zero, based on the assumption, and the last integral from (6.1.27) tends to zero based on Theorem 6.1.6. Also, we deduce that both limits are satisfied uniformly with respect to x, on compact sets from Ω, based on the assumption and on the fact that the result from Theorem 6.1.6 was obtained in the same way.

(ii). The result is obtained in a similar manner. ■

6.2 The Method of Green's Function

We will obtain, first, Green's formula for the equation of heat. To this end we define the operators $\mathcal{L}_{(\tau,\xi)}$ and $\mathcal{M}_{(\tau,\xi)}$ by

$$\mathcal{L}_{(\tau,\xi)}u = \Delta_\xi u - \frac{\partial u}{\partial \tau},$$
$$\mathcal{M}_{(\tau,\xi)}v = \Delta_\xi v + \frac{\partial v}{\partial \tau}. \tag{6.2.1}$$

Let Ω be a bounded domain whose boundary $\partial\Omega$ has a tangent plane continuously varying almost everywhere.

In the following, we will use the function $u(t, x)$ which satisfies the following standard hypotheses:

- $u \in C(\overline{\mathcal{T}_T} \times \overline{\Omega})$;
- $u_{x_i x_i},\ u_t \in C(\mathcal{T}_T \times \Omega)$, for $0 \leq \tau < t \leq T$.

If we multiply (6.2.1)$_1$ with $v(\tau, \xi)$ and (6.2.1)$_2$ with $u(\tau, \xi)$, we obtain

$$v\mathcal{L}u - u\mathcal{M}v = v\Delta_\xi u - u\Delta_\xi v - v\frac{\partial u}{\partial \tau} - u\frac{\partial v}{\partial \tau}$$

that is

$$v\mathcal{L}u - u\mathcal{M}v = v\Delta_\xi u - u\Delta_\xi v - \frac{\partial}{\partial \tau}(uv). \tag{6.2.2}$$

Proposition 6.2.1 *Assume that the above hypotheses are satisfied on the domain Ω and for the function u. If the function v satisfies the hypotheses of the function u, then the following Green's formula holds true:*

$$\int_\Omega \int_0^t [v\mathcal{L}u - u\mathcal{M}v]\mathrm{d}\tau\mathrm{d}\xi = \int_{\partial\Omega} \int_0^t \left[v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right]\tau\mathrm{d}\sigma_\xi$$
$$- \int_\Omega u(t,\xi)v(t,\xi)\mathrm{d}\xi + \int_\Omega u(0,\xi)v(0,\xi)\mathrm{d}\xi. \tag{6.2.3}$$

Proof By integrating on the set $\Omega \times [0, t]$ the equality (6.2.2), we get

$$\int_\Omega \int_0^t [v\mathcal{L}u - u\mathcal{M}v]\mathrm{d}\tau\mathrm{d}\xi = \int_\Omega \int_0^t [v\Delta_\xi u - u\Delta_\xi v]\mathrm{d}\tau\mathrm{d}\xi$$

$$- \int_\Omega \int_0^t \frac{\partial}{\partial \tau}(uv)\,\mathrm{d}\tau\mathrm{d}\xi. \tag{6.2.4}$$

By using the Gauss–Ostrogradsky formula, we deduce that

$$\int_{\Omega}\int_0^t v\Delta_\xi u \mathrm{d}\tau\mathrm{d}\xi = \int_{\Omega}\int_0^t v\sum_{i=1}^n \frac{\partial^2 u}{\partial\xi_i^2}\mathrm{d}\tau\mathrm{d}\xi$$
$$= \int_{\Omega}\int_0^t \sum_{i=1}^n v\frac{\partial}{\partial\xi_i}\left(\frac{\partial u}{\partial\xi_i}\right)\mathrm{d}\tau\mathrm{d}\xi = \int_{\partial\Omega}\int_0^t v\sum_{i=1}^n \frac{\partial u}{\partial\xi_i}\cos\alpha_i\mathrm{d}\tau\mathrm{d}\sigma_\xi$$
$$= \int_{\partial\Omega}\int_0^t v\frac{\partial u}{\partial\nu_\xi}\mathrm{d}\tau\mathrm{d}\sigma_\xi,$$

where ν is the outside unit normal to the surface $\partial\Omega$.

The following equality is obtained analogously:

$$\int_{\Omega}\int_0^t u\Delta_\xi v\mathrm{d}\tau\mathrm{d}\xi = \int_{\partial\Omega}\int_0^t u\frac{\partial v}{\partial\nu_\xi}\mathrm{d}\tau\mathrm{d}\sigma_\xi.$$

Then

$$\int_{\Omega}\int_0^t \frac{\partial}{\partial\tau}(uv)\,\mathrm{d}\tau\mathrm{d}\xi = \int_{\partial\Omega} uv|_0^t\,\mathrm{d}\xi$$
$$= \int_{\partial\Omega}[u(t,\xi)v(t,\xi) - u(0,\xi)v(0,\xi)]\mathrm{d}\xi.$$

If we use these evaluations in (6.2.4), we obtain Green's formula. ■

Green's formula (6.2.3) can be generalized in the sense that, in the form (6.2.1) of the operators $\mathcal{L}$ and $\mathcal{M}$, instead of the Laplacian Δ we can take an arbitrary operator, which is a second-order linear operator.

We define thus, the operator L and its adjoint M by

$$Lu = \sum_{i=1}^n\sum_{j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$
$$Mv = \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2(a_{ij}(x)v)}{\partial x_i\partial x_j} - \sum_{i=1}^n \frac{\partial(b_i(x)v)}{\partial x_i} + c(x)u, \tag{6.2.5}$$

in which $a_{ij} = a_{ji} \in C^2(\Omega),\ b_i \in C^1(\Omega)$, and $c \in C^0(\Omega)$.

By analogy with (6.2.1), we will build the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A}u = Lu - \frac{\partial u}{\partial t},$$
$$\mathcal{B}v = Mv + \frac{\partial v}{\partial t}. \tag{6.2.6}$$

Proposition 6.2.2 *Assume that the hypotheses from Proposition 6.2.1 are satisfied on the domain Ω and for the functions u and v. In addition, suppose that L is an elliptic operator. Then the following Green's formula holds true:*

$$\int_\Omega \int_0^t [v\mathcal{A}u - u\mathcal{B}v] \mathrm{d}\tau \mathrm{d}\xi = \int_{\partial\Omega} \int_0^t \left\{ \gamma \left[v\frac{\partial u}{\partial \tau} - u\frac{\partial v}{\partial \tau} \right] + buv \right\} \mathrm{d}\tau \mathrm{d}\sigma_\xi$$
$$- \int_\Omega u(t, \xi)v(t, \xi)\mathrm{d}\xi + \int_\Omega u(0, \xi)v(0, \xi)\mathrm{d}\xi. \qquad (6.2.7)$$

Proof We multiply $(6.2.6)_1$ with v and $(6.2.6)_2$ with u and the obtained relations are subtracted member by member. We obtain the equality

$$\begin{aligned} v\mathcal{A}u - u\mathcal{B}v &= vLu - uMv - v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t} \\ &= vLu - uMv - \frac{\partial}{\partial t}(uv). \end{aligned}$$

We integrate this equality on the set $\Omega \times [0, t]$ and after we use the Gauss–Ostrogradsky formula, we easily deduce the formula of Green (6.2.7). ∎

We now consider again the operators $\mathcal{L}$ and $\mathcal{M}$ defined in (6.2.1). Accordingly, we will use Green's formula in the form (6.2.3). Starting from this form of Green's formula, we want to find form of the Riemann–Green's formula. For this, we use again the function $V(t, \tau, x, \xi)$ defined by

$$V(t, \tau, x, \xi) = \frac{1}{\left(2\sqrt{\pi}\right)^n \left(\sqrt{t-\tau}\right)^n} \exp\left(-\frac{\sum_{i=1}^n (x_i - \xi_i)^2}{4(t-\tau)} \right). \qquad (6.2.8)$$

The unique singularity of the function $V(t, \tau, x, \xi)$ is reached for $(t, x) = (\tau, \xi)$. To avoid this singularity, we will consider the domain

$$\{\tau;\ 0 \le \tau \le t - \delta,\ \delta > 0\} \times \Omega.$$

On this domain, we write Green's formula (6.2.3) for the pair of the functions (v, u), where $v = V(t, \tau, x, \xi)$ and $u = u(\tau, \xi)$

$$\begin{aligned} &\int_\Omega \int_0^{t-\delta} [V(t, \tau, x, \xi)\mathcal{L}u(\tau, \xi) - u(\tau, \xi)\mathcal{M}V(t, \tau, x, \xi)] \,\mathrm{d}\tau \mathrm{d}\xi \\ &= \int_{\partial\Omega} \int_0^{t-\delta} \left[V(t, \tau, x, \xi)\frac{\partial u}{\partial \nu}(\tau, \xi) - u(\tau, \xi)\frac{\partial V(t, \tau, x, \xi)}{\partial \nu} \right] \mathrm{d}\tau \mathrm{d}\sigma_\xi \qquad (6.2.9) \\ &- \int_\Omega V(t, t-\tau, x, \xi)u(t-\tau, \xi)\mathrm{d}\xi + \int_\Omega V(t, 0, x, \xi)u(0, \xi)\mathrm{d}\xi. \end{aligned}$$

In this equality, we pass to the limit with $\delta \to 0$ and we use Theorem 6.1.7 from Sect. 6.1. Thus, if $x \in \Omega$, we deduce that

$$u\ (t,x) = -\int_{\Omega}\int_0^t V(t,\tau,x,\xi)\mathcal{L}u(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi + \int_{\partial\Omega}\int_0^t \left[V(t,\tau,x,\xi)\frac{\partial u}{\partial\nu}(\tau,\xi) - u(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi + \int_{\Omega} V(t,0,x,\xi)u(0,\xi)\mathrm{d}\xi. \tag{6.2.10}$$

The result proven here can be summarized as in the following theorem.

Theorem 6.2.1 *For the equation of heat, the Riemann–Green's formula has the form (6.2.10), in which the operators $\mathcal{L}$ and $\mathcal{M}$ are defined in (6.2.1) and the function $V(t,\tau,x,\xi)$ has the form (6.2.8).*

Observation 6.2.1 *If $x \in \mathbb{R}^n \setminus \overline{\Omega}$ then passing to the limit in (6.2.9) with $\delta \to 0$ and using the second part of Theorem 6.1.7 (from Sect. 6.1), we deduce that*

$$0 = -\int_{\Omega}\int_0^t V(t,\tau,x,\xi)\mathcal{L}u(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi + \int_{\partial\Omega}\int_0^t \left[V(t,\tau,x,\xi)\frac{\partial u}{\partial\nu}(\tau,\xi) - u(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi + \int_{\Omega} V(t,0,x,\xi)u(0,\xi)\mathrm{d}\xi.$$

Consider now the following initial-boundary value problem:

$$\begin{aligned}
&\mathcal{L}u(t,x) = f(t,x),\ \forall(t,x) \in \mathcal{T}_T \times \Omega,\\
&u(0,x) = \varphi(x),\ \forall x \in \overline{\Omega},\\
&u(t,y) = \alpha(t,y),\ \forall(t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega,\\
&\frac{\partial u}{\partial\nu}u(t,y) = \beta(t,y),\ \forall(t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega.
\end{aligned}$$

Then Riemann–Green's formula receives the form

$$u(t,x) = -\int_{\Omega}\int_0^t V(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi + \int_{\partial\Omega}\int_0^t V(t,\tau,x,\xi)\beta(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi - \int_{\partial\Omega}\int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\alpha(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi + \int_{\Omega} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi. \tag{6.2.11}$$

The integrals from the right-hand side of the formula (6.2.11) are the potentials associated to the problem of heat, namely,

- the thermal potential of volume

$$I_1 = -\int_{\Omega}\int_0^t V(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi;$$

- the thermal potential of the surface of a single layer

$$I_2 = \int_{\partial\Omega}\int_0^t V(t,\tau,x,\xi)\beta(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi;$$

- the thermal potential of the surface of a double layer

$$I_3 = -\int_{\partial\Omega}\int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\alpha(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi;$$

- the temporal thermal potential

$$I_4 = \int_{\Omega} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi.$$

Thus, formula (6.2.11) is also called the formula of the thermal potentials. As in the case of elliptic equations, the thermal potentials are used to solve the initial-boundary value problems, in the context of parabolic equations. More precisely, the thermal potentials allow the transformation of these problems in integral equations of Fredholm type.

Consider the Dirichlet problem

$$\begin{aligned}
&\Delta_\xi u(\tau,\xi) - \frac{\partial u}{\partial\tau}(\tau,\xi) = f(\tau,\xi),\ \forall(\tau,\xi)\in\mathcal{T}_T\times\Omega,\\
&u(0,\xi) = \varphi(\xi),\ \forall\xi\in\overline{\Omega},\\
&u(\tau,\eta) = \alpha(\tau,\eta),\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega,
\end{aligned} \tag{6.2.12}$$

where Ω is a bounded domain with boundary $\partial\Omega$ having a tangent plane continuously varying almost everywhere. We denote by $\mathcal{T}_T$ the interval $(0, T]$ and by $\overline{\mathcal{T}_T}$ the closed interval $[0, T]$. The functions f, φ and α are given and are continuous on the specified domains. The condition $(6.2.12)_3$ is called Dirichlet's condition. In a problem of Neumann type, the condition $(6.2.12)_3$ is replaced by the boundary condition of Neumann

$$\frac{\partial u}{\partial\nu}(\tau,\eta) = \beta(\tau,\eta),\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega.$$

Definition 6.2.1 The function $G(t, \tau, x, \xi)$ is called the function of Green attached to the domain Ω, to the operator $\mathcal{L}$, and to the Dirichlet boundary condition (6.2.12)$_3$, if it is defined by

$$G(t, \tau, x, \xi) = V(t, \tau, x, \xi) + g(t, \tau, x, \xi), \tag{6.2.13}$$

where the function $V(t, \tau, x, \xi)$ is defined in (6.2.8) and the function $g(t, \tau, x, \xi)$ has the following properties:

- $g(t, \tau, x, \xi)$ is continuous with regards to the variables t, τ, x, and ξ on the set $\overline{\mathcal{T}_T} \times \overline{\mathcal{T}_T} \times \Omega \times \Omega$;
- the derivatives $g_{x_i x_i}$ and g_t are continuous on the set $\mathcal{T}_T \times \mathcal{T}_T \times \Omega \times \Omega$;
- $g(t, \tau, x, \xi)$ satisfies the homogeneous adjoint equation of heat

$$\mathcal{M}g(t, \tau, x, \xi) = \Delta_\xi g(t, \tau, x, \xi) + \frac{\partial}{\partial \tau} g(t, \tau, x, \xi) = 0;$$

- $g(t, \tau, x, \xi)$ satisfies the condition $g(t, t, x, \xi) = 0$.

The function of Green $G(t, \tau, x, \xi)$ satisfies, by definition, the homogeneous boundary condition of Dirichlet type

$$G(t, \tau, x, \eta) = 0, \ \forall (\tau, \eta) \in \overline{\mathcal{T}_T} \times \partial\Omega.$$

In the following theorem, we prove that if the problem of Dirichlet (6.2.12) admits a classical solution, then this solution can be expressed with the help of the function of Green.

Theorem 6.2.2 *If we assume that the problem of Dirichlet (6.2.12) admits a classical solution, then this has the form*

$$\begin{aligned} u(t, x) = &- \int_\Omega \int_0^t G(t, \tau, x, \xi) f(\tau, \xi) \mathrm{d}\tau \mathrm{d}\xi \\ &- \int_{\partial\Omega} \int_0^t \frac{\partial G(t, \tau, x, \eta)}{\partial \nu} \alpha(\tau, \eta) \mathrm{d}\tau d\mathrm{d}\sigma_\eta + \int_\Omega G(t, 0, x, \xi) \varphi(\xi) \mathrm{d}\xi. \end{aligned} \tag{6.2.14}$$

Proof We write Green's formula (6.2.7) for the pair of functions $v = g(t, \tau, x, \xi)$ and $u = u(\tau, \xi)$, where $u(\tau, \xi)$ is the solution of the problem (6.2.12)

$$\begin{aligned} 0 = &- \int_\Omega \int_0^t g(t, \tau, x, \xi) f(\tau, \xi) \mathrm{d}\tau \mathrm{d}\xi + \int_\Omega \int_0^t u(\tau, \xi) \mathcal{M} g(t, \tau, x, \xi) \mathrm{d}\tau \mathrm{d}\xi \\ &+ \int_{\partial\Omega} \int_0^t \left[g(t, \tau, x, \xi) \frac{\partial u(\tau, \xi)}{\partial \nu} - u(\tau, \xi) \frac{\partial g(t, \tau, x, \xi)}{\partial \nu} \right] \mathrm{d}\tau \mathrm{d}\sigma_\xi \\ &- \int_\Omega g(t, \tau, x, \xi) u(\tau, \xi) \mathrm{d}\xi + \int_\Omega g(t, 0, x, \xi) \varphi(\xi) \mathrm{d}\xi. \end{aligned}$$

Based on the hypotheses imposed on the function g, this equality becomes

$$0=-\int_{\Omega}\int_0^t g(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi+\int_{\Omega}g(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi \\ +\int_{\partial\Omega}\int_0^t\left[g(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu}-\alpha(\tau,\xi)\frac{\partial g(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_{\xi}. \tag{6.2.15}$$

We now write Riemann–Green's formula (6.2.10) for the pair of functions $v = V(t,\tau,x,\xi)$ and $u = u(\tau,\xi)$, where $u(\tau,\xi)$ is the solution of the problem (6.2.12)

$$u(t,x)=-\int_{\Omega}\int_0^t V(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi+\int_{\Omega}V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi \\ +\int_{\partial\Omega}\int_0^t\left[V(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu}-\alpha(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_{\xi}. \tag{6.2.16}$$

If we add term by term the formulas (6.2.15) and (6.2.16), we are led to

$$u(t,x)=-\int_{\Omega}\int_0^t G(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi \\ +\int_{\partial\Omega}\int_0^t G(t,\tau,x,\xi)\frac{u(\tau,\xi)}{\partial\nu}\mathrm{d}\tau\mathrm{d}\sigma_{\xi}-\int_{\partial\Omega}\int_0^t\frac{\partial G(t,\tau,x,\xi)}{\partial\nu}\alpha(\tau,\eta)\mathrm{d}\tau d\mathrm{d}\sigma_{\eta} \\ +\int_{\Omega}G(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi.$$

Since the function of Green $G(t,\tau,x,\xi)$ becomes null on the boundary (because, by definition, $G(t,\tau,x,\xi)$ satisfies the homogeneous Dirichlet condition), we deduce that the second integral from the right-hand side of the equality above disappears, and the formula which remains is just (6.2.14). ■

We will now perform analogous considerations for the Neumann problem which is obtained from the problem of Dirichlet (6.2.12) by replacing the condition $(6.2.12)_3$ with the condition

$$\frac{\partial u(\tau,\eta)}{\partial\nu}=\beta(\tau,\eta),\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega. \tag{6.2.17}$$

The function of Green for the domain Ω, the operator $\mathcal{L}$ and the Neumann condition (6.2.17) is given in the formula (6.2.13) from the definition 2.1, but the last condition from this definition is replaced by

$$\frac{\partial G(t,\tau,x,\eta)}{\partial\nu}=0,\ \forall(\tau,\eta)\in\overline{\mathcal{T}_T}\times\partial\Omega, \tag{6.2.18}$$

that is, the function G satisfies the homogeneous Neumann condition.

Proposition 6.2.3 *Suppose that the Neumann problem (6.2.12)$_1$, (6.2.12)$_2$, (6.2.17) admits a classical solution. Then it can be expressed with the help of the function of Green in the form*

$$u(t,x) = -\int_{\Omega}\int_0^t G(t,\tau,x,\xi)f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi$$
$$+\int_{\partial\Omega}\int_0^t G(t,\tau,x,\eta)\beta(\tau,\eta)\mathrm{d}\tau d\mathrm{d}\sigma_\eta + \int_{\Omega} G(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi. \tag{6.2.19}$$

Proof We will use the same reasoning from the proof of the formula (6.2.14). We can write, first, Green's formula for the pair of the functions $v = g(t,\tau,x,\xi)$ and $u = u(\tau,\xi)$, where $u(\tau,\xi)$ is the solution of the above Neumann problem. Then we write Riemann–Green's formula for the pair of functions $v = V(t,\tau,x,\xi)$ and $u = u(\tau,\xi)$, where $u(\tau,\xi)$ is the solution of the Neumann problem. If we add term by term the two obtained relations and take into account the conditions imposed on the functions $g(t,\tau,x,\xi)$ and $G(t,\tau,x,\xi)$, we obtain formula (6.2.19). ■

Analyzing the formulas (6.2.14) and (6.2.19) we deduce that, if they exist, the solutions of the Dirichlet and Neumann problems, respectively, if they exists, can be expressed uniquely with the help of the function of Green. Because the function $V(t,\tau,x,\xi)$ from the definition of the function of Green is given in (6.2.8), we deduce that the problem of determining the function of Green is reduced to determining of the function $g(t,\tau,x,\xi)$. Apparently, the problem determining the function $g(t,\tau,x,\xi)$ is as difficult as properly determining the solution of Dirichlet's or Neumann's problem, especially due to the conditions of regularity imposed on the function $g(t,\tau,x,\xi)$, which are reminiscent of the conditions imposed on a classical solution.

But, in contrast to the classical solution u, the function $g(t,\tau,x,\xi)$ satisfies both in the case of Dirichlet's problem and also in the case of the Neumann problem, a homogeneous equation of heat. Then, if in the case of the Dirichlet problem the solution u satisfies a boundary condition with α arbitrarily chosen, and in the case of the problem of Neumann, with β arbitrarily chosen, the function $g(t,\tau,x,\xi)$ satisfies a boundary condition in which the right-hand side is perfectly determined, because

$$g(t,\tau,x,\eta) = -V(t,\tau,x,\eta),\ \forall(\tau,\eta) \in \overline{\mathcal{T}_T} \times \partial\Omega,$$

and

$$\frac{\partial g(t,\tau,x,\eta)}{\partial\nu} = -\frac{\partial V(t,\tau,x,\eta)}{\partial\nu},\ \forall(\tau,\eta) \in \overline{\mathcal{T}_T} \times \partial\Omega,$$

respectively, where $V(t,\tau,x,\eta)$ is given in (6.2.8).

These commentaries prove that the method of Green's function can be successfully used in solving initial-boundary value problems, from the theory of parabolic equations.

In the considerations from this paragraph, the method of Green's function was used for solving linear problems. But this method can be used also for nonlinear problems. Let us mention that for determining the function of Green we can use the Laplace transform. By applying the Laplace transform on the parabolic equation and on the boundary and initial conditions, an elliptic problem with boundary conditions is obtained, because the Laplace transform acts on the time variable. Also, the initial-boundary value problem for parabolic equations is subject to some simplifications if we apply the Fourier transform on the spatial variables.

Consider now the nonlinear problem

$$\begin{aligned}
&\Delta u - \frac{\partial u}{\partial t} = F(t, x, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}),\ \forall (t, x) \in \mathcal{T}_T \times \Omega, \\
&u(0, x) = \varphi(x),\ \forall x \in \overline{\Omega}, \\
&u(t, y) = \alpha(t, y),\ \forall (t, y) \in \overline{\mathcal{T}_T} \times \partial\Omega.
\end{aligned} \tag{6.2.20}$$

In the approach of the problem (6.2.20), we can proceed as in the case of the linear problems. First, the function of Green attached to the domain Ω, to the linear operator $\Delta u - u_t$, and to the boundary condition $(6.2.20)_3$ is determined. Assuming that the problem (6.2.20) admits a classical solution, then this solution can be expressed with the help of the function of Green in the form

$$\begin{aligned}
u(t, x) = &-\int_\Omega \int_0^t G(t, \tau, x, \xi) F(\tau, \xi, u, u_{\xi_1}, u_{\xi_2}, \ldots, u_{\xi_n}) \mathrm{d}\tau \mathrm{d}\xi \\
&-\int_{\partial\Omega} \int_0^t \frac{\partial G(t, \tau, x, \eta)}{\partial \nu} \alpha(\tau, \eta) \mathrm{d}\tau d \mathrm{d}\sigma_\eta + \int_\Omega G(t, 0, x, \xi) \varphi(\xi) \mathrm{d}\xi.
\end{aligned} \tag{6.2.21}$$

Then we have to determine the conditions that must be imposed on the functions F, α and φ so that the function u from (6.2.21) is an effective solution of the problem (6.2.20). We can prove a result according to which, if the function F is continuous in all its variables and satisfies a condition of Lipschitz type with regards to the variables $u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}$, then u from (6.2.21) is an effective solution of the problem (6.2.20).

6.3 The Cauchy Problem

In the initial-boundary value problems for the equation of heat considered in the previous paragraphs, it was essential to know the temperature on the surface of the body on which the problem was formulated.

In the present paragraph, we consider that the surface is at a very great distance, so that instead of a bounded domain we will consider the whole space $\mathbb{R}^n$. Therefore the boundary condition disappears and then we have the Cauchy problem

$$\begin{aligned} &\Delta u(t,x) - u_t(t,x) = f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n, \\ &u(0,x) = \varphi(x),\ \forall x \in \mathbb{R}^n, \end{aligned} \tag{6.3.1}$$

in which $\mathcal{T}_T$ is the interval of time $(0, T]$ and the functions f and φ are given and are continuous on $\mathcal{T}_T \times \mathbb{R}^n$ and on $\mathbb{R}^n$, respectively.

The problem (6.3.1) is complete if we know the behavior of the function u at infinity. Two types of behaviors at infinity are known, namely,

- it is required that the function u is bounded;
- the function u tends asymptotically to zero.

In the following, we will assume that the function u is bounded at infinity.

It is called *a classical solution* of the Cauchy problem, a function u which satisfies the conditions:

- $u \in C(\overline{\mathcal{T}_T} \times \mathbb{R}^n)$;
- u and u_{x_i} are bounded functions on $\mathcal{T}_T \times \mathbb{R}^n$;
- $u_{x_i x_i},\ u_t \in C(\mathcal{T}_T \times \mathbb{R}^n)$;
- u satisfies Eq. $(6.3.1)_1$ and the initial condition $(6.3.1)_2$.

In the approach of the Cauchy problem (6.3.1), we will go through two steps. In the first step, assuming that the problem admits a classical solution, we will find its form with the help of the Riemann–Green formula.

In the second step, we will show that in certain conditions of regularity imposed on the functions f and φ, the formula found for the function u is an effective solution of the problem (6.3.1).

We recall that the fundamental solution $V(t, \tau, x, \xi)$ is given by

$$V(t,\tau,x,\xi) = \frac{1}{\left(2\sqrt{\pi}\right)^n \left(\sqrt{t-\tau}\right)^n} \exp\left(-\frac{\sum_{i=1}^{n} (x_i - \xi_i)^2}{4(t-\tau)} \right). \tag{6.3.2}$$

Theorem 6.3.1 *Suppose that the Cauchy problem (6.3.1) admits a classical solution. Then this solution admits the representation*

$$u(t,x) = -\int_{\mathbb{R}^n} \int_0^t V(t,\tau,x,\xi) f(\tau,\xi) \mathrm{d}\tau \mathrm{d}\xi + \int_{\mathbb{R}^n} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi. \tag{6.3.3}$$

Proof We arbitrarily fix $x \in \mathbb{R}^n$ and we take the ball $B(0, R)$ with the center in the origin and the radius R sufficiently big so that the ball contains the point x. Write then

Riemann–Green's formula on this ball, for the pairs of the functions $v = V(t, \tau, x, \xi)$ and $u = u(t, x)$, where $u(t, x)$ is the solution of the problem (6.3.1)

$$u(t,x) = -\int_{B(0,R)}\int_0^t V(t,\tau,x,\xi) f(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi + \int_{B(0,R)} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi \\ + \int_{\partial B(0,R)}\int_0^t \left[V(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu} - u(\tau,\xi)\frac{\partial V(t,\tau,x,\xi)}{\partial\nu}\right]\mathrm{d}\tau\mathrm{d}\sigma_\xi. \tag{6.3.4}$$

We assumed that u and u_{x_i} are bounded functions (because u is a classical solution for the problem (6.3.1)). Then, having in mind also the properties of the function $V(t, \tau, x, \xi)$, we can show that if $R \to \infty$, then the last integral from (6.3.4) tends to zero. To this aim, we write the last integral from (6.3.4) in the form

$$\int_{\partial B(0,R)}\int_0^t V(t,\tau,x,\xi)\frac{\partial u(\tau,\xi)}{\partial\nu}\mathrm{d}\tau\mathrm{d}\sigma_\xi \\ - \int_{\partial B(0,R)}\int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial\nu} u(\tau,\xi)\mathrm{d}\tau\mathrm{d}\sigma_\xi = I_1 + I_2. \tag{6.3.5}$$

Then

$$|I_1| \le c_0 \int_{\partial B(0,R)}\int_0^t \frac{1}{\left(\sqrt{t-\tau}\right)^n}\exp\left(-\frac{\sum_{i=1}^n (x_i-\xi_i)^2}{4(t-\tau)}\right)\mathrm{d}\tau\mathrm{d}\sigma_\xi,$$

where $c_0 = \frac{1}{(2\sqrt{\pi})^n}\sup\frac{\partial u}{\partial\nu}$ and this supremum exists because u is a bounded function.

It is clear that

$$|x_k - \xi_k| \le r = |\xi x| = \sqrt{\sum_{i=1}^n (x_i-\xi_i)^2}.$$

We can choose the radius of the ball R so that for an arbitrarily fixed $x, x \in int B(0, R)$ and $\xi \in \partial B(0, R)$ we have $|\xi x| > R/2$.

With these evaluations, for I_1 we obtain

$$|I_1| \le c_0 \int_{\partial B(0,R)}\int_0^t \frac{1}{\left(\sqrt{t-\tau}\right)^n} e^{-\frac{R^2}{16(t-\tau)}}\mathrm{d}\tau\mathrm{d}\sigma_\xi.$$

For the derivative of the function $V(t, \tau, x, \xi)$ in the direction of normal, we have the bound

$$\left|\frac{\partial V}{\partial \nu}\right| = \left|\sum_{k=1}^{n} \frac{\partial V}{\partial x_k} \cos \alpha_k\right| \leq \left|\sum_{k=1}^{n} \frac{\partial V}{\partial x_k}\right|,$$

so that for I_2 we obtain

$$|I_2| \leq c_1 \int_{\partial B(0,R)} \int_0^t \sum_{i=1}^{n} |x_i - \xi_i| \, (t-\tau)^{-\frac{n+2}{2}} \exp\left(-\frac{\sum_{i=1}^{n} (x_i - \xi_i)^2}{4(t-\tau)}\right) \mathrm{d}\tau \mathrm{d}\sigma_\xi,$$

and with the evaluations above, we deduce that

$$\begin{aligned} |I_2| &\leq c_2 \int_{\partial B(0,R)} \int_0^t \frac{r}{(t-\tau)^{(n+2)/2}} e^{-\frac{r^2}{4(t-\tau)}} \mathrm{d}\tau \mathrm{d}\sigma_\xi \\ &\leq c_2 R \int_{\partial B(0,R)} \int_0^t \frac{1}{(t-\tau)^{(n+2)/2}} e^{-\frac{R^2}{16(t-\tau)}} \mathrm{d}\tau \mathrm{d}\sigma_\xi, \end{aligned}$$

in which c_1 comes from the supremum of the function u and $c_2 = nc_1$.

We make the change of variables

$$t - \tau = \frac{R^2}{16\sigma^2} \Rightarrow \mathrm{d}\tau = \frac{R^2}{8\sigma^3} d\sigma.$$

Then for the upper bound of I_2, we have

$$\begin{aligned} |I_2| &\leq c_3 \frac{1}{R^{n-1}} \int_{\partial B(0,R)} \int_{\frac{R}{4\sqrt{t}}}^{\infty} \sigma^{n-1} e^{-\sigma^2} d\sigma \mathrm{d}\sigma_\xi \\ &= c_3 \omega_n \int_{\frac{R}{4\sqrt{t}}}^{\infty} \sigma^{n-1} e^{-\sigma^2} d\sigma \mathrm{d}\sigma_\xi. \end{aligned}$$

An analogous bound is obtained also for I_1, using the same change of variables. By integrating, $n-1$ times, by parts, we can show that

$$\lim_{R\to\infty} \int_{\frac{R}{4\sqrt{t}}}^{\infty} \sigma^{n-1} e^{-\sigma^2} d\sigma \mathrm{d}\sigma_\xi = 0.$$

Then I_1 and I_2 tend to zero, as $R \to \infty$. If we pass to the limit with $R \to \infty$ in (6.3.5), then we obtain that the integral on the left-hand side tends to zero. With this observation, we pass to the limit with $R \to \infty$ in (6.3.4) and we obtain formula (6.3.3). ∎

Formula (6.3.3) is called *the formula of Poisson* for the representation of the solution of the Cauchy problem (6.3.1). With the help of the formula of Poisson, we can prove the uniqueness of a classical solution of the problem (6.3.1).

Theorem 6.3.2 *The Cauchy problem for heat conduction (6.3.1) admits at most a classical solution.*

Proof Assume, by absurd, that the problem (6.3.1) admits two classical solutions, $u_1(t,x)$ and $u_2(t,x)$, which are bounded, that is,

$$\Delta u_i(t,x) - \frac{\partial u_i}{\partial t}(t,x) = f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n,$$
$$u_i(0,x) = \varphi(x),\ \forall x \in \mathbb{R}^n,$$

where $i = 1, 2$. We define the function $v(t,x)$ by $v(t,x) = u_1(t,x) - u_2(t,x)$. Then

$$\Delta v(t,x) - \frac{\partial v}{\partial t}(t,x) = f(t,x) - f(t,x) = 0,$$
$$v(0,x) = u_1(0,x) - u_2(0,x) = \varphi(x) - \varphi(x) = 0. \tag{6.3.6}$$

We obtained a new Cauchy problem with $f \equiv 0$ and $\varphi \equiv 0$. According to Theorem 6.3.1, if a Cauchy problem admits a solution, then the solution necessarily has the form (6.3.3). If we write the formula (6.3.3) and take into account that $f \equiv 0$ and $\varphi \equiv 0$, then we obtain $v(t,x) = 0,\ \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n$ such that $u_1(t,x) = u_2(t,x)$. ∎

It remains only to be proven that the function u from (6.3.3) is an effective solution of the Cauchy problem 6.3.1. This is the object of the theorem of existence that follows.

Theorem 6.3.3 *Assume that the following conditions are satisfied:*

(i) the functions $f(t,x)$, $\frac{\partial f}{\partial x_i}(t,x)$, $\frac{\partial^2 f}{\partial x_i^2}(t,x)$ *are continuous and bounded on* $\mathcal{T}_T \times \mathbb{R}^n$, *that is,*

$$f(t,x),\ \frac{\partial f}{\partial x_i}(t,x),\ \frac{\partial^2 f}{\partial x_i^2}(t,x) \in C(\mathcal{T}_T \times \mathbb{R}^n) \cap B(\mathcal{T}_T \times \mathbb{R}^n);$$

(ii) the functions $\varphi(t,x)$, $\frac{\partial \varphi}{\partial x_i}(t,x)$, $\frac{\partial^2 \varphi}{\partial x_i^2}(t,x)$ *are continuous and bounded on* $\mathcal{T}_T \times \mathbb{R}^n$, *that is,*

$$\varphi(t,x),\ \frac{\partial \varphi}{\partial x_i}(t,x),\ \frac{\partial^2 \varphi}{\partial x_i^2}(t,x) \in C(\mathcal{T}_T \times \mathbb{R}^n) \cap B(\mathcal{T}_T \times \mathbb{R}^n).$$

Then the function u from (6.3.3) is an effective solution of the Cauchy problem (6.3.1), namely, a bounded solution on $\mathcal{T}_T \times \mathbb{R}^n$.

Proof We define the integral I_1 by

$$I_1 = \int_{\mathbb{R}^n} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi,$$

and show that I_1 verifies the problem

$$\Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) = 0,\ \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n,$$
$$u(0,x) = \varphi(x),\ \forall x \in \mathbb{R}^n. \tag{6.3.7}$$

Then we show that the integral I_2

$$I_2 = \int_{\mathbb{R}^n} \int_0^t V(t,\tau,x,\xi) f(\tau,\xi) \mathrm{d}\tau \mathrm{d}\xi,$$

verifies the problem

$$\Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) = f(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \mathbb{R}^n,$$
$$u(0,x) = 0,\ \forall x \in \mathbb{R}^n.$$

Thus it will be obvious that $I_1 + I_2$, that is, u from (6.3.3), verifies the Cauchy problem (6.3.1).

Because φ is a bounded and continuous function, we have

$$|I_1| \le \|\varphi\| \int_{\mathbb{R}^n} V(t,0,x,\xi)\mathrm{d}\xi = \|\varphi\|,$$

which proves that the integral I_1 is convergent and therefore we can differentiate under the integral sign. Then

$$\Delta I_1 - \frac{\partial I_1}{\partial t} = \int_{\mathbb{R}^n} \left(\Delta V - \frac{\partial V}{\partial t} \right) \varphi(\xi)\mathrm{d}\xi = 0,$$

taking into account the properties of the function $V(t,\tau,x,\xi)$.

On the other hand, also using the properties of the function $V(t,\tau,x,\xi)$, we have

$$\lim_{t\to 0} I_1 = \lim_{t\to 0} \lim_{R\to\infty} \int_{\mathbb{R}^n} V(t,0,x,\xi)\varphi(\xi)\mathrm{d}\xi = \varphi(x),$$

because the ball $B(0,R)$ has the radius R sufficient big, so that the point x is inside the ball.

As in the case of I_1, we can show that the integral from I_2 is convergent and then we can differentiate under the integral sign and therefore we have

$$\Delta_x I_2 = -\int_0^t \int_{\mathbb{R}^n} \Delta_x V(t,\tau,x,\xi) f(\tau,\xi)\mathrm{d}\xi\mathrm{d}\tau. \tag{6.3.8}$$

In the case of the derivative with respect to t, we differentiate an integral with parameter

$$\frac{\partial I_2}{\partial t} = -\int_{\mathbb{R}^n} V(t,t,x,\xi) f(t,\xi) \mathrm{d}\xi - \int_{\mathbb{R}^n} \int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial \tau} f(\tau,\xi) \mathrm{d}\xi \mathrm{d}\tau. \tag{6.3.9}$$

For the first integral from the right-hand side of the relation (6.3.9) we have, in fact,

$$\lim_{\tau \to t^-} \lim_{R \to \infty} \int_{B(0,R)} V(t,\tau,x,\xi) f(\tau,\xi) \mathrm{d}\xi = f(t,x),$$

according to the first part of Theorem 6.1.7 (Sect. 6.1). Thus (6.3.9) becomes

$$\frac{\partial I_2}{\partial t} = -f(t,x) - \int_{\mathbb{R}^n} \int_0^t \frac{\partial V(t,\tau,x,\xi)}{\partial \tau} f(\tau,\xi) \mathrm{d}\xi \mathrm{d}\tau,$$

relation which, together with (6.3.8), leads to

$$\Delta_x I_2 - \frac{\partial I_2}{\partial t} = f(t,x) \\ - \int_{\mathbb{R}^n} \int_0^t \left[\Delta_x V(t,\tau,x,\xi) - \frac{\partial V(t,\tau,x,\xi)}{\partial \tau} \right] f(\tau,\xi) \mathrm{d}\tau \mathrm{d}\xi.$$

But

$$\Delta_x V(t,\tau,x,\xi) - \frac{\partial V(t,\tau,x,\xi)}{\partial \tau} = 0,$$

and then the previous relation becomes

$$\Delta_x I_2 - \frac{\partial I_2}{\partial t} = f(t,x).$$

Then it is clear that

$$\lim_{t \to 0} I_2 = \int_0^0 \int_{\mathbb{R}^n} V(t,\tau,x,\xi) f(\tau,\xi) \mathrm{d}\xi \mathrm{d}\tau = 0,$$

and this ends the proof of the theorem. ■

At the end of the paragraph, we will solve a Cauchy problem, attached to the equation of heat, in a particular case.

Let us consider the strip $B = [0,T] \times (-\infty,\infty)$, where T is a fixed positive number, which can be also ∞. Consider the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \ \forall (t, x) \in B. \tag{6.3.10}$$

If the function $u(t, x)$, defined on the strip B has the derivatives $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ continuous inside of the strip and $u(t, x)$ satisfies Eq. (6.3.10), we say that the function $u(t, x)$ is a *regular solution* of Eq. (6.3.10).

The Cauchy problem consists in determining a regular solution of Eq. (6.3.10) which satisfies the initial condition

$$u(0, x) = \varphi(x), \tag{6.3.11}$$

where $\varphi(x)$ is a real given function, which is continuous and bounded for any $x \in (-\infty, -\infty)$.

We will prove that the function $u(t, x)$, defined by

$$u(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(\xi - x)^2}{4t}} \mathrm{d}\xi, \tag{6.3.12}$$

is a solution of the Cauchy problem (6.3.10), (6.3.11).

It is known from classical mathematical analysis that the integral from (6.3.12) is uniformly convergent in a neighborhood of any point (t, x) from inside the strip B. If we make the change of variables $\xi = x + 2\eta\sqrt{t}$, then formula (6.3.12) becomes

$$u(t, x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x + 2\eta\sqrt{t}) e^{-\eta^2} \mathrm{d}\eta. \tag{6.3.13}$$

Because φ is continuous and bounded, we have $\sup\limits_{-\infty < x < \infty} |\varphi(x)| < M, \ M > 0$. The integral from (6.3.13) is absolutely convergent and then

$$|u(t, x)| < \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} \mathrm{d}\eta = \frac{M}{\sqrt{\pi}} \sqrt{\pi} = M.$$

The integrals which are obtained by differentiating under the integral sign in (6.3.12), with respect to x and with respect to t, are uniformly convergent.

On the other hand, the function

$$\frac{1}{\sqrt{t}} e^{-\frac{(\xi - x)^2}{4t}}, \ t > 0,$$

obviously satisfies Eq. (6.3.10). With these arguments, we get that the function u defined in (6.3.12) satisfies Eq. (6.3.10).

Also, the uniform convergence of the integral in a neighborhood of any point (t, x), with $t > 0$, from inside the strip B, allows passing to the limit, with $t \to 0$, in (6.3.13), so that we obtain

$$\lim_{t \to 0} u(t, x) = \varphi(x).$$

The uniqueness and stability of the regular solution of our Cauchy problem are immediately obtained. We can show that the regular solution of Eq. (6.3.10), satisfies the inequality

$$m \leq u(t, x) \leq M,$$

where $m = \inf u(0, x)$ and $M = \sup u(0, x)$, $x \in (-\infty, \infty)$.

We can then use the function $v(t, x) = 2t + x^2$, which obviously is a particular solution of Eq. (6.3.10).

Chapter 7
Hyperbolic Equations

7.1 The Problem of the Infinite Oscillating Chord

The prototype of hyperbolic equations is considered to be the equation of the oscillating chord, also known as the wave equation.

We will address, first, the case of the infinite oscillating chord. In fact, the chord is not infinite, but its longitudinal dimension is infinitely bigger than its cross section.

Mainly, in this section we will consider the following initial value problem, attached to the equation of the infinite oscillating chord:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t, x), \ \forall (t, x) \in (-\infty, +\infty), \ \forall t > 0, \\
&u(0, x) = \varphi(x), \ \forall x \in (-\infty, +\infty), \\
&\frac{\partial u}{\partial t}(0, x) = \psi(x), \ \forall x \in (-\infty, +\infty),
\end{aligned} \tag{7.1.1}$$

in which the functions $f(t, x)$, $\varphi(x)$ and $\psi(x)$ are given and are continuous, on their domain of definition. The function $u = u(t, x)$ is the unknown function of the problem and represents the amplitude of the chord at the moment t, in the point x. The positive constant a is catalogued with respect to the type of material from which the chord is made.

We will decompose the Cauchy problem (7.1.1) into two other problems, the first one which is homogeneous with respect to the right-hand side, and the second one which is homogeneous with respect to the initial conditions

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \ \forall (t, x) \in (-\infty, +\infty), \ \forall t > 0, \\
&u(0, x) = \varphi(x), \ \forall x \in (-\infty, +\infty), \\
&\frac{\partial u}{\partial t}(0, x) = \psi(x), \ \forall x \in (-\infty, +\infty),
\end{aligned} \tag{7.1.2}$$

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and

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(t,x),\ \forall (t,x) \in (-\infty,+\infty),\ \forall t>0,\\
&u(0,x) = 0,\ \forall x \in (-\infty,+\infty), \qquad (7.1.3)\\
&\frac{\partial u}{\partial t}(0,x) = 0,\ \forall x \in (-\infty,+\infty),
\end{aligned}$$

respectively.

Proposition 7.1.1 *If the function $u_1(t,x)$ is a solution of the problem (7.1.2) and the function $u_2(t,x)$ is a solution of the problem (7.1.3), then the function*

$$u(t,x) = u_1(t,x) + u_2(t,x),\ \forall (t,x) \in (-\infty,+\infty),\ \forall t>0, \qquad (7.1.4)$$

is a solution of the problem (7.1.1).

Proof We verify, first, the initial conditions

$$\begin{aligned}
&u(0,x) = u_1(0,x) + u_2(0,x) = \varphi(x) + 0 = \varphi(x),\\
&\frac{\partial u}{\partial t}(0,x) = \frac{\partial u_1}{\partial t}(0,x) + \frac{\partial u_2}{\partial t}(0,x) = \psi(x) + 0 = \psi(x),
\end{aligned}$$

in which we take into account the initial conditions $(7.1.2)_2$ and $(7.1.3)_2$, respectively $(7.1.2)_3$ and $(7.1.3)_3$.

Using the linearity of the derivative and by differentiating (7.1.4), we obtain

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial^2 u_1}{\partial t^2} - a^2 \frac{\partial^2 u_1}{\partial x^2}\right) + \left(\frac{\partial^2 u_2}{\partial t^2} - a^2 \frac{\partial^2 u_2}{\partial x^2}\right)\\
&= 0 + f(t,x) = f(t,x),
\end{aligned}$$

in which we take into account Eqs. $(7.1.2)_1$ and $(7.1.3)_1$. ■

Let us solve, in the following, the problems (7.1.2) and (7.1.3). Based on Proposition 7.1.1 we obtain the solution of the problem (7.1.1).

In relation to the problem (7.1.3), we have the following result.

Theorem 7.1.1 *The function $U(t,x)$ defined by*

$$U(t,x) = \frac{1}{2a} \int_0^t \left\{ \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi) \mathrm{d}\xi \right\} \mathrm{d}\tau, \qquad (7.1.5)$$

is a solution of the problem (7.1.3).

Proof It is clear that

$$U(0,x)=\frac{1}{2a}\int_0^0\left\{\int_{x-a(0-\tau)}^{x+a(0-\tau)} f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau=0.$$

Then using the rule of differentiating an integral with a parameter, we deduce that

$$\begin{aligned}\frac{\partial U(t,x)}{\partial t}&=\frac{1}{2a}\int_x^x f(t,\xi)\mathrm{d}\xi+\frac{1}{2a}\int_0^t\frac{\partial}{\partial t}\left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau\\&=\frac{1}{2a}\int_0^t a[f(\tau,x+a(t-\tau))+f(\tau,x-a(t-\tau))]\mathrm{d}\tau\\&\quad+\frac{1}{2a}\int_0^t\left[\int_{x-a(t-\tau)}^{x+a(t-\tau)}\frac{\partial f(\tau,\xi)}{\partial\tau}\mathrm{d}\xi\right]\mathrm{d}\tau\\&=\frac{1}{2}\int_0^t[f(\tau,x+a(t-\tau))+f(\tau,x-a(t-\tau))]\mathrm{d}\tau.\end{aligned}$$

Then it is clear that

$$\frac{\partial U}{\partial t}(0,x)=\frac{1}{2}\int_0^0[f(\tau,x+a(0-\tau))+f(\tau,x-a(0-\tau))]\mathrm{d}\tau=0.$$

We differentiate the previous relation again with respect to t and so we get

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2}(t,x)&=\frac{1}{2}[f(t,x+a.0)+f(t,x-a.0)]\\&\quad+\frac{1}{2}\int_0^t\frac{\partial}{\partial\tau}\left[f(\tau,x+a(t-\tau))+f(\tau,x-a(t-\tau))\right]\mathrm{d}\tau,\end{aligned}$$

that is,

$$\frac{\partial^2 U}{\partial t^2}(t,x)=f(t,x)+\frac{a}{2}\int_0^t\left[\frac{\partial f(\tau,x+a(t-\tau))}{\partial(x+a(t-\tau))}-\frac{\partial f(\tau,x-a(t-\tau))}{\partial(x-a(t-\tau))}\right]\mathrm{d}\tau. \tag{7.1.6}$$

We differentiate now (7.1.5) with respect to x, using again the rule of differentiating of an integral with a parameter

$$\begin{aligned}\frac{\partial U(t,x)}{\partial x}&=\frac{1}{2a}\int_0^t\frac{\partial}{\partial x}\left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau\\&=\frac{1}{2a}\int_0^t[f(\tau,x+a(t-\tau))-f(\tau,x-a(t-\tau))]\mathrm{d}\tau.\end{aligned}$$

Here we differentiate again with respect to x, so that we are led to

$$\frac{\partial^2 U(t,x)}{\partial x^2} = \frac{1}{2a}\int_0^t \frac{\partial}{\partial x}\left[f(\tau, x+a(t-\tau)) - f(\tau, x-a(t-\tau))\right] \mathrm{d}\tau$$
$$= \frac{1}{2a}\int_0^t \left[\frac{\partial f(\tau, x+a(t-\tau))}{\partial(x+a(t-\tau))}\,\frac{\partial(x+a(t-\tau))}{\partial x}\right.$$
$$\left. -\frac{\partial f(\tau, x-a(t-\tau))}{\partial(x-a(t-\tau))}\,\frac{\partial(x-a(t-\tau))}{\partial x}\right] \mathrm{d}\tau,$$

and therefore

$$\frac{\partial^2 U(t,x)}{\partial x^2} = \frac{1}{2a}\int_0^t \left[\frac{\partial f(\tau, x+a(t-\tau))}{\partial(x+a(t-\tau))} - \frac{\partial f(\tau, x-a(t-\tau))}{\partial(x-a(t-\tau))}\right] \mathrm{d}\tau. \tag{7.1.7}$$

Then from (7.1.6) and (7.1.7) we obtain

$$\frac{\partial^2 U(t,x)}{\partial t^2} - a^2\frac{\partial^2 U(t,x)}{\partial x^2} = f(t,x),$$

that is, $U(t,x)$ verifies Eq. $(7.1.3)_1$. ■

We intend now to solve the problem (7.1.2).

Theorem 7.1.2 *The solution of the problem (7.1.2) is given by*

$$u(t,x) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\mathrm{d}s.$$

Proof First, we intend to obtain the canonical form of Eq. $(7.1.2)_1$.

Using the considerations from Sect. 1.1 from Chap. 1, we deduce that in the present case, the characteristic equation is

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - a^2 = 0.$$

We can observe that $\Delta = a^2 > 0$ and therefore we are, indeed, in the case of hyperbolic equations. The following prime integrals are immediately obtained:

$$x + at = C_1,$$
$$x - at = C_2,$$

in which C_1 and C_2 are arbitrary constants. Then we make the change of variables

$$\xi = x + at,$$
$$\eta = x - at. \tag{7.1.8}$$

Note that the transformation (7.1.8) is non-singular, because its Jacobian is not null. Indeed, we have

$$\left|\frac{\partial(\xi,\eta)}{\partial(t,x)}\right| = \begin{vmatrix} a & -a \\ 1 & 1 \end{vmatrix} = 2a > 0.$$

With the change of variables (7.1.8), the canonical form of Eq. $(7.1.2)_1$ becomes

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0,$$

that is,

$$\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}\right) = 0 \Rightarrow \frac{\partial u}{\partial\xi} = \gamma(\xi),\;\; \gamma \in C^1((-\infty,+\infty)).$$

We integrate again so that we obtain

$$u(\xi,\eta) = \int \gamma(\xi)\mathrm{d}\xi + \beta(\eta) = \alpha(\xi) + \beta(\eta), \tag{7.1.9}$$

where α is a primitive function of the arbitrary function γ.

If we suppose that α and β are functions of class C^1, then the order of differentiation above does not matter, according to the classic criterion of Schwartz. But, in order to verify the equation with partial derivatives of second order, α and β must be functions of class C^2.

We introduce (7.1.8) in (7.1.9) and so we deduce that

$$u(t,x) = \alpha(x+at) + \beta(x-at), \tag{7.1.10}$$

in which the functions α and β will be determined with the help of the initial conditions

$$\varphi(x) = u(0,x) = \alpha(x) + \beta(x),$$
$$\psi(x) = \frac{\partial u}{\partial t}(0,x) = a\alpha'(x) - a\beta'(x).$$

This system is equivalent to

$$\alpha(x) + \beta(x) = \varphi(x),$$
$$\alpha(x) - \beta(x) = \frac{1}{a}\int_0^x \psi(s)\mathrm{d}s + C,$$

where C is an arbitrary constant of integration. The solution of this system is

$$\alpha(x) = \frac{\varphi(x)}{2} + \frac{1}{2a}\int_0^x \psi(s)\mathrm{d}s + \frac{C}{2},$$
$$\beta(x) = \frac{\varphi(x)}{2} - \frac{1}{2a}\int_0^x \psi(s)\mathrm{d}s - \frac{C}{2},$$

and then from (7.1.10), we obtain

$$\begin{aligned} u(t,x) =& \frac{\varphi(x+at)}{2} + \frac{1}{2a}\int_0^{x+at} \psi(s)\mathrm{d}s + \frac{C}{2} \\ &+ \frac{\varphi(x-at)}{2} - \frac{1}{2a}\int_0^{x-at} \psi(s)\mathrm{d}s - \frac{C}{2} \\ &= \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\mathrm{d}s, \end{aligned}$$

that is, just the desired result. ■

Observation 7.1.1 *Based on the results from Theorems 7.1.1, 7.1.2 and Proposition 7.1.1, we deduce that the solution of the problem (7.1.1) is*

$$\begin{aligned} u(t,x) = & \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\mathrm{d}s \\ &+ \frac{1}{2a}\int_0^t \left\{ \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\mathrm{d}\xi \right\} \mathrm{d}\tau. \end{aligned} \tag{7.1.11}$$

We thus proved the following result of existence.

Theorem 7.1.3 (of existence) *If the given function $f(t,x)$ is supposed to be of class $C^0((0,\infty)\times(-\infty,+\infty))$, the given function $\varphi(x)$ is of class $C^2(-\infty,+\infty)$ and the given function $\psi(x)$ is of class $C^1(-\infty,+\infty)$, then the nonhomogeneous problem of the infinite oscillating chord admits the classical solution (7.1.11).*

A *classical solution* is a function $u = u(t,x)$ of class C^2 with respect to $x \in (-\infty,+\infty)$ and $t > 0$, which verifies the initial conditions $(7.1.1)_2$ and $(7.1.1)_3$, and replaced in Eq. $(7.1.1)_1$, transforms it into an identity.

Observation 7.1.2 *The form from (7.1.11) of the solution of the problem (7.1.10) is also called the formula of D'Alembert for the nonhomogeneous problem of the infinite oscillating chord.*

In the following theorem, we prove the uniqueness of the solution of the Cauchy problem (7.1.1).

Theorem 7.1.4 (of uniqueness) *The only classical solution of the nonhomogeneous problem of the infinite oscillating chord is that given in (7.1.11).*

Proof Assume the absurd case that the problem (7.1.1) admits two classical solutions $u_1(t,x)$ and $u_2(t,x)$ and then

$$\begin{aligned}
&\frac{\partial^2 u_i}{\partial t^2} - a^2 \frac{\partial^2 u_i}{\partial x^2} = f(t,x),\ \forall (t,x) \in (-\infty,+\infty),\ \forall t > 0,\\
&u_i(0,x) = \varphi(x),\ \forall x \in (-\infty,+\infty),\\
&\frac{\partial u_i}{\partial t}(0,x) = \psi(x),\ \forall x \in (-\infty,+\infty),
\end{aligned} \tag{7.1.12}$$

where $i = 1, 2$. We define the function $v(t,x)$ by

$$v(t,x) = u_1(t,x) - u_2(t,x).$$

Then

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} =& \frac{\partial^2 u_1}{\partial t^2} - a^2 \frac{\partial^2 u_1}{\partial x^2} = \left(\frac{\partial^2 u_2}{\partial t^2} - a^2 \frac{\partial^2 u_2}{\partial x^2} \right)\\
&= f(t,x) - f(t,x) = 0,
\end{aligned}$$

in which we used $(7.1.12)_1$.

Therefore

$$\begin{aligned}
&v(0,x) = u_1(0,x) - u_2(0,x) = \varphi(x) - \varphi(x) = 0,\\
&\frac{\partial v}{\partial t}(0,x) = \frac{\partial u_1}{\partial t}(0,x) - \frac{\partial u_2}{\partial t}(0,x) = \psi(x) - \psi(x) = 0,
\end{aligned}$$

in which we used the initial conditions $(7.1.12)_2$ and $(7.1.12)_3$.

Thus, the function v satisfies a problem of the form (7.1.1) in which $f(t,x) = \varphi(x) = \psi(x) = 0$ and then, according to (7.1.11), we have

$$v(t,x) = 0 \Rightarrow u_1(t,x) = u_2(t,x),$$

and this ends the proof of the theorem. ■

To obtain a result of stability with respect to the right-hand side and with respect to the initial conditions, for the problem (7.1.1), we ask that the time variable belongs to a finite interval, say $t \in (0, T]$, where T is a conveniently chosen moment.

Theorem 7.1.5 (of stability) *Denote by $u_1(t,x)$ and by $u_2(t,x)$, respectively, the unique solutions of the problems*

$$\begin{aligned}
&\frac{\partial^2 u_i}{\partial t^2} - a^2 \frac{\partial^2 u_i}{\partial x^2} = f_i(t,x),\ \forall (t,x) \in (-\infty,+\infty),\ \forall t > 0,\\
&u_i(0,x) = \varphi_i(x),\ \forall x \in (-\infty,+\infty),
\end{aligned} \tag{7.1.13}$$

$$\frac{\partial u_i}{\partial t}(0, x) = \psi_i(x), \ \forall x \in (-\infty, +\infty),$$

where $i = 1, 2$ and T is fixed in a sense which will be seen later. Then for any $\varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ so that if

$$\begin{aligned} |f(t, x)| &= |f_1(t, x) - f_2(t, x)| < \delta, \\ |\varphi(t, x)| &= |\varphi_1(t, x) - \varphi_2(t, x)| < \delta, \\ |\psi(t, x)| &= |\psi_1(t, x) - \psi_2(t, x)| < \delta, \end{aligned} \tag{7.1.14}$$

then

$$|u(t, x)| = |u_1(t, x) - u_2(t, x)| < \varepsilon.$$

Proof Based on Theorems 7.1.3 and 7.1.4, the only classical solutions of the problems (7.1.13) are the functions $u_i(t, x)$ given by

$$u_i(t, x) = \frac{1}{2}[\varphi_i(x + at) + \varphi_i(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi_i(s)\mathrm{d}s + \frac{1}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} f_i(\tau, \xi)\mathrm{d}\xi\right\}\mathrm{d}\tau,$$

where $i = 1, 2$. We subtract these two solutions

$$\begin{aligned} u_1(t, x) - u_2(t, x) &= \frac{1}{2}\left[\varphi_1(x + at) - \varphi_1(x + at)\right] \\ &+ \frac{1}{2}\left[\varphi_1(x - at) - \varphi_2(x - at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \left[\psi_1(s) - \psi_2(s)\right]\mathrm{d}s \\ &+ \frac{1}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} \left[f_1(\tau, \xi) - f_2(\tau, \xi)\right]\mathrm{d}\xi\right\}\mathrm{d}\tau. \end{aligned}$$

We pass to the absolute value in this equality and using the triangle inequality, we obtain that the absolute value of the the right-hand side is lower, at most equal, than the sum of the absolute values. We use then the fact that the absolute value of an integral is lower than the integral of the absolute value

$$\begin{aligned} |u_1(t, x) - u_2(t, x)| &\le \frac{1}{2}\left|\varphi_1(x + at) - \varphi_1(x + at)\right| \\ &+ \frac{1}{2}\left|\varphi_1(x - at) - \varphi_2(x - at)\right| + \frac{1}{2a}\int_{x-at}^{x+at} \left|\psi_1(s) - \psi_2(s)\right|\mathrm{d}s \\ &+ \frac{1}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} \left|f_1(\tau, \xi) - f_2(\tau, \xi)\right|\mathrm{d}\xi\right\}\mathrm{d}\tau. \end{aligned}$$

If we take into account (7.1.14), this inequality leads to

$$|u_1(t,x) - u_2(t,x)| \le \frac{\delta}{2} + \frac{\delta}{2} + \frac{\delta}{2a}\int_{x-at}^{x+at} \mathrm{d}s$$
$$+ \frac{\delta}{2a}\int_0^t \left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)} \mathrm{d}\xi\right\}\mathrm{d}\tau = \delta + \delta t + \frac{\delta}{2a}\int_0^t 2a(t-\tau)\mathrm{d}\tau$$
$$= \delta\left(1 + t + \frac{t^2}{2}\right) \le \delta\left(1 + T + \frac{T^2}{2}\right).$$

If we choose T so that

$$1 + T + \frac{T^2}{2} < \frac{\varepsilon}{\delta},$$

we obtain

$$|u_1(t,x) - u_2(t,x)| < \varepsilon,$$

and this ends the proof the theorem. ■

At the end of the paragraph, we make some comments on the results of existence, uniqueness, and stability from Theorems 7.1.3, 7.1.4, and 7.1.5.

In the case of a problem with initial conditions, or with boundary conditions, or, more generally, in the case of a mixed initial boundary value problem, there is the concept of a *well-posed problem*, used the first time by J. Hadamard. According to this concept, a problem, of any of the abovementioned types, is called a "well-posed problem" if there exists a theorem of uniqueness of the solution for the respective problem.

A theorem of uniqueness can be proven only for a certain class of functions. In the case of the problem of the infinite oscillating chord, studied above, we cannot have classical solutions without assuming that the given functions f, φ, and ψ are continuous. Therefore the class of continuous functions is the class in which we approach the problem of the uniqueness of the solution.

If we want to prove only the uniqueness of the solution, then it is enough to suppose that the functions f, φ, and ψ are of class C^0 on their domain of definition.

But to prove the existence of the solution, it is required to suppose that the functions φ and ψ are of class C^1.

Thus, the concept of class of correctness for the initial conditions and the boundary conditions appears. This is that class of the functions to which the initial conditions and the boundary conditions, respectively, belong and for which the uniqueness of the solution of the respective problem is guaranteed.

When we have a theorem of existence and a theorem of uniqueness, we can talk about the existence and the uniqueness of the solutions of the problems for which the function from the right-hand side is given and likewise, the functions from the initial conditions and the boundary conditions are prescribed.

A *particular solution* of a given problem is that solution which corresponds uniquely (in virtue of a theorem of existence and uniqueness) to a right-hand side function, to some boundary conditions, and to fixed initial conditions. Therefore to each fixed right-hand side and to fixed initial and boundary conditions, it corresponds to a particular solution. In this context, *the general solution* will be conceived as the family of all particular solutions.

In some cases, there may be some solutions for which a theorem of existence and uniqueness is not proven. These solutions are obtained by direct calculation and are called *singular solutions*.

The functions which define the the right-hand side and the initial conditions and the boundary conditions are supplied by experiments. In the case of the problem of the infinite oscillating chord, for the functions f_1, φ_1, and ψ_1 provided by a researcher we have a uniquely determined solution, u_1. If another researcher provides other data f_2, φ_2, and ψ_2, for the same phenomenon, the problem will admit the uniquely determined solution u_2. We will talk about the *stability of the solution* in the following case: if the data f_1, φ_1, and ψ_1 differ sufficiently little from the data f_2, φ_2, and ψ_2, then the corresponding solutions u_1 and u_2, respectively differ sufficiently little.

7.2 Problem with Initial and Boundary Conditions

Let Ω be a bounded domain from the space $\mathbb{R}^n$ with the boundary $\partial\Omega$ having tangent plane continuously varying almost everywhere. As usual, we denote by $\mathcal{T}_T$ the interval of time $\mathcal{T}_T = (0, T]$ and $\overline{\mathcal{T}_T} = [0, T]$, where $T > 0$.

Consider the initial-boundary value problem, attached to the equation of waves

$$\begin{aligned}
&\Delta u(t,x) - u_{tt}(t,x) = f(t,x), \ \forall (t,x) \in \mathcal{T}_T \times \Omega, \\
&u(t,y) = \alpha(t,y), \ \forall x \in \overline{\mathcal{T}_T} \times \partial\Omega, \\
&u(0,x) = \varphi(x), \ \forall x \in \overline{\Omega}, \\
&u_t(0,x) = \psi(x), \ \forall x \in \overline{\Omega},
\end{aligned} \tag{7.2.1}$$

where the functions f, α, φ, and ψ are given and are continuous on their domain of definition.

Definition 7.2.1 A classical solution of the problem (7.2.1) is the function $u = u(t,x)$ which satisfies the conditions

- u is a continuous function on $\overline{\mathcal{T}_T} \times \overline{\Omega}$;
- the derivatives $u_{x_i x_i}$ and u_{tt} are continuous functions on $\mathcal{T}_T \times \Omega$;
- u satisfies Eq. $(7.2.1)_1$, the boundary condition $(7.2.1)_2$ and the initial conditions $(7.2.1)_3$ and $(7.2.1)_4$.

We will use an energy method to show that the problem (7.2.1) can only have one solution.

Theorem 7.2.1 *The mixed initial boundary values problem (7.2.1) has at most one classical solution.*

Proof Assume, the absurd case that the problem (7.2.1) admits two classical solutions, $u_1(t,x)$ and $u_2(t,x)$. We define the function v by

$$v(t,x) = u_1(t,x) - u_2(t,x).$$

It can be verified, immediately, that the function v satisfies the conditions imposed on a classical solution, because the solutions $u_1(t,x)$ and $u_2(t,x)$ are assumed to be classical solutions. Also, the function v satisfies the problem (7.2.1) in its homogeneous form

$$\begin{aligned}
&\Delta v(t,x) - v_{tt}(t,x) = 0,\ \forall (t,x) \in \mathcal{T}_T \times \Omega,\\
&v(t,y) = 0,\ \forall (t,y) \in \overline{\mathcal{T}_T} \times \partial\Omega,\\
&v(0,x) = 0,\ \forall x \in \overline{\Omega},\\
&v_t(0,x) = 0,\ \forall x \in \overline{\Omega}.
\end{aligned} \tag{7.2.2}$$

To the function v, we attach the function E defined by

$$E(t) = \frac{1}{2}\int_\Omega \left[v_t^2(t,\xi) + \sum_{i=1}^n v_{x_i}^2(t,\xi) \right] \mathrm{d}\xi, \tag{7.2.3}$$

which will be called *the integral of energy*.

We will write the proof in two steps. In the first step, we will show that $E(0) = 0$, and in the second step, we will prove that

$$\frac{dE(t)}{\mathrm{d}t} = 0,$$

from where we will be led to the conclusion that, in fact, $E(t)$ is a constant. But, according to the first step, $E(0) = 0$ and then the conclusion will be that $E \equiv 0$. We will couple this conclusion with the definition (7.2.3) of the function E and we will deduce that

$$v_t = 0,\ v_{x_i} = 0,\ i = 1, 2, \ldots, n,$$

which proves that v is a constant. But on the boundary, the function v is null and then we deduce that this constant is null, that is, $v \equiv 0$ and therefore $u_1 \equiv u_2$.

We will prove immediately the first step. We directly replace $t = 0$ and we obtain

$$E(0) = \frac{1}{2}\int_\Omega \left[v_t^2(0,\xi) + \sum_{i=1}^n v_{x_i}^2(0,\xi) \right] \mathrm{d}\xi = 0,$$

in which we used the initial conditions $(7.2.2)_3$ and $(7.2.2)_4$.

We approach now the second step. Due to the conditions of regularity which are satisfied by the function v, we can differentiate in (7.2.3) under the integral sign

$$\frac{dE(t)}{\mathrm{d}t} = \int_\Omega \left[v_t(t,\xi)v_{tt}(t,\xi) + \sum_{i=1}^{n} v_{x_i}(t,\xi)v_{tx_i}(t,\xi) \right] \mathrm{d}\xi. \tag{7.2.4}$$

But we have

$$\begin{aligned}\int_\Omega v_{x_i}(t,\xi)v_{tx_i}(t,\xi)\mathrm{d}\xi &= \int_\Omega \frac{\partial}{\partial x_i}\left[v_{x_i}(t,\xi)v_t(t,\xi)\right]\mathrm{d}\xi \\ -\int_\Omega v_t(t,\xi)v_{x_ix_i}(t,\xi)\mathrm{d}\xi &= \int_{\partial\Omega} v_{x_i}(t,\xi)v_t(t,\xi)\cos\alpha_i \mathrm{d}\sigma_\xi \\ -\int_\Omega v_t(t,\xi)v_{x_ix_i}(t,\xi)\mathrm{d}\xi &= -\int_\Omega v_t(t,\xi)v_{x_ix_i}(t,\xi)\mathrm{d}\xi,\end{aligned} \tag{7.2.5}$$

in which we used, first, Gauss–Ostrogradski's formula (and this was possible, since the surface $\partial\Omega$ admits a tangent plane that is continuously varying almost everywhere). Then we used the boundary condition $(7.2.2)_2$.

From (7.2.5) we get

$$\int_\Omega \sum_{i=1}^{n} v_{x_i}(t,\xi)v_{x_ix_i}(t,\xi)\mathrm{d}\xi = -\int_\Omega v_t(t,\xi)\Delta v(t,\xi)\mathrm{d}\xi,$$

and therefore (7.2.4) becomes

$$\frac{dE(t)}{\mathrm{d}t} = \int_\Omega v_t(t,\xi)\left[v_{tt}(t,\xi) - \Delta v(t,\xi)\right]\mathrm{d}\xi = 0,$$

because v satisfies the homogeneous equation $(7.2.2)_1$. ■

In the following, we will prove a result of stability for the solution of the problem (7.2.1), with respect to the right-hand side and the initial conditions.

Theorem 7.2.2 *Let $u_1(t,x)$ and $u_2(t,x)$ be the solutions of the problems*

$$\begin{aligned}&\Delta u_i(t,x) - \frac{\partial^2 u_i}{\partial t^2}(t,x) = f_i(t,x),\ \forall (t,x) \in \mathcal{T}_T \times \Omega,\\ &u_i(t,y) = \alpha(t,y),\ \forall x \in \overline{\mathcal{T}_T} \times \partial\Omega,\\ &u_i(0,x) = \varphi_i(x),\ \forall x \in \overline{\Omega},\\ &\frac{\partial u_i}{\partial t}(0,x) = \psi_i(x),\ \forall x \in \overline{\Omega},\end{aligned}$$

where $i = 1, 2$.

Suppose that for $\forall\varepsilon > 0,\ \exists\delta = \delta(\varepsilon) > 0$ *such that*

$$
\begin{aligned}
&|f_1(t,x) - f_2(t,x)| < \delta,\\
&|\varphi_1(t,x) - \varphi_2(t,x)| < \delta,\\
&\left|\frac{\partial\varphi_1}{\partial x_i}(t,x) - \frac{\partial\varphi_1}{\partial x_i}(t,x)\right| < \delta,\\
&|\psi_1(t,x) - \psi_2(t,x)| < \delta.
\end{aligned}
$$

Then we have

$$|u_1(t,x) - u_2(t,x)| < \varepsilon.$$

Proof Denote by $u(t,x)$ the difference between the two solutions

$$u(t,x) = u_1(t,x) - u_2(t,x).$$

Now, we attach to the function $u(t,x)$ the integral of energy

$$E(t) = \frac{1}{2}\int_\Omega \left[u_t^2(t,\xi) + \sum_{i=1}^n u_{x_i}^2(t,\xi)\right] \mathrm{d}\xi. \tag{7.2.6}$$

Due to the conditions of regularity satisfied by the function u, we can differentiate under the integral sign in relation (7.2.6)

$$
\begin{aligned}
\frac{dE(t)}{\mathrm{d}t} = &\int_\Omega u_t(t,\xi)\left[u_{tt}(t,\xi) - \Delta u(t,\xi)\right] \mathrm{d}\xi\\
&+ \int_{\partial\Omega} u_t(t,\xi) \sum_{i=1}^n u_{x_i}(t,\xi)\cos\alpha_i \mathrm{d}\sigma_\xi,
\end{aligned}
\tag{7.2.7}
$$

where we used Gauss–Ostrogradski's formula, as in the proof of Theorem 7.2.1. But on the boundary we have

$$\frac{\partial u}{\partial x_i} = \frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i} = \frac{\partial \alpha}{\partial x_i} - \frac{\partial \alpha}{\partial x_i} = 0. \tag{7.2.8}$$

Also

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \Delta u = \frac{\partial^2 u_1}{\partial t^2} - \Delta u_1 - \frac{\partial^2 u_2}{\partial t^2} + \Delta u_2 =\\
- f_1(t,x) + f_2(t,x).
\end{aligned}
$$

If we denote by $f(t,x) = f_1(t,x) - f_2(t,x)$ and if we take into account (7.2.8), then the derivative from (7.2.7) becomes

$$\frac{dE(t)}{\mathrm{d}t} = -\int_{\Omega} u_t(t,\xi) f(t,\xi)\mathrm{d}\xi. \tag{7.2.9}$$

It is elementary to prove the inequality

$$\pm ab \le \frac{a^2}{2} + \frac{b^2}{2}, (*)$$

so that from (7.2.9) we deduce that

$$\frac{dE(t)}{\mathrm{d}t} = \frac{1}{2}\int_{\Omega} u_t^2(t,\xi)\mathrm{d}\xi + \frac{1}{2}\int_{\Omega} f^2(t,\xi)\mathrm{d}\xi. \tag{7.2.10}$$

Based on the assumption that

$$|f(t,x)| = |f_1(t,x) - f_2(t,x)| < \delta$$

we deduce that the last integral from (7.2.10) is as small as possible. We use the notation

$$A(t) = \frac{1}{2}\int_{\Omega} f^2(t,\xi)\mathrm{d}\xi.$$

Taking into account (7.2.6), it is clear that

$$\frac{1}{2}\int_{\Omega} u_t^2(t,\xi)\mathrm{d}\xi \le E(t),$$

and then (7.2.10) becomes

$$\frac{dE(t)}{\mathrm{d}t} \le E(t) + A(t), \tag{7.2.11}$$

so that if we multiply in both members with e^{-t}, we are led to the following inequality:

$$\frac{d}{\mathrm{d}t}\left[e^{-t}E(t)\right] \le A(t)e^{-t}.$$

We integrate here on the interval $[0,t]$ so that we obtain

$$e^{-t}E(t) \le E(0) + \int_0^t e^{-\tau}A(\tau)\mathrm{d}\tau,$$

and this relation can be rewritten in the form

$$E(t) \le e^t E(0) + \int_0^t e^{t-\tau}A(\tau)\mathrm{d}\tau.$$

Because $t \in (0, T]$, the last inequality leads to

$$E(t) \leq e^T E(0) + \int_0^T e^{T-\tau} A(\tau)\mathrm{d}\tau. \tag{7.2.12}$$

Using the hypotheses of the theorem, we deduce that $E(0)$ is as small as possible and because also $A(t)$ is no matter how small, we deduce that also the integral from (7.2.12) is as small as possible. Therefore the function $E(t)$ is dominated by a constant which can be made as small as possible. To show that u is as small as possible, we define the function $E_1(t)$ by

$$E_1(t) = \frac{1}{2} \int_\Omega u^2(t, \xi)\mathrm{d}\xi. \tag{7.2.13}$$

Based on the hypotheses of regularity of the function u, we deduce that we can differentiate under the integral sign in (7.2.13) and then we obtain

$$\begin{aligned} \frac{dE_1(t)}{\mathrm{d}t} &= \int_\Omega u(t, \xi) u_t(t, \xi)\mathrm{d}\xi \\ &\leq \frac{1}{2} \int_\Omega u_t^2(t, \xi)\mathrm{d}\xi + \frac{1}{2} \int_\Omega u^2(t, \xi)\mathrm{d}\xi, \end{aligned}$$

where we used again the above elementary inequality (*).

In this way, we proved that

$$\frac{dE_1(t)}{\mathrm{d}t} \leq E_1(t) + E(t),$$

and we will proceed analogously as for the inequality (7.2.11), so that we obtain

$$E_1(t) \leq e^T E_1(0) + \int_0^T e^{T-\tau} E(\tau)\mathrm{d}\tau.$$

Because $E_1(0)$ is as small as possible, and because we proved that $E(t)$ is as small as possible, we deduce that $E_1(t)$ is arbitrarily small and then u is as small as possible. ■

7.3 The Cauchy Problem

The mixed initial boundary values problems from the previous paragraph include the conditions imposed on the surface which borders the body on which a problem is formulated. In this paragraph, it is assumed that the surface is at an appreciably high distance so that we can consider that the domain on which we formulated the problem is the whole space. Therefore the boundary condition disappears from the formulation of the problem.

We will consider the problem with initial conditions, that is, the Cauchy problem, in the Euclidean space with three dimensions $\mathbb{R}^3$.

Let us consider the problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x,y,z)-a^2\Delta u(t,x,y,z)=f(t,x,y,z),\ \forall(t,x,y,z)\in(0,+\infty)\times\mathbb{R}^3,\\
&u(0,x,y,z)=\varphi(x,y,z),\ \forall(x,y,z)\in\mathbb{R}^3,\\
&\frac{\partial u}{\partial t}(0,x,y,z)=\psi(x,y,z),\ \forall(x,y,z)\in\mathbb{R}^3,
\end{aligned} \tag{7.3.1}$$

in which the functions f, φ, and ψ are given and are continuous on their domain of definition and a is a positive known constant of the material.

A function $u=u(t,x,y,z)$ is called *a classical solution* of the problem (7.3.1) if it satisfies the conditions

- u and all its derivatives of the first order are continuous functions on $[0,+\infty)\times\mathbb{R}^3$;
- the homogeneous derivatives of second order of the function u are continuous functions on $(0,+\infty)\times\mathbb{R}^3$;
- u verifies Eq. $(7.3.1)_1$ and satisfies the boundary conditions $(7.3.1)_2$ and $(7.3.1)_3$.

We define the function $u(t,x,y,z)$ by

$$u(t,x,y,z)=U_f(t,x,y,z)+W_\psi(t,x,y,z)+V_\varphi(t,x,y,z), \tag{7.3.2}$$

where the functions $U_f(t,x,y,z)$, $W_\psi(t,x,y,z)$, $V_\varphi(t,x,y,z)$ have, by definition, the expressions

$$\begin{aligned}
U_f(t,x,y,z)&=\frac{1}{4\pi a^2}\int_{B(x,y,z,at)}\frac{f(\xi,\eta,\zeta,t-r/a)}{r}\mathrm{d}\xi\mathrm{d}\eta d\zeta,\\
W_\psi(t,x,y,z)&=\frac{1}{4\pi a^2}\int_{\partial B(x,y,z,at)}\frac{\psi(\xi,\eta,\zeta)}{t}\mathrm{d}\sigma_{at},\\
V_\varphi(t,x,y,z)&=\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\left(\int_{\partial B(x,y,z,at)}\frac{\psi(\xi,\eta,\zeta)}{t}\mathrm{d}\sigma_{at}\right),
\end{aligned} \tag{7.3.3}$$

where

$$r=|\xi x|=\sqrt{\sum_{i=1}^{3}(x_i-\xi_i)^2}=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}.$$

Also, in the formulas (7.3.3), $B(x, y, z, at)$ is a ball with the center in the point of coordinates (x, y, z) having the radius at. The surface $\partial B(x, y, z, at)$ is the boundary of this ball, that is, the sphere with the same center and the same radius.

In the following theorem, we will prove the central result of this paragraph. Namely, we show that the function u defined in (7.3.2) is an effective classical solution of the Cauchy problem (7.3.1).

Theorem 7.3.1 *If $f \in C^2((0, +\infty))$, $\varphi \in C^3(\mathbb{R}^3)$ and $\psi \in C^3(\mathbb{R}^3)$, then the function u defined in (7.3.2) is a classical solution of the Cauchy problem (7.3.1).*

Proof We write the proof in three steps. In the first step, we prove that the function W_ψ from $(7.3.3)_2$ is a solution of the problem

$$\begin{aligned}
&\frac{\partial^2 W_\psi}{\partial t^2}(t, x, y, z) - a^2 \Delta W_\psi(t, x, y, z) = 0, \ \forall (t, x, y, z) \in (0, +\infty) \times \mathbb{R}^3, \\
&W_\psi(0, x, y, z) = 0, \ \forall (x, y, z) \in \mathbb{R}^3, \\
&\frac{\partial W_\psi}{\partial t}(0, x, y, z) = \psi(x, y, z), \ \forall (x, y, z) \in \mathbb{R}^3.
\end{aligned} \tag{7.3.4}$$

In the second step, we prove that the function V_φ from $(7.3.3)_3$ is a solution of the problem

$$\begin{aligned}
&\frac{\partial^2 V_\varphi}{\partial t^2}(t, x, y, z) - a^2 \Delta V_\varphi(t, x, y, z) = 0, \ \forall (t, x, y, z) \in (0, +\infty) \times \mathbb{R}^3, \\
&V_\varphi(0, x, y, z) = \varphi(x, y, z), \ \forall (x, y, z) \in \mathbb{R}^3, \\
&\frac{\partial V_\varphi}{\partial t}(0, x, y, z) = 0, \ \forall (x, y, z) \in \mathbb{R}^3,
\end{aligned} \tag{7.3.5}$$

and in the third step, we prove that the function U_f from $(7.3.3)_1$ is a solution of the problem

$$\begin{aligned}
&\frac{\partial^2 U_f}{\partial t^2}(t, x, y, z) - a^2 \Delta U_f(t, x, y, z) = f(t, x, y, z), \ \forall (t, x, y, z) \in (0, +\infty) \times \mathbb{R}^3 \\
&V_\varphi(0, x, y, z) = \varphi(x, y, z), \ \forall (x, y, z) \in \mathbb{R}^3 \\
&\frac{\partial V_\varphi}{\partial t}(0, x, y, z) = 0, \ \forall (x, y, z) \in \mathbb{R}^3.
\end{aligned} \tag{7.3.6}$$

If the three steps are proved, then, by taking into account (7.3.2), we deduce that

$$\frac{\partial^2 u}{\partial t^2} - a^2 \Delta u = \frac{\partial^2 U_f}{\partial t^2} - a^2 \Delta U_f + \frac{\partial^2 W_\psi}{\partial t^2} - a^2 \Delta W_\psi$$
$$+ \frac{\partial^2 V_\varphi}{\partial t^2} - a^2 \Delta V_\varphi = f(t, x, y, z) + 0 + 0 = f(t, x, y, z).$$

Then

$$u(0, x, y, z) = U_f(0, x, y, z) + W_\psi(0, x, y, z)$$
$$+ V_\varphi(0, x, y, z) = 0 + 0 + \varphi(x, y, z) = \varphi(x, y, z),$$

and, finally,

$$\frac{\partial u}{\partial t}(0, x, y, z) = \frac{\partial U_f}{\partial t}(0, x, y, z) + \frac{\partial W_\psi}{\partial t}(0, x, y, z)$$
$$+ \frac{\partial V_\varphi}{\partial t}(0, x, y, z) = 0 + \psi(x, y, z) + 0 = \psi(x, y, z),$$

that is, the function u from (7.3.2) actually verifies the problem (7.3.1) and the proof will be complete.

Step I.

Denote by M the points of the coordinates (x, y, z), and then we can write W_ψ in the form

$$W_\psi(t, x, y, z) = \frac{1}{4\pi a^2 t} \int_{\partial B(M,at)} \psi(\xi, \eta, \zeta) \mathrm{d}\sigma_{at},$$

where $\mathrm{d}\sigma_{at}$ is the element of the area on the sphere of radius at.

We make the change of variables $(\xi, \eta, \zeta) \to (\alpha, \beta, \gamma)$ given by

$$\begin{aligned} \xi &= x + \alpha a t, \\ \eta &= y + \beta a t, \\ \zeta &= z + \gamma a t. \end{aligned} \tag{7.3.7}$$

Then we obtain

$$\alpha^2 + \beta^2 + \gamma^2 = \frac{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}{a^2 t^2} = 1,$$

that is, the point of coordinates (α, β, γ) is on the unit sphere $B(M, 1)$. Consequently, the function W_ψ receives the form

$$W_\psi(t, x, y, z) = \frac{t}{4\pi} \int_{\partial B(M,1)} \psi(x + \alpha a t, y + \beta a t, z + \gamma a t) \mathrm{d}\sigma_1. \tag{7.3.8}$$

Taking into account that the function ψ was assumed to be of class C^2 and that we are on a compact set (the unit sphere), we have

$$\left|W_\psi(t,x,y,z)\right| \le \frac{|t|c_0}{4\pi}\int_{\partial B(M,1)} \mathrm{d}\sigma_1 = tc_0,$$

and then $W_\psi(t,x,y,z) \to 0$, for $t \to 0^+$, uniformly with respect to x, y, z, that is, W_ψ satisfies the condition $(7.3.4)_2$.

We differentiate now in relation (7.3.8), with respect to t

$$\begin{aligned}\frac{\partial W_\psi}{\partial t}(t,x,y,z) &= \frac{1}{4\pi}\int_{\partial B(M,1)} \psi(x+\alpha at, y+\beta at, z+\gamma at)\mathrm{d}\sigma_1 \\ &+ \frac{at}{4\pi}\int_{\partial B(M,1)}\left[\alpha\frac{\partial\psi(x+\alpha at, y+\beta at, z+\gamma at)}{\partial(x+\alpha at)}\right. \\ &\left. + \beta\frac{\partial\psi(x+\alpha at, y+\beta at, z+\gamma at)}{\partial(y+\beta at)} + \gamma\frac{\partial\psi(x+\alpha at, y+\beta at, z+\gamma at)}{\partial(z+\zeta at)}\right]\mathrm{d}\sigma_1.\end{aligned} \tag{7.3.9}$$

We denote by I_2 the last integral from (7.3.9) and we notice that the integrand is the derivative in the direction of the normal. Then we can deduce

$$\begin{aligned}|I_2| &\le \frac{at}{4\pi}\int_{\partial B(M,1)}\left|\frac{\partial\psi(x+\alpha at, y+\beta at, z+\gamma at)}{\partial\nu}\right|\mathrm{d}\sigma_1 \\ &\le \frac{atc_1}{4\pi}\int_{\partial B(M,1)}\mathrm{d}\sigma_1 = \frac{atc_1}{4\pi}4\pi = atc_1,\end{aligned}$$

in which c_1 is the supremum of the derivative in the direction of the normal, which exists due to the conditions of regularity imposed on the function ψ.

Then $I_2 \to 0$, as $t \to 0^+$, uniformly with respect to x, y, z.

For the first integral from the right-hand side of the relation (7.3.9), which will be denoted by I_1, we will apply the mean value theorem.

According to this, there is a point $(\alpha^*, \beta^*, \gamma^*) \in \partial B(M,1)$ so that

$$\begin{aligned}I_1 &= \frac{1}{4\pi}\int_{\partial B(M,1)}\psi(x+\alpha at, y+\beta at, z+\gamma at)\mathrm{d}\sigma_1 \\ &= \frac{1}{4\pi}\psi(x+\alpha^* at, y+\beta^* at, z+\gamma^* at)\int_{\partial B(M,1)}\mathrm{d}\sigma_1 \\ &= \psi(x+\alpha^* at, y+\beta^* at, z+\gamma^* at).\end{aligned}$$

Then, clearly, $I_1 \to \psi(x,y,z)$, as $t \to 0^+$, uniformly with respect to x, y, z. In conclusion, if we pass to the limit in (7.3.9) with $t \to 0^+$, we obtain

$$\lim_{t\to 0^+} \frac{\partial W_\psi}{\partial t}(t, x, y, z) = \psi(x, y, z),$$

that is, W_ψ verifies the initial condition (7.3.4)$_3$.

We should note that (7.3.9) can be rewritten in the form

$$\frac{\partial W_\psi(t, x, y, z)}{\partial t} = \frac{W_\psi(t, x, y, z)}{t} + \frac{1}{4\pi at} \int_{\partial B(M,at)} \frac{\partial \psi(\xi, \eta, \zeta)}{\partial \nu} \mathrm{d}\sigma_{at}, \tag{7.3.10}$$

where we returned to the variables (ξ, η, ζ).

In the integral from (7.3.10), we apply Gauss–Ostrogradski's formula and then (7.3.10) becomes

$$\frac{\partial W_\psi(t, x, y, z)}{\partial t} = \frac{W_\psi(t, x, y, z)}{t} + \frac{1}{4\pi at} \int_{\partial B(M,at)} \Delta\psi(\xi, \eta, \zeta)\mathrm{d}\xi\mathrm{d}\eta d\zeta. \tag{7.3.11}$$

We denote by $I(t)$ the integral from (7.3.11) so that (7.3.11) can be written in the form

$$\frac{\partial W_\psi(t, x, y, z)}{\partial t} = \frac{W_\psi(t, x, y, z)}{t} + \frac{1}{4\pi at} I(t),$$

a relation which, after we differentiate with respect to t, leads to

$$\frac{\partial^2 W_\psi(t, x, y, z)}{\partial t^2} = \frac{1}{4\pi at} I'(t). \tag{7.3.12}$$

In order to differentiate easily in the integral $I(t)$ we use, first, the spherical coordinates

$$\begin{aligned} I(t) &= \int_{\partial B(M,at)} \Delta\psi(\xi, \eta, \zeta)\mathrm{d}\xi\mathrm{d}\eta d\zeta \\ &= \int_0^{at} \int_0^{\pi} \int_0^{2\pi} \Delta\psi(r, \theta, \varphi) r \sin\theta \mathrm{d}r\mathrm{d}\theta d\varphi. \end{aligned}$$

Then

$$\begin{aligned} I'(t) &= a^3 t^2 \int_0^{\pi} \int_0^{2\pi} \Delta\psi(r, \theta, \varphi) \sin\theta \mathrm{d}\theta d\varphi \\ &= a^3 t^2 \int_{\partial B(M,1)} \Delta\psi \mathrm{d}\sigma_1 = a \int_{\partial B(M,at)} \Delta\psi(\xi, \eta, \zeta)\mathrm{d}\sigma_{at} \end{aligned}$$

and thus (7.3.12) becomes

$$\frac{\partial^2 W_\psi(t,x,y,z)}{\partial t^2} = \frac{1}{4\pi t}\int_{\partial B(M,at)} \Delta\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at} = a^2\Delta W_\psi, \tag{7.3.13}$$

taking into account the definition $(7.3.3)_2$ for W_ψ and the fact that we can differentiate under the integral sign with respect to (ξ, η, ζ), based on the conditions of regularity of the function ψ.

The relation (7.3.13) shows that W_ψ satisfies Eq. $(7.3.4)_1$ and the first step is fully demonstrated.

Step II.

First, we observe that

$$V_\varphi(t,x,y,z) = \frac{\partial W_\varphi(t,x,y,z)}{\partial t}, \tag{7.3.14}$$

by taking into account the definition $(7.3.3)_3$ for V_φ and the definition $(7.3.3)_2$ written for W_φ (instead of W_ψ).

Then, we deduce that

$$V_\varphi(0,x,y,z) = \frac{\partial W_\varphi(0,x,y,z)}{\partial t} = \varphi(x,y,z),$$

by taking into account the first step, that is, V_φ verifies the initial condition $(7.3.5)_2$.

If we differentiate with respect to t in (7.3.14), we obtain

$$\frac{\partial V_\varphi(t,x,y,z)}{\partial t} = \frac{\partial^2 W_\varphi(t,x,y,z)}{\partial t^2} = \frac{1}{4\pi a t} I'(t), \tag{7.3.15}$$

and in the deduction of this relation we used the equality (7.3.12).

Based on the proof from step I, we have

$$I'(t) = a\int_{\partial B(M,at)} \Delta\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at},$$

and then (7.3.15) becomes

$$\begin{aligned}\frac{\partial V_\varphi(t,x,y,z)}{\partial t} &= \frac{1}{4\pi t}\int_{\partial B(M,at)} \Delta\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at}\\ &= \frac{a^2 t}{4\pi}\int_{\partial B(M,1)} \Delta\psi(x+\alpha a t, y+\beta a t, z+\gamma a t)\mathrm{d}\sigma_1.\end{aligned}$$

Thus we deduce that

$$\frac{\partial V_\varphi(t,x,y,z)}{\partial t} \to 0, \text{ as } t \to 0^+,$$

because the integral

$$\int_{\partial B(M,1)} \Delta\psi(x+\alpha at, y+\beta at, z+\gamma at)\mathrm{d}\sigma_1$$

is a bounded function, based on the conditions of regularity satisfied by the function φ.

Therefore V_φ satisfies the initial condition $(7.3.5)_3$.

Taking into account (7.3.14), we deduce that

$$\begin{aligned}\frac{\partial^2 V_\varphi}{\partial t^2} - a^2\Delta V_\varphi &= \frac{\partial^2}{\partial t^2}\left(\frac{\partial W_\varphi}{\partial t}\right) - a^2\Delta\frac{\partial W_\varphi}{\partial t}\\ &= \frac{\partial}{\partial t}\left(\frac{\partial^2 W_\varphi}{\partial t^2} - a^2\Delta W_\varphi\right) = 0,\end{aligned}$$

because in the first step we proved that

$$\frac{\partial^2 W_\varphi}{\partial t^2} - a^2\Delta W_\varphi = 0.$$

In conclusion, V_φ satisfies Eq. $(7.3.5)_1$ and the proof of the second step is finished.

Step III.

First, from $(7.3.3)_1$ we deduce immediately that

$$\lim_{t\to 0^+} U_f(t,x,y,z) = \frac{1}{4\pi a^2}\lim_{t\to 0^+}\int_{B(M,at)} \frac{f(\xi,\eta,\zeta,t-r/a)}{r}\mathrm{d}\xi\mathrm{d}\eta d\zeta = 0,$$

by taking into account the regularity of the function f and the fact that, at the limit, the ball $B(M,at)$ is reduced to the point (x,y,z).

Therefore the function U_f satisfies the initial condition $(7.3.6)_2$. We now write U_f in the form

$$\begin{aligned}U_f(t,x,y,z) &= \frac{1}{4\pi a^2}\int_0^t\left\{\int_{\partial B(M,\varrho)}\frac{f(\xi,\eta,\zeta,t-\varrho/a)}{\varrho}d\sigma_\varrho\right\}d\varrho\\ &= \frac{1}{4\pi a^2}\int_0^t\left\{\int_{\partial B(0,1)} f(x+\alpha\varrho, y+\beta\varrho, z+\gamma\varrho, t-\varrho)d\sigma_\varrho\right\}\varrho d\varrho.\end{aligned} \quad (7.3.16)$$

then

$$\begin{aligned}\frac{\partial U_f(t,x,y,z)}{\partial t} &= \frac{1}{4\pi a^2}\int_{\partial B(0,1)} f(x+\alpha\varrho, y+\beta\varrho, z+\gamma\varrho, t-\varrho)t d\sigma_\varrho\\ &+ \frac{1}{4\pi a^2}\int_0^t\left\{\int_{\partial B(0,1)} f(x+\alpha\varrho, y+\beta\varrho, z+\gamma\varrho, t-\varrho)d\sigma_\varrho\right\}\varrho d\varrho.\end{aligned} \quad (7.3.17)$$

The second integral from (7.3.17) disappears for $t = 0$. For the first integral, we use the mean value theorem and then this integral becomes the product between t and a bounded constant and therefore tends to zero, as $t \to 0^+$, that is,

$$\lim_{t\to 0^+} \frac{\partial U_f(t, x, y, z)}{\partial t} = 0,$$

the limit taking place uniformly with respect to (x, y, z). Therefore U_f also satisfies the initial condition $(7.3.6)_3$.

It only remains to prove that U_f verifies Eq. $(7.3.6)_1$. For this, we introduce the notation

$$\begin{aligned} &U_1(t, \tau, x, y, z) = \\ &= \frac{t-\tau}{4\pi a^2} \int_{\partial B(0,1)} f(x+(t-\tau)\xi, y+(t-\tau)\eta, z+(t-\tau)\zeta, t-\varrho)\mathrm{d}\sigma_1. \end{aligned} \tag{7.3.18}$$

Then (7.3.16) becomes

$$U_f(t, x, y, z) = \int_0^t U_1(t, \tau, x, y, z)\mathrm{d}\tau. \tag{7.3.19}$$

Starting from (7.3.18), we obtain without difficulty the relations

$$\begin{aligned} &\frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2} - \Delta U_1(t, \tau, x, y, z) = 0, \\ &U_1(t, t, x, y, z) = 0, \\ &\frac{\partial U_1(t, t, x, y, z)}{\partial t} = f(t, x, y, z). \end{aligned} \tag{7.3.20}$$

Then from (7.3.19) we deduce that

$$\begin{aligned} \frac{\partial^2 U_f(t, x, y, z)}{\partial t^2} &= \frac{\partial U_1(t, t, x, y, z)}{\partial t} + \int_0^t \frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2}\mathrm{d}\tau \\ &= f(t, x, y, z) + \int_0^t \frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2}\mathrm{d}\tau. \end{aligned} \tag{7.3.21}$$

On the other hand, we have

$$\begin{aligned} \Delta U_f(t, x, y, z) &= \int_0^t \Delta U_1(t, \tau, x, y, z)\mathrm{d}\tau \\ &= \int_0^t \frac{\partial^2 U_1(t, \tau, x, y, z)}{\partial t^2}\mathrm{d}\tau, \end{aligned} \tag{7.3.22}$$

in which we take into account $(7.3.20)_1$.

From (7.3.21) and (7.3.22), by subtracting member by member, we deduce that

$$\frac{\partial^2 U_f(t,x,y,z)}{\partial t^2} - \Delta U_f(t,x,y,z) = f(t,x,y,z),$$

that is, the function U_f verifies the nonhomogeneous equation $(7.3.6)_1$ and the proof of the third step is finished and therewith, the proof of the theorem is concluded. ■

Formula (7.3.2) which gives the form of the solution of the Cauchy problem (7.3.1) is known under the name of *the formula of Kirchhoff*.

The formula of Kirchhoff can be used to prove the uniqueness of the solution of the Cauchy problem. Indeed, if the problem (7.3.1) admits two solutions, say $u_1(t,x,y,z)$ and $u_2(t,x,y,z)$, then we denote by $u(t,x,y,z)$ their difference, $u(t,x,y,z) = u_1(t,x,y,z) - u_2(t,x,y,z)$. Then $u(t,x,y,z)$ satisfies the Cauchy problem of the form (7.3.1) in which $f \equiv 0$, $\varphi \equiv 0$ and $\psi \equiv 0$. If we write the formula of Kirchhoff for u, obviously we obtain $u \equiv 0$ from where we deduce that $u_1 \equiv u_2$.

Also, with the help of the formula of Kirchhoff, we can prove a result of stability for the solution of the Cauchy problem (7.3.1), with respect to the right-hand side and the initial conditions.

Part II
Solutions in Distributions

Chapter 8
Elements of Distributions

8.1 Spaces of Distributions

One of the essential reasons for which the concept of distribution was introduced is the necessity to solve a differential equation in weaker conditions of regularity.

In the chapters that follows, the main types of partial differential equations will be solved both in the classic manner and also in the context of distributions.

To achieve the relative independence of our exposure, we will present in the beginning some notions of the theory of distributions.

Let Ω be an open set (bounded or not) from the space $\mathbb{R}^n$. We will denote, as usual, by $\overline{A}$ the closure of the set A. We introduce the notion of *the support of a function* φ, defined on Ω by

$$\text{supp } \varphi(x) = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}.$$

Also, the following notations are well known:

$$C^{\infty}(\Omega) = \{\varphi = \varphi(x) : D^k \varphi \in C(\Omega)\},$$
$$C_0^{\infty}(\Omega) = \{\varphi \in C^{\infty}(\Omega) : \text{supp } \varphi = \text{compact } \subset \Omega)\}, \tag{8.1.1}$$

where $k = (k_1, k_2, \dots, k_n)$, $k_i \in \mathbb{N}$, $i = 1, 2, \dots, n$ is called *a multi-index* and by D^k we denote

$$D^k \varphi(x) = \frac{\partial^{k_1 k_2 \dots k_n} \varphi(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} = \frac{\partial^{|k|} \varphi(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}},$$
$$|k| = |(k_1, k_2, \dots, k_n)| = \sum_{i=1}^{n} k_i.$$

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
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If the set Ω is bounded, then any of its subset is bounded and because supp φ is a closed set, we deduce that supp φ is a compact set and then the two sets from (8.1.1) are equivalent.

Analogously, for a fixed natural number m, which can have also the value zero, we will use the notations

$$C^m(\Omega) = \{\varphi = \varphi(x) : D^k\varphi \in C(\Omega),\ |k| \leq m\},$$
$$C_0^m(\Omega) = \{\varphi \in C^m(\Omega) : \text{supp}\ \varphi(x) = \text{compact}\ \subset \Omega)\}. \tag{8.1.2}$$

In the following, we will denote the sets $C_0^\infty(\Omega)$ from (8.1.1) and $C_0^m(\Omega)$ from (8.1.2), by $\mathcal{D}(\Omega)$ and $\mathcal{D}^m(\Omega)$, respectively.

We want to introduce a topology on $\mathcal{D}^m(\Omega)$. To this end, we consider the sequence $\{Q_n\}_{n\geq 1}$ of compact sets from Ω, having the properties

$$Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_n \subseteq \ldots$$
$$\Omega = \bigcup_{i=1}^{n} Q_i. \tag{8.1.3}$$

On each compact set Q_i, we define a semi-norm as follows:

$$|\varphi|_{Q_{i,j}} = \sup_{x\in Q_i, |k|\leq j} \left|D^k\varphi(x)\right|,\ \forall \varphi \in \mathcal{D}^m(\Omega), \tag{8.1.4}$$

for i and j arbitrarily fixed, $i = 1, 2, \ldots,\ j = 1, 2, \ldots$

We can verify, without difficulty, the fact that, in fact, for i and j fixed in (8.1.4) we have just a norm.

Starting from (8.1.4), we define

$$d_i(\varphi, \psi) = \sum_{j=1}^{m} \frac{1}{2^j} \frac{|\varphi - \psi|_{Q_{i,j}}}{1 + |\varphi - \psi|_{Q_{i,j}}} \tag{8.1.5}$$

so that for a fixed i we obtain a metric on the space $\mathcal{D}^m(\Omega)$. Therefore if we consider a fixed compact set, by using (8.1.3), we obtain a topology of the metric space. As it is known, on each compact set Q_i, we have the topology of a locally convex space.

If we compare the semi-norms introduced in (8.1.4) for two compact sets which are different, we obtain

$$|\varphi|_{Q_{i+1,j}} \geq |\varphi|_{Q_{i,j}};\ Q_i \subseteq Q_{i+1}. \tag{8.1.6}$$

But the restriction of the topology defined on Q_{i+1} to Q_i coincides with the topology defined on Q_i. Because the set $\mathcal{D}^m(\Omega)$ can be represented as a reunion of sets of the functions defined on compact sets Q_i, we deduce that we can talk of *the topology of the inductive limit* defined with the help of the semi-norms (8.1.4) on Ω.

Thus, a neighborhood of an element in $\mathcal{D}^m(\Omega)$ is any subset of $\mathcal{D}^m(\Omega)$ which has the property that if it is intersected with each of the locally convex spaces corresponding to a compact set Q_i, with i fixed, then a neighborhood of the respective locally convex space is obtained.

We want to outline that this topology is one of the strict inductive limits, because

1. $\mathcal{D}^m(\Omega)$ is represented as a reunion of the subspaces which correspond to each Q_i, with i fixed;

2. The restriction of the topology defined by the semi-norms which correspond to the compact set Q_{i+1} at the corresponding space of Q_i coincides with the topology of Q_i.

Moreover, we want to outline that, in the particular case $m = 0$, therefore in the case of the space $\mathcal{D}^0(\Omega)$, the semi-norms defined in (8.1.4) depend only on compact sets Q_i, because we have no ordering with respect to j, which is, commonly, called the *order of derivation*.

Finally, in the case of the set $\mathcal{D}^\infty(\Omega) = C_0^\infty(\Omega)$ a topology of the strict inductive limit can be defined analogously. In this case, analogously with (8.1.5), we have the following metric:

$$d_i(\varphi, \psi) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\varphi - \psi|_{Q_{i,j}}}{1 + |\varphi - \psi|_{Q_{i,j}}}. \tag{8.1.7}$$

Taking into account the topologies that were introduced, the definition of convergence for each space is natural.

Definition 8.1.1 In the space $\mathcal{D}(\Omega)$, we say that the sequence $\{\varphi_\nu\}_{\nu \geq 1}$ is convergent to φ, where $\varphi, \ \varphi_\nu \in \mathcal{D}(\Omega)$, $\nu = 1, 2, \ldots$, if, by definition, the following two conditions are met:

1. There is a compact set Q suitable for all indices ν, so that

$$supp\ \varphi_\nu \subset Q \subset \Omega, \ supp\ \varphi \subset Q \subset \Omega;$$

2. $D^k \varphi_\nu(x) \to D^k \varphi(x)$, uniformly on Q, $\forall k = (k_1, k_2, \ldots, k_n)$.

Analogously, in the case of the space $\mathcal{D}^m(\Omega)$, where m is a fixed natural number, we have that $\mathcal{D}^m(\Omega) \ni \varphi_\nu \to \varphi \in \mathcal{D}^m(\Omega)$ if, by definition

1. There is a compact set Q, suitable for all indices ν, so that

$$supp\ \varphi_\nu \subset Q \subset \Omega, \ supp\ \varphi \subset Q \subset \Omega;$$

2. $D^k \varphi_\nu(x) \to D^k \varphi(x)$, uniformly on Q, $\forall k = (k_1, k_2, \ldots, k_n)$, $|k| \leq m$.

The spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}^m(\Omega)$, equipped with the mentioned topologies, are called *the fundamental space of distributions* and *the fundamental space of distributions of order m*, respectively.

In the particular case $m = 0$, we have the space $\mathcal{D}^0(\Omega)$ which is called *the fundamental space of distributions of order 0* or *the fundamental space of measures*.

One uses also the name of the space *of test functions*, in the case of the space $\mathcal{D}(\Omega)$, and *of the test functions of order m* in the case of the space $\mathcal{D}^m(\Omega)$, respectively.

Example 8.1.1 We define the function $\varphi(x, \varepsilon)$ by

$$\varphi(x,\varepsilon) = \begin{cases} e^{-\varepsilon^2/(\varepsilon^2-r^2)} & , \text{ if } r < \varepsilon \\ 0 & , \text{ if } r \geq \varepsilon, \end{cases}$$

where $r = |\overline{0x}| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

We can show, without difficulty, that for $\forall \varepsilon > 0$, the function $\varphi(x, \varepsilon)$ is of class C^∞ with respect to x, that is, $D_x^k \varphi(x, \varepsilon) \in C^\infty(\mathbb{R}^n)$.

Also, note that $supp\ \varphi(x, \varepsilon) = \overline{B(0, \varepsilon)}$, from where we are led to the conclusion that $\varphi(x, \varepsilon) \in \mathcal{D}(\mathbb{R}^n)$.

In the proposition which follows, we highlight a property of the notion of convergence, introduced in Definition 8.1.1.

Proposition 8.1.1 *If the sequence $\{\varphi_\nu\}_{\nu \geq 1}$ is convergent to φ in the sense of the space $\mathcal{D}(\mathbb{R}^n)$, then for any function $\psi \in C^\infty(\mathbb{R}^n)$ we have that the sequence $\{\psi\varphi_\nu\}_{\nu\geq 1}$ is convergent to $\psi\varphi$ in the sense of $\mathcal{D}(\mathbb{R}^n)$.*

Proof 1^o The compact set Q, which is suitable for all indices of the sequence $\{\varphi_\nu\}_{\nu \geq 1}$, is obviously suitable also for all indices of the sequence $\{\psi\varphi_\nu\}_{\nu\geq 1}$ because $supp\ (\psi\varphi_\nu) \subseteq supp\ \varphi_\nu$.

2^o Since $\psi \in C^\infty(\mathbb{R}^n)$ and $D^k\varphi_\nu(x) \to D^k\varphi(x)$, uniformly on $\mathbb{R}^n$, taking into account the classic rule of differentiation of the product of functions, we immediately obtain that $D^k((\psi\varphi_\nu)(x)) \to D^k(\psi\varphi)(x)$. ∎

Example 8.1.2 For exemplifying the convergence on the space $\mathcal{D}(\Omega)$, consider the sequence $\{\varphi_\nu(x)\}_{\nu\geq 1}$ defined by

$$\varphi_\nu(x) = \frac{1}{\nu}\varphi(x, \varepsilon),\ \ \nu = 1, 2, \ldots,$$

where $\varphi(x, \varepsilon)$ is the test function defined in Example 8.1.1. It can be proved without difficulty that $\varphi_\nu(x) \to 0$ in the sense of $\mathcal{D}(\Omega)$.

But, if we consider the sequence $\{\psi_\nu(x)\}_{\nu\geq 1}$ defined by

$$\psi_\nu(x) = \frac{1}{\nu}\varphi\left(\frac{x}{\nu}, \varepsilon\right),$$

then it can be immediately deduced that all derivatives are convergent uniformly on compact sets which are different, but no compact set exists that is suitable for all indices ν so that we have $supp\ \psi_\nu \subset Q$. Therefore, the sequence $\{\psi_\nu\}_{\nu\geq 1}$ is not convergent in the sense of $\mathcal{D}(\Omega)$.

In the following, we will make some considerations in the case when $\Omega = \mathbb{R}^n$. The notions introduced above will be adapted accordingly.

Definition 8.1.2 We call a distribution (of order ∞) any linear and continuous functional defined on the space $\mathcal{D}(\mathbb{R}^n)$ with real values.

Observation 8.1.1 1^o *The continuity from the definition of a distribution must be understood in the sense of continuity of an application defined on a linear topological space.*

2^o *Denote by $\mathcal{D}'(\mathbb{R}^n)$ the set of all distributions defined on $\mathcal{D}(\mathbb{R}^n)$.*

The notation is suggested by the fact that, according to the Definition 8.1.3, the space $\mathcal{D}'(\mathbb{R}^n)$ represents the dual of the space $\mathcal{D}(\mathbb{R}^n)$.

3^o *If T is a distribution, $T \in \mathcal{D}'(\mathbb{R}^n)$, we make the convention to use the writing (T, φ) to designate the value of distribution T computed for the test function φ.*

4^o *$\mathcal{D}^{m\prime}(\mathbb{R}^n)$ can be defined analogously as the space of distributions of order m that is the dual of the space $\mathcal{D}^m(\mathbb{R}^n)$. In the particular case $m = 0$, we have the space of distributions of order zero, which are also called measures.*

Definition 8.1.3 We say that two distributions $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n)$ are equal if

$$(T_1, \varphi) = (T_2, \varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We provide now two simple, but very helpful, examples of distributions.

Example 8.1.3 Let f be a function, $f \in L^1_{loc}(\mathbb{R}^n)$, that is, f is measurable and integrable in the sense of Lebesgue on any compact set from $\mathbb{R}^n$. We define the application

$$(T_f, \varphi) \stackrel{def}{=} \int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x, \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n), \tag{8.1.8}$$

which is obviously well defined because φ is null outside a compact set from $\mathbb{R}^n$, and on each compact set $\varphi(x)$ is bounded because φ is of class C^∞. Similarly, f is a bounded function, since it is from $L^1_{loc}(\mathbb{R}^n)$.

Proposition 8.1.2 *The functional T_f defined by the correspondence (8.1.8) is a distribution.*

Proof 1^o The linearity. $\forall \lambda_1, \lambda_2 \in \mathbb{R}$ and $\forall \varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(T, \lambda_1\varphi_1 + \lambda_2\varphi_2) = \int_{\mathbb{R}^n} f(x)\left[\lambda_1\varphi_1 + \lambda_2\varphi_2\right]\mathrm{d}x$$
$$= \lambda_1 \int_{\mathbb{R}^n} f(x)\varphi_1(x)\mathrm{d}x + \lambda_2 \int_{\mathbb{R}^n} f(x)\varphi_2(x)\mathrm{d}x.$$

2^o The continuity. We will consider the sequence $\{\varphi_\nu\}_{\nu \geq 1}$, $\varphi_\nu \in \mathcal{D}(\mathbb{R}^n)$, $\nu = 1, 2, \ldots,$ which is convergent (in the sense of $\mathcal{D}(\mathbb{R}^n)$) to the function $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We have therefore

$$\lim_{\nu\to\infty}\left(T_f,\varphi_\nu\right)=\lim_{\nu\to\infty}\int_{\mathbb{R}^n} f(x)\varphi_\nu(x)\mathrm{d}x$$
$$=\int_{\mathbb{R}^n} f(x)\lim_{\nu\to\infty} f(x)\varphi_\nu(x)\mathrm{d}x=\int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x=(T_f,\varphi),$$

that is, T_f is commutative with the limit and therefore is continuous. ■

Definition 8.1.4 The distribution generated by a function which is locally integrable is called the distribution of the function type or generalized function.

Let us consider the step function, called also *the function of Heaviside*, defined by

$$u(x)=\begin{cases}0\ , & \text{if } x\le 0\\ 1\ , & \text{if } x>0.\end{cases}$$

Obviously, the function $u\in L^1_{loc}(R)$, since it is a continuous function on $\mathbb{R}\setminus\{0\}$. Then $\forall\varphi\in\mathcal{D}(\mathbb{R})$ we have

$$(T_u,\varphi)=\int_{\mathbb{R}} u(x)\varphi(x)\mathrm{d}x=\int_{-\infty}^{0} u(x)\varphi(x)\mathrm{d}x+$$
$$+\int_0^\infty u(x)\varphi(x)\mathrm{d}x=\int_0^\infty \varphi(x)\mathrm{d}x.$$

Therefore, we can talk of the distribution of the function type, which is generated by the function of Heaviside, which will be called *the distribution of Heaviside*.

It naturally occurs the question if perhaps all distributions are generated by functions which are locally integrable. We will show that the answer is negative by a counterexample.

Example 8.1.4 We define the functional δ by

$$(\delta,\varphi)=\varphi(0),\ \ \forall\varphi\in\mathcal{D}(\mathbb{R}^n). \tag{8.1.9}$$

Proposition 8.1.3 *The functional δ defined in (8.1.9) is a distribution and is called the distribution of Dirac.*

Proof 1^o The linearity. $\forall\lambda_1,\lambda_2\in R$,$\forall\varphi_1,\varphi_2\in\mathcal{D}(\mathbb{R}^n)$

$$(\delta,\lambda_1\varphi_1+\lambda_2\varphi_2)=[\lambda_1\varphi_1(x)+\lambda_2\varphi_2(x)]\,(0)=\lambda_1\varphi(0)+\lambda_2\varphi(0)$$
$$=\lambda_1(\delta,\varphi_1)+\lambda_2(\delta,\varphi_2),\ \ \forall\lambda_1,\lambda_2\in\mathbb{R},\ \ \forall\varphi_1,\varphi_2\in\mathcal{D}(\mathbb{R}^n).$$

2^o The continuity. Consider the sequence $\{\varphi_\nu(x)\}_{\nu\ge 1}$ of elements from $\mathcal{D}(\mathbb{R}^n)$, which is supposed to be convergent to $\varphi(x)\in\mathcal{D}(\mathbb{R}^n)$, in the sense of convergence from the space $\mathcal{D}(\mathbb{R}^n)$. We have

$$\lim_{\nu\to\infty} (\delta, \varphi_\nu(x)) = \lim_{\nu\to\infty} \varphi_\nu(0) = \varphi(0) = (\delta, \lim_{\nu\to\infty} \varphi_\nu(0)),$$

that is, δ is commutative with the limit and therefore is continuous. ■

With the help of the distribution of Dirac, we can argue that not any distribution is a generalized function.

Proposition 8.1.4 *The distribution δ of Dirac is not a distribution of the function type.*

Proof We assume, by contradiction, that δ is a generalized function and therefore there is a function $f \in L^1_{loc}(\mathbb{R}^n)$ so that

$$\delta = T_f,$$

the equality taking place in the sense of $\mathcal{D}'(\mathbb{R}^n)$. Thus,

$$(\delta, \varphi) = (T_f, \varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x, \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \qquad (8.1.10)$$

If the equality (8.1.10) holds true for $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, then it takes place also for the test $\varphi(x, \varepsilon)$ defined in Example 8.1.1, so that

$$(\delta, \varphi(x, \varepsilon)) = \int_{\mathbb{R}^n} f(x)\varphi(x, \varepsilon)\mathrm{d}x = \int_{B(0,\varepsilon)} f(x)\varphi(x, \varepsilon)\mathrm{d}x.$$

On the other hand, according to (8.1.9) we have

$$(\delta, \varphi(x, \varepsilon)) = \varphi(0, \varepsilon) = \frac{1}{e}.$$

We equalize the two expressions for $(\delta, \varphi(x, \varepsilon))$

$$\int_{B(0,\varepsilon)} f(x)\varphi(x, \varepsilon)\mathrm{d}x = \frac{1}{e}, \ \forall \varepsilon > 0.$$

We now pass to the limit in this last equality with $\varepsilon \to 0$ and we obtain $0 = e^{-1}$, and this is absurd. ■

We intend to equip the space $\mathcal{D}'(\mathbb{R}^n)$ with the structure of algebra. The sum of two distributions as well as the product of a distribution with a scalar can be defined in a natural way as follows:

$$(T_1 + T_2, \varphi) = (T_1, \varphi) + (T_2, \varphi), \ \forall T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$
$$(\lambda T, \varphi) = (T, \lambda\varphi) = \lambda(T, \varphi), \ \forall T \in \mathcal{D}'(\mathbb{R}^n), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n), \ \forall \lambda \in \mathbb{R}.$$

These two operations are well defined taking into account that they are reduced to summation and multiplication of real numbers.

We can verify immediately that the set $\mathcal{D}'(\mathbb{R}^n)$ together with these two operations becomes a real vector space. We obviously pose the problem if the usual product of two distributions is also a distribution. The answer is negative. Schwartz has proven that *the product of two distributions is not a distribution*.

In order to obtain a structure of algebra on the set $\mathcal{D}'(\mathbb{R}^n)$, we introduce the product of a distribution by a function of class C^∞.

Definition 8.1.5 For a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ and a function $g \in C^\infty(\mathbb{R}^n)$, the product gT can be defined by

$$(gT, \varphi) = (T, g\varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{8.1.11}$$

It is clear that $g\varphi \in \mathcal{D}(\mathbb{R}^n)$ because $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$. In addition $supp\ (g\varphi) \subseteq supp\ \varphi$.

Proposition 8.1.5 *The functional gT defined by means of the correspondence (8.1.11) is a distribution.*

Proof Let us observe, first, that the product gT defines a functional because

$$\mathcal{D}(\mathbb{R}^n) \ni \varphi \xrightarrow{gT} (T, g\varphi) \in \mathbb{R}.$$

To prove the linearity, we notice that

$$\begin{aligned}(gT, \lambda_1\varphi_1 + \lambda_2\varphi_2) &= (T, g\,(\lambda_1\varphi_1 + \lambda_2\varphi_2)) = (T, \lambda_1(g\varphi_1) + \lambda_2(g\varphi_2)) \\ &= \lambda_1\,(T, g\varphi_1) + \lambda_2\,(T, g\varphi_2) = \lambda_1\,(gT, \varphi_1) + \lambda_2\,(gT, \varphi_2)\,.\end{aligned}$$

To prove the continuity, consider the sequence of test functions $\{\varphi_\nu\}_{\nu\geq 1}$ so that $\varphi_\nu \to \varphi$, in $\mathcal{D}(\mathbb{R}^n)$. Then

$$\begin{aligned}\lim_{\nu\to\infty}(gT, \varphi_{nu}) &= \lim_{\nu\to\infty}(T, g\varphi_\nu) = \left(T, \lim_{\nu\to\infty} g\varphi_{nu}\right) \\ &= (T, g\varphi) = (gT, \varphi) = \left(gT, \lim_{\nu\to\infty}\varphi_\nu\right).\end{aligned}$$

and this finished the proof. ■

Example 8.1.5 1^o. For a real number, arbitrarily fixed, we define the distribution of Dirac in the form

$$(\delta_a, \varphi) = \varphi(a), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{8.1.12}$$

It is clear that for $a = 0$, we obtain the distribution δ defined in Example 8.1.4. If $g \in C^\infty(\mathbb{R}^n)$ then

$$(g\delta_a, \varphi) = (\delta_a, g\varphi) = g(a)\varphi(a) = g(a)\,(\delta_a, \varphi)\,, \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

From here we deduce that the distribution $g\delta_a$ is equal to the product between the real number $g(a)$ and the distribution δ_a.

2^o If h is a locally integrable function on $\mathbb{R}^n$ and g is a infinitely differentiable function on $\mathbb{R}^n$, then the product gh is a locally integrable function on $\mathbb{R}^n$ and then according to Example 8.1.3 we can talk of the distribution of the function type generated by the function $gh \in L^1_{loc}(\mathbb{R}^n)$:

$$(gT_h, \varphi) = (T_h, gf) = \int_{\mathbb{R}^n} h(x)g(x)\varphi(x)\mathrm{d}x = \left(T_{gh}, f\right), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Thus, the distribution gT_h is equal to the distribution T_{gh}, the equality is taking place, obviously, in the sense of the distributions.

We finish the considerations on the structure of the space of the distributions by introducing a topological structure on this space. In this sense we equip the space $\mathcal{D}'(\mathbb{R}^n)$ with the following family of semi-norms:

$$\left\{p_\varphi : \varphi \in \mathcal{D}(\mathbb{R}^n)\right\}, \ p_\varphi(T) = |(T, \varphi)|, \ T \in \mathcal{D}'(\mathbb{R}^n).$$

Then we say that the sequence of the distributions $\{T_\nu\}_{\nu \geq 1}$ is convergent to the distribution T if the numerical sequence (T_ν, φ) is convergent to the number (T, φ) for $\forall \varphi \in \mathcal{D}(\mathbb{R})$.

$$T_\nu \xrightarrow{\mathcal{D}'(\mathbb{R}^n)} T \overset{def}{\Leftrightarrow} (T_\nu, \varphi) \xrightarrow{\mathbb{R}} (T, \varphi).$$

Example 8.1.6 Consider the sequence of the functions $\{\pi_\nu\}_{\nu \geq 1}$ defined by

$$\pi_\nu(x) = \begin{cases} 0 \ , \text{ if } |x| \geq \frac{1}{2\nu}, \\ \nu \ , \text{ if } |x| < \frac{1}{2\nu}. \end{cases}$$

We can easily verify the fact that the functions $\pi_\nu \in L^1_{loc}(R), \ \forall \nu = 1, 2, \ldots$

Then we can talk of the distribution generated by π_ν, for each $\nu = 1, 2, \ldots$, in the sense of Definition 8.1.5

$$\left(T_{\pi_\nu}, \varphi\right) = \nu \int_{-\frac{1}{2\nu}}^{\frac{1}{2\nu}} \varphi(x)\mathrm{d}x, \ \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

By using a primitive of φ, we obtain

$$\left(T_{\pi_\nu}, \varphi\right) \to \varphi(0), \text{ for } \nu \to +\infty,$$

that is, the sequence of the distributions $\left(T_{\pi_\nu}\right)_\nu$ is convergent, in the sense of the distributions, to the distribution $\delta_0 = \delta$. A brief analysis of this example shows that though the distribution of Dirac is not a distribution of the function type, it is the limit, in the sense of the distributions, of a sequence of the distributions of the function type.

In fact, more generally, we will show that any distribution on $\mathbb{R}^n$ is the limit in $\mathcal{D}'(\mathbb{R}^n)$ of a sequence of the test functions, if we accept the convention to identify a function from $L^1_{loc}(\mathbb{R}^n)$ with the distribution generated by it.

For the moment, we illustrate this statement with one more example.

Starting from the test function φ from Example 8.1.1 for $\varepsilon = 1$, consider the function $\psi_1 = c\varphi$, where the constant c is chosen so that $c\int_{|x|\leq 1}\varphi(x)\mathrm{d}x = 1$, and we define the sequence

$$\psi_\nu(x) = \nu^n \psi_1(\nu x). \tag{8.1.13}$$

From the definition, it is immediately certified that $\psi_\nu \in \mathcal{D}(\mathbb{R}^n)$. Also, it is obvious that $supp\ \psi_\nu = \overline{B(0, \frac{1}{\nu})}$ and

$$\int_{|x|\leq \frac{1}{\nu}} \psi_\nu(x)\mathrm{d}x = 1.$$

It is clear that as $\nu \to \infty$, the sequence $\{\psi_\nu\}_\nu$ is convergent in $\mathcal{D}'(\mathbb{R}^n)$ to δ

$$\lim_{\nu\to\infty} (\psi_\nu, \varphi) = \lim_{\nu\to\infty} \int_{|x|\leq \frac{1}{\nu}} \psi_\nu(x)\varphi(x)\mathrm{d}x = \varphi(0),$$

in which we used the definition of a distribution of the function type and the properties of the test functions $\psi_\nu(x),\ \nu = 1, 2, \ldots$.

The considerations regarding the convergence of the sequence of distributions can be transposed without difficulty on the convergence of series of the distributions.

Definition 8.1.6 Consider the series $\sum\limits_{\nu=1}^{\infty} T_\nu$, where T_ν are arbitrary distributions, $T_\nu \in \mathcal{D}'(\mathbb{R}^n)$ for $\nu = 1, 2, \ldots$. We say that this series is convergent in $\mathcal{D}'(\mathbb{R}^n)$ if the sequence of partial sums, attached to the series, is convergent in the sense of $\mathcal{D}'(\mathbb{R}^n)$

$$\left(\sum_{\nu=1}^{m} T_\nu, \varphi\right) \xrightarrow{\mathbb{R}} (T, \varphi),\ \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

The distribution T, which is the limit of the sequence of partial sums, will be called the sum of the series and can be written as

$$T = \sum_{\nu=1}^{\infty} T_\nu,$$

the equality taking place in the sense of the distributions.

8.2 The Derivative of a Distribution

The incontestable advantages of the operation of differentiation of a distribution, which will be seen in the following, constitute without doubt the main reason why the concept of the distribution was introduced and then extended.

As we shall see, any distribution has a derivative and the derivative of a distribution is in turn a new distribution, which, therefore, has in turn a derivative, and so on. In short, a distribution is differentiable of any order, in a sense which will be introduced immediately.

Definition 8.2.1 For an arbitrary distribution $T \in \mathcal{D}'(\mathbb{R}^n)$, we define its partial derivative, $\partial T/\partial x_j$, by

$$\left(\frac{\partial T}{\partial x_j}, \varphi\right) = \left(T, -\frac{\partial \varphi}{\partial x_j}\right) = -\left(T, \frac{\partial \varphi}{\partial x_j}\right), \forall j = 1, 2, \ldots, n, \forall \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{8.2.1}$$

Proposition 8.2.1 *If T is a distribution, $T \in \mathcal{D}'(\mathbb{R}^n)$, then the derivative defined as in (8.2.1) is also a distribution.*

Proof We should note, first, the well definiteness. Indeed, because $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we deduce that $\frac{\partial \varphi}{\partial x_j} \in \mathcal{D}(\mathbb{R}^n)$ and because T is a distribution we deduce that $\left(T, \frac{\partial \varphi}{\partial x_j}\right) \in \mathbb{R}$.

Now, we prove the linearity of the functional $\partial T/\partial x_j$

$$\left(\frac{\partial T}{\partial x_i}, \lambda_1\varphi_1 + \lambda_2\varphi_2\right) = -\left(T, \frac{\partial}{\partial x_j}(\lambda_1\varphi_1 + \lambda_2\varphi_2)\right)$$
$$= -\left(T, \lambda_1\frac{\partial \varphi_1}{\partial x_j} + \lambda_2\frac{\partial \varphi_2}{\partial x_j}\right) = -\lambda_1\left(T, \frac{\partial \varphi_1}{\partial x_j}\right) - \lambda_2\left(T, \frac{\partial \varphi_2}{\partial x_j}\right)$$
$$= \lambda_1\left(\frac{\partial T}{\partial x_j}, \varphi_1\right) + \lambda_2\left(\frac{\partial T}{\partial x_j}, \varphi_2\right), \forall \lambda_1, \lambda_2 \in \mathbb{R}, \forall \varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^n).$$

We used here the property of linearity of the operation of differentiation in the case of classical functions.

For the proof of continuity, we choose the sequence of test functions $\{\varphi_\nu\}_\nu$, $\varphi_\nu \in \mathcal{D}(\mathbb{R}^n)$, $\nu = 1, 2, \ldots$ so that $\varphi_\nu \to \varphi$, in $\mathcal{D}(\mathbb{R}^n)$. Then

$$\lim_{\nu\to\infty}\left(\frac{\partial T}{\partial x_j}, \varphi_\nu\right) = \lim_{\nu\to\infty}\left(T, \frac{\partial \varphi_\nu}{\partial x_j}\right) = -\left(T, \lim_{\nu\to\infty}\frac{\partial \varphi_\nu}{\partial x_j}\right)$$
$$= -\left(T, \frac{\partial}{\partial x_j}\left(\lim_{\nu\to\infty}\varphi_\nu\right)\right) = -\left(T, \frac{\partial \varphi_\nu}{\partial x_j}\right) = \left(\frac{\partial T}{\partial x_j}, \varphi\right) = \left(\frac{\partial T}{\partial x_j}, \lim_{\nu\to\infty}\varphi_\nu\right).$$

In the deduction of these relations, we used the fact that since T is a distribution, it is continuous and therefore it is commutative with the limit. ∎

Example 8.2.1 1^o. Consider the distribution of Heaviside T_u (generated by the function of Heaviside u) in the one-dimensional case. According to the Definition 8.2.1, we have

$$\left(\frac{dT_u}{\mathrm{d}x}, \varphi\right) = -\left(T_u, \frac{d\varphi}{\mathrm{d}x}\right) = -\int_0^\infty \varphi'(x)\mathrm{d}x = \varphi(x)|_0^\infty = \varphi(0) = (\delta_0, \varphi)\,.$$

Thus, we can deduce that the derivative of the distribution of Heaviside is the distribution of Dirac. The step function of Heaviside is not differentiable in the origin, and otherwise it has null derivative. The jump of u in $x = 0$ is obtained by differentiation in the sense of the distributions and is given by the distribution δ_0.

2^o. Let us consider the function $h : \mathbb{R} \to \mathbb{R}_+,\ h(x) = |x|$. It is clear that $h \in L^1_{loc}(\mathbb{R})$, since it is continuous on $\mathbb{R}$. Let us compute the derivative of the distribution generated by the function h

$$\left(T_h', \varphi\right) = -\left(T_h, \varphi'\right) = -\int_R |x| f'(x)\mathrm{d}x = \int_{-\infty}^0 x f'(x)\mathrm{d}x - \int_0^\infty x f'(x)\mathrm{d}x.$$

If integrated by parts, then we obtain

$$\left(T_h', \varphi\right) = -\int_{-\infty}^0 f(x)\mathrm{d}x + \int_0^\infty f(x)\mathrm{d}x = \int_{\mathbb{R}} h'(x) f(x)\mathrm{d}x,$$

where

$$h'(x) = \begin{cases} -1\ , & a.e.\ x \in (-\infty, 0) \\ \ \ 1\ , & a.e.\ x \in (0, +\infty). \end{cases}$$

It is clear that h is differentiable on $\mathbb{R}^*$ and has the derivative h'.

It can be shown, without difficulty, that a criterion of Schwartz type about the commutativity of mixed derivatives holds true. In contrast to the classic case of the functions, in the case of the distributions this criterion takes place for any distribution

$$\frac{\partial^{k_1+k_2} T}{\partial x_1^{k_1} \partial x_2^{k_2}} = \frac{\partial^{k_1+k_2} T}{\partial x_2^{k_2} \partial x_1^{k_1}},\ \forall k_1, k_2 \in \mathbb{N},\ \forall T \in \mathcal{D}'(\mathbb{R}^n).$$

Moreover, for any multi-index $k = (k_1, k_2, \ldots, k_n),\ k_i \in \mathbb{N}$, we have

$$\begin{aligned}\left(D^k T, \varphi\right) &= \left(\frac{\partial^{k_1+k_2+\cdots+k_n} T}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n}}, \varphi\right) \\ &= \left(\frac{\partial^{k_2+k_3+\cdots+k_n} T}{\partial x_2^{k_2} ... \partial x_n^{k_n}}, (-1)^{k_1} \frac{\partial^{k_1} \varphi}{\partial x_1^{k_1}}\right) \\ = ... &\left(T, (-1)^{k_1+k_2+\cdots+k_n} \frac{\partial^{k_1+k_2+\cdots+k_n} \varphi}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n}}\right),\end{aligned}$$

that is,

$$\left(D^k T, \varphi\right) = (-1)^{|k|} \left(T, D^k \varphi\right), \ \forall T \in \mathcal{D}'(\mathbb{R}^n), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$

where $|k| = k_1 + k_2 + \cdots + k_n$.

It is interesting to see the definition of derivative of a distribution in the case of a distribution generated by a locally integrable function.

Let us consider a set $\Omega \subset \mathbb{R}^n$ and denote by T_f the distribution generated by the function f, where $f \in L^1_{loc}(\Omega)$ and the distribution $T_{\partial f/\partial x_j}$ generated by the function $\partial f/\partial x_j \in L^1_{loc}(\Omega)$. Then we can write

$$\left(T_f, -\frac{\partial \varphi}{\partial x_j}\right) = \int_\Omega f(x) \left(-\frac{\partial \varphi}{\partial x_j}\right)(x)\mathrm{d}x = -\int_\Omega f(x) \frac{\partial \varphi}{\partial x_j}(x)\mathrm{d}x$$
$$= -f(x)\varphi(x)|_{\partial\Omega} + \int_\Omega \frac{\partial f}{\partial x_j}(x)\varphi(x)\mathrm{d}x = \left(T \frac{\partial f}{\partial x_j}, \varphi\right), \ \forall \varphi \in \mathcal{D}(\Omega).$$

We used here the fact that $F(x)\varphi(x)|_{\partial\Omega} = 0$ because $supp\ \varphi \subset \Omega$.

These formulas can constitute a justification for the definition of the derivative of a distribution, that is, Definition 8.2.1. In fact to the origin of the notion of the distribution stands the formula of integration by parts.

We can verify the fact that the properties of the operation of differentiation in the case of the distributions are analogous to those from the case of classic functions.

Proposition 8.2.2 *The following statements hold true:*

1^o. *Differentiation in the context of the distributions is a linear operator*

$$\frac{\partial}{\partial x_j}\left(\lambda_1 T_1 + \lambda_2 T_2\right) = \lambda_1 \frac{\partial T_1}{\partial x_j} + \lambda_2 \frac{\partial T_2}{\partial x_j}, \ \forall \lambda_1, \lambda_2 \in \mathbb{R}, \ \forall T_1, T_2 \in \mathcal{D}'(\mathbb{R}^n),$$

where the equality is taking place in the sense of the distributions.

2^o. *For any function $g \in C^\infty(\mathbb{R}^n)$ and any distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ we have*

$$\frac{\partial}{\partial x_j}(gT) = \frac{\partial g}{\partial x_j} T + g \frac{\partial T}{\partial x_j}$$

where the equality is taking place in the sense of the distributions.

Proof 1^o. The proof of this statement is immediately obtained based on the linearity of the derivative of the test functions.

2^o. Taking into account the definition of the product of a distribution with a function of class C^∞, we deduce that $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\left(\frac{\partial(gT)}{\partial x_j}, \varphi\right) = -\left(gT, \frac{\partial \varphi}{\partial x_j}\right) = -\left(T, g\frac{\partial \varphi}{\partial x_j}\right)$$
$$= -\left(T, \frac{\partial}{\partial x_j}(g\varphi)\right) + \left(T, \varphi\frac{\partial g}{\partial x_j}\right) = \left(g\frac{\partial T}{\partial x_j}, \varphi\right) + \left(\frac{\partial g}{\partial x_j}T, \varphi\right),$$

and this ends the proof. ■

We will make some comments about the derivative of a distribution of the function type and about the distribution of order zero (measures). To this end, we recall some results which are specific to mathematical analysis on the set of real numbers.

Let us consider the function $f : [a, b] \to \mathbb{R}$ and τ an arbitrary division of the interval $[a, b]$ to which we attach the sum

$$\bigvee_{\tau} = \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| . \tag{8.2.2}$$

Definition 8.2.2 If the set of sums $\bigvee_{\tau}$ of the form (8.2.2), which corresponds to all divisions τ of the interval $[a, b]$, is bounded, then f is called a function of bounded variation and we write $f \in B.V.([a, b])$.

For a function of bounded variation, the number $\bigvee_{a}^{b} f = \sup_{\tau} \bigvee_{\tau}$ is called the total variation of the function f.

Definition 8.2.3 We call the Stieltjes measure, a measure which is generated by a function of bounded variation.

Definition 8.2.4 If M is a set from an algebra, we say that the function f is absolutely continuous on M (and write $f \in A.C.(M)$) if $\mu(M) = 0$ implies $f(A) = 0$, where μ is a measure, that is, a function of sets which is positive and countably additive.

Theorem 8.2.1 *Let $I = (a, b)$ be an interval on the real axis and g a distribution, $g \in \mathcal{D}'(I)$. The necessary and sufficient condition that the derivative in the sense of the distributions of the distribution g is a measure, therefore a distribution of order zero, is that g must be generated by a function from $B.V.(Q)$, for any compact set Q from the interval I.*

Moreover, if the distribution g is generated by a function from $B.V.(Q)$, for any compact set Q from I, then its derivative in the sense of the distributions is a Stieltjes measure dg generated by the function g.

Theorem 8.2.2 *Let $I = (a, b)$ be an interval on real axis and g a distribution $g \in \mathcal{D}'(I)$. The necessary and sufficient condition that the derivative in the sense of the distributions of the distribution g is a distribution generated by a locally integrable function on I is that the distribution g is generated by an absolutely continuous function on any compact subinterval from I. Moreover, if the distribution g is generated by an absolutely continuous function on any compact subinterval from*

I then the classic derivative, in the sense of the functions, of the function g, that is, g', generates the distribution g' (the derivative in the sense of the distributions).

In theorem 0.1 and in theorem 0.2s, we denoted by g both the function and the distribution generated by this function. The proofs of these two theorems are laborious, having a very technical character, based on elements which are specific to the absolutely continuous functions, and to the functions of bounded variation, respectively. For this reason, we do not write these proofs. Moreover, we recall, in this context, a result which will be given without proof as well and, which is due to Vulih.

Theorem 8.2.3 *The general form of a linear and continuous functional defined on the space $C[a, b]$ is given by the Stieltjes integral*

$$(f, x) = \int_a^b f(t)x(t)dg(t),$$

where g is an arbitrary function of bounded variation. Moreover, $\|f\| = \bigvee_a^b g$ and the functional f defines uniquely the function g.

Consider now a function f which is continuous almost everywhere. We restrict our attention to a function which is continuous except at point x_0 in which f has a discontinuity of the first order. Denote by s_0 the jump of f in x_0, that is

$$s_0 = l_d(x_0) - l_s(x_0) = f(x_0 + 0) - f(x_0 - 0).$$

As far as of the derivative in the sense of the distributions for the distribution generated by such a function is concerned, we have the result from the following theorem.

Theorem 8.2.4 *If we denote by $f'(x)$ the derivative in the sense of the distributions and with $\tilde{f}'(x)$ the derivative in the sense of classic functions, then we have*

$$f'(x) = \tilde{f}'(x) + s_0\delta_{x_0}(x). \tag{8.2.3}$$

Proof For an arbitrary test function φ, we have the following calculations:

$$\begin{aligned}
\big(f'(x), \varphi(x)\big) &= \big(f(x), -\varphi'(x)\big) = -\big(f(x), \varphi'(x)\big) \\
&= -\lim_{\varepsilon \searrow 0}\left[\int_{-\infty}^{x_0-\varepsilon} f(x)\varphi'(x)\mathrm{d}x + \int_{x_0+\varepsilon}^{\infty} f(x)\varphi'(x)\mathrm{d}x\right] \\
&= -\lim_{\varepsilon \searrow 0}\left[f(x)\varphi(x)|_{-\infty}^{x_0-\varepsilon} - \int_{-\infty}^{x_0-\varepsilon} f'(x)\varphi(x)\mathrm{d}x\right. \\
&\qquad \left. + F(x)\varphi(x)|_{x_0+\varepsilon}^{+\infty} - \int_{x_0+\varepsilon}^{\infty} \tilde{f}'(x)\varphi(x)\right]\mathrm{d}x \\
&= -\lim_{\varepsilon \searrow 0}[f(x_0-\varepsilon)\varphi(x_0-\varepsilon) - f(x_0+\varepsilon)\varphi(x_0+\varepsilon)
\end{aligned}$$

$$+\int_{-\infty}^{x_0-\varepsilon}\tilde{f}'(x)\varphi(x)\mathrm{d}x+\int_{x_{0+\varepsilon}}^{\infty}\tilde{f}'(x)\varphi(x)\mathrm{d}x\Bigg]$$
$$=[f(x_0+0)-f(x_0-0)]\,\varphi(x_0)+\int_{-\infty}^{x_0}\tilde{f}'(x)\varphi(x)\mathrm{d}x$$
$$+\int_{x_0}^{\infty}\tilde{f}'(x)\varphi(x)\mathrm{d}x=s_0\varphi(x_0)+\int_{-\infty}^{\infty}\tilde{f}'(x)\varphi(x)\mathrm{d}x$$
$$=s_0\left(\partial_{x_0}(x),\varphi(x)\right)+\left(\tilde{f}'(x),\varphi(x)\right)=\left(\tilde{f}'(x)+s_0\delta_{x_0}(x),\varphi(x)\right).$$

Finally, because the test function φ is arbitrarily chosen, we obtain the equality (8.2.3). ■

Here and in the following, the notation $f(x)$, where f is a distribution, has only the role to state the fact that the distribution f is applied to a test function of variable x. Thus, the derivative in the sense of the distributions is the sum between the derivative in the sense of the functions and the product of the jump of f in the point of discontinuity x_0 by the distribution of Dirac, concentrated in x_0.

It is clear that in the case of differentiable functions, the derivative in the sense of the distributions coincides with the derivative in the sense of functions.

Example 8.2.2 In quantum mechanics, we frequently meet the distribution $V_p 1/x$, which in the context of functions has the significance of the main value in sense of Cauchy of the function $1/x$.

The function $f(x)=1/x,\ x\neq 0$ is not locally integrable because it is not integrable in any neighborhood of the origin. In mathematical analysis, in order to avoid the effect of a function not being integrable in any neighborhood of the origin, we use the notion of the main value in sense of Cauchy, defined by

$$V_p\int_a^b\frac{1}{x}\mathrm{d}x=\lim_{\varepsilon\searrow 0}\left(\int_{-a}^{-\varepsilon}\frac{1}{x}\mathrm{d}x+\int_{\varepsilon}^{b}\frac{1}{x}\mathrm{d}x\right),\ a,b>0.$$

By direct calculation, we obtain

$$\lim_{\varepsilon\searrow 0}\left(\int_{-a}^{-\varepsilon}\frac{1}{x}\mathrm{d}x+\int_{\varepsilon}^{b}\frac{1}{x}\mathrm{d}x\right)=\lim_{\varepsilon\searrow 0}\left(\ln\frac{\varepsilon}{a}+\ln\frac{b}{\varepsilon}\right)=\ln\frac{b}{a}.$$

We chose the interval $[-a,b]$ to give a sense of the integral on a neighborhood of the origin. In particular, we have

$$V_p\int_{-a}^{a}\frac{1}{x}\mathrm{d}x=0,\ V_p\int_{-\infty}^{\infty}\frac{1}{x}\mathrm{d}x=0.$$

Instead of the function $1/x$, consider the function $V_p 1/x$ which will generate a distribution which will be denoted also by $V_p 1/x$ and which is defined by

$$\left(V_p\frac{1}{x},\varphi\right) = V_p\int_{R^1}\frac{\varphi(x)}{x}\mathrm{d}x,\ \forall\varphi\in\mathcal{D}(\mathbb{R}^1).$$

Proposition 8.2.3 *The above correspondence $V_p\frac{1}{x}$ is a distribution.*

Proof We should note, first, the fact that

$$\left(V_p\frac{1}{x},\varphi\right) = V_p\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}\mathrm{d}x = \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{\varepsilon}\frac{\varphi(x)}{x}\mathrm{d}x + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x}\mathrm{d}x\right)$$

and this proves the well definiteness of the functional $V_p\frac{1}{x}$.

Indeed, $\varphi\in C_0^\infty(\mathbb{R}^1)$, therefore more than continuous, and $supp\ \varphi$ is compact. In addition x does not touch the origin. The linearity can be deduced from the linearity of the integral

$$\begin{aligned}\left(V_p\frac{1}{x},\lambda_1\varphi_1+\lambda_2\varphi_2\right) &= V_p\int_{-\infty}^{\infty}\frac{1}{x}\left(\lambda_1\varphi_1+\lambda_2\varphi_2\right)(x)\mathrm{d}x\\ &= \lambda_1 V_p\int_{-\infty}^{\infty}\frac{\varphi_1}{x}\mathrm{d}x + \lambda_2 V_p\int_{-\infty}^{\infty}\frac{\varphi_2}{x}\mathrm{d}x\\ &= \lambda_1\left(V_p\frac{1}{x},\varphi_1\right)+\lambda_2\left(V_p\frac{1}{x},\varphi_2\right),\ \forall\lambda_1,\lambda_2\in\mathbb{R}^1,\forall\varphi_2,\varphi_2\in\mathcal{D}(\mathbb{R}^1).\end{aligned}$$

To prove the continuity we will use the sequence $\{\varphi_k\}_k\subset\mathcal{D}(\mathbb{R}^1)$ so that $\varphi_k\to\varphi$, as $k\to\infty$, where $\varphi\in\mathcal{D}(\mathbb{R}^1)$. Then we have

$$\begin{aligned}\lim_{k\to\infty}\left(V_p\frac{1}{x},\varphi_k\right) &= \lim_{k\to\infty}V_p\int_{R^1}\frac{\varphi_k(x)}{x}\mathrm{d}x\\ &= \lim_{k\to\infty}\lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi_k(x)}{x}\mathrm{d}x + \int_{\varepsilon}^{\infty}\frac{\varphi_k(x)}{x}\mathrm{d}x\right)\\ &= \lim_{\varepsilon\searrow 0}\lim_{k\to\infty}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi_k(x)}{x}\mathrm{d}x + \int_{\varepsilon}^{\infty}\frac{\varphi_k(x)}{x}\mathrm{d}x\right)\\ &= \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon}\frac{\varphi_k(x)}{x}\mathrm{d}x + \int_{\varepsilon}^{\infty}\frac{\varphi_k(x)}{x}\mathrm{d}x\right)\\ &= \left(V_p\frac{1}{x},\varphi\right) = \left(V_p\frac{1}{x},\lim_{k\to\infty}\varphi_k\right),\end{aligned}$$

and this ends the proof. ■

We proved that the product of a function of class C^∞ with a distribution is a distribution. If we consider the function $g(x)=x$, $x\in\mathbb{R}^1$, which obvious is of class C^∞, and the distribution $V_p\frac{1}{x}$, we deduce that the product $x\cdot V_p\frac{1}{x}$ is a distribution. Moreover, we have the following result.

Proposition 8.2.4 *In the set of the distributions $\mathcal{D}'(\mathbb{R}^1)$ we have the equality*

$$x \cdot V_p \frac{1}{x} = 1.$$

Proof For an arbitrary test, $\varphi \in \mathcal{D}(\mathbb{R}^1)$, by direct calculation we deduce that

$$\left(x V_p \frac{1}{x}, \varphi(x)\right) = \left(V_p \frac{1}{x}, x\varphi(x)\right) = V_p \int_{-\infty}^{\infty} \frac{x\varphi(x)}{x} \mathrm{d}x$$
$$= V_p \int_{-\infty}^{\infty} \varphi(x)\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\mathrm{d}x = \int_{-\infty}^{\infty} 1\varphi(x)\mathrm{d}x = (1, \varphi).$$

Due to the fact that $\varphi \in \mathcal{D}(\mathbb{R}^1)$ is an arbitrary test function, we deduce the equality from the statement of the proposition. ■

From Proposition 8.2.4, we deduce that the distribution $x V_p \frac{1}{x}$ is a distribution of the function type and is generated by the constant function 1.

8.3 The Primitive of a Distribution

In the set of the distributions, we will introduce the notion of primitive by using a procedure analogous to that from the case of classic functions. Thus, we will see that any distribution admits a primitive which is also a distribution. Moreover, for a fixed distribution we have a whole family of primitive so that any two distributions from this family differ by a constant, that is, a distribution generated by a constant function.

To simplify the expressions, we make the considerations only in the one-dimensional case. Passing to the n-dimensional case does not create problems as far as the results are concerned, only the ordinary derivative is replaced by the partial derivative.

Definition 8.3.1 Let us consider the distribution $f \in \mathcal{D}'(\mathbb{R}^1)$. We say that the distribution $T \in \mathcal{D}'(\mathbb{R}^1)$ is a primitive of the distribution f if

$$\frac{dT}{\mathrm{d}x} = f, \text{ in } \mathcal{D}'(\mathbb{R}^1) \Leftrightarrow \left(\frac{dT}{\mathrm{d}x}, \varphi\right) = (f, \varphi), \ \forall \varphi \in \mathcal{D}'(\mathbb{R}^1). \tag{8.3.1}$$

Observation 8.3.1 *Taking into account the definition of the derivative of a distribution, we can rewrite (8.3.1) in the form*

$$(T, -\varphi') = (f, \varphi), \ \forall \varphi \in \mathcal{D}'(\mathbb{R}^1),$$

from where we deduce that the primitive of a distribution is not defined for any test function.

In the following, we will show that this restriction is not essential. We introduce the notation

$$\Phi_0 = \left\{ \varphi_0 \in \mathcal{D}'(R^1) \,:\, \exists \psi \in \mathcal{D}'(R^1) \text{ so that } \varphi_0(t) = \frac{d\psi(t)}{\mathrm{d}t} \right\}. \tag{8.3.2}$$

Theorem 8.3.1 *The necessary and sufficient condition that a test function $\psi_0 \in \mathcal{D}(\mathbb{R}^1)$, is from the set Φ_0, defined in (8.3.2), is*

$$\int_{-\infty}^{\infty} \psi_0(\tau)\mathrm{d}\tau = 0. \tag{8.3.3}$$

Proof *The necessity*. Suppose that $\psi_0 \in \Phi_0$. Then we know that there is the test function $\psi \in \mathcal{D}(\mathbb{R}^1)$ so that

$$\psi_0 = \frac{d\psi(t)}{\mathrm{d}t},$$

and then

$$\int_{-\infty}^{\infty} \psi_0(\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} \frac{d\psi(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = \psi(\infty) - \psi(-\infty) = 0,$$

in which we used the fact that $\psi \in \mathcal{D}(\mathbb{R}^1)$ and therefore $supp\ \psi \subset \mathbb{R}^1$.

The sufficiency. Let ψ_0 be a test function, $\psi_0 \in \mathcal{D}(\mathbb{R}^1)$ which satisfies (8.3.3) and we must prove that ψ_0 is the derivative in the sense of classic functions for another test function from $\mathcal{D}(\mathbb{R}^1)$. We define the function $\chi(t)$ by

$$\chi(t) = \int_{-\infty}^{t} \psi_0(\tau)\mathrm{d}\tau. \tag{8.3.4}$$

Because ψ_0 satisfies (8.3.3), we obtain that $\chi(\infty) = 0$. Then $\chi(-\infty) = 0$ and therefore χ has compact support in $\mathbb{R}^1$. Because ψ_0 is of class C^∞ we obtain, from (8.3.4), that χ is of class C^∞ and also from (8.3.4) it is certified that $\frac{d\chi(t)}{\mathrm{d}t} = \psi_0(t)$. ■

We intend to show now that any test function from $D(\mathbb{R}^1)$ can be projected uniquely on the set Φ_0, defined in (8.3.2).

Theorem 8.3.2 *Let φ_1 be a fixed test function, $\varphi_1 \in \mathcal{D}(\mathbb{R}^1)$, having the property*

$$\int_{-\infty}^{\infty} \varphi_1(\sigma)d\sigma = 1. \tag{8.3.5}$$

Then, for any test function $\varphi \in \mathcal{D}(\mathbb{R}^1)$, there is a function $\varphi_0 \in \Phi_0$ so that we can write

$$\varphi(t) = \varphi_1(t) \int_{-\infty}^{\infty} \varphi(\tau) \mathrm{d}\tau + \varphi_0(t). \tag{8.3.6}$$

Proof We want to mention, first, the fact that the function φ_0, which is uniquely determined so that the relation (8.3.6) holds true, is called the projection of the test function φ on the set Φ_0. We prove now that the existence of a test function φ_1, which satisfies (8.3.5) is realistic, that is, the set of the functions φ_1 with the property (8.3.5), is non-empty. Indeed, if we take the function φ_2 so that $\int_{-\infty}^{\infty} \varphi_2(\sigma) d\sigma \neq 0$, then we can consider the function φ_1, defined by

$$\varphi_1(t) = \frac{\varphi_2(t)}{\int_{-\infty}^{\infty} \varphi_2(\tau) \mathrm{d}\tau}.$$

Then it is clear that

$$\int_{-\infty}^{\infty} \varphi_1(t) \mathrm{d}t = \frac{1}{\int_{-\infty}^{\infty} \varphi_2(\tau) \mathrm{d}\tau} \int_{-\infty}^{\infty} \varphi_2(t) \mathrm{d}t = 1.$$

With the fixed function φ_1, to satisfy (8.3.6), we consider the function φ_0 defined by

$$\varphi_0(t) = \varphi(t) - \varphi_1(t) \int_{-\infty}^{\infty} \varphi(\tau) \mathrm{d}\tau, \tag{8.3.7}$$

and so show that φ_0 belongs to the set Φ_0. We have

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_0(\tau) \mathrm{d}\tau &= \int_{-\infty}^{\infty} \varphi(t) \mathrm{d}t - \int_{-\infty}^{\infty} \varphi_1(t) \left(\int_{-\infty}^{\infty} \varphi(\tau) \mathrm{d}\tau \right) \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \varphi(t) \mathrm{d}t - \int_{-\infty}^{\infty} \varphi(\tau) \mathrm{d}\tau . \int_{-\infty}^{\infty} \varphi_1(t) \mathrm{d}t \\ &= \left(1 - \int_{-\infty}^{\infty} \varphi_1(t) \mathrm{d}t \right) . \int_{-\infty}^{\infty} \varphi(\tau) \mathrm{d}\tau = 0, \end{aligned}$$

in which we used (8.3.5). According to Theorem 8.3.1, we deduce that $\varphi_0 \in \Phi_0$. From the construction of the function φ_0 in (8.3.7), we can immediately deduce that the function φ_0 is uniquely determined. ■

The result which follows proves the analogy between the primitives of the distributions and the primitives of the functions.

Theorem 8.3.3 *Any primitive of null distribution is a constant distribution, that is, the distribution generated by a constant function. In short, the primitives of the distribution* $0 \in \mathcal{D}'(\mathbb{R}^n)$ *are constants.*

Proof Let us consider $0 \in \mathcal{D}'(\mathbb{R}^1)$ and $T \in \mathcal{D}'(\mathbb{R}^1)$ so that

$$\frac{dT}{\mathrm{d}t} = 0 \Longleftrightarrow 0 = (0, \varphi) = \left(\frac{dT}{\mathrm{d}t}, \varphi \right) = - \left(T, \frac{d\varphi}{\mathrm{d}t} \right), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^1).$$

Therefore

$$\left(T, \frac{d\varphi}{\mathrm{d}t}\right) = 0, \ \forall \varphi \in \mathcal{D}(\mathbb{R}^1). \tag{8.3.8}$$

Note that relation (8.3.8) holds true for test functions from Φ_0. Therefore we can rewrite (8.3.8) in the form

$$(T, \psi_0) = 0, \ \forall \psi_0 \in \Phi_0. \tag{8.3.9}$$

We apply now the distribution T to decomposition (8.3.6), so that, based on the linearity of T, we can write

$$\begin{aligned}(T, \varphi) &= \left(T, \varphi_1 \int_{-\infty}^{\infty} \varphi(\tau)\mathrm{d}\tau\right) + (T, \varphi_0) \\ &= (T, \varphi_1) \int_{-\infty}^{\infty} \varphi(\tau)\mathrm{d}\tau + (T, \varphi_0) = C_1 \int_{-\infty}^{\infty} \varphi(\tau)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} C_1 \varphi(\tau)\mathrm{d}\tau = (C_1, \varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^1).\end{aligned}$$

We used here the fact that $(T, \varphi_0) = 0$ (according to (8.3.9)) as well as the fact that the test function φ_1 is fixed and therefore (T, φ_1) is a constant (a fixed number), which has been denoted here by C_1. ■

It is clear that for each fixed test function φ_1 we obtain a constant C_1 and therefore C_1 is an arbitrary constant. A more general result than those from Theorem 8.3.3 is the following.

Theorem 8.3.4 *Any two primitives of a fixed distribution differ by a constant distribution, that is, generated by a constant.*

Proof Let us consider the fixed distribution f, $f \in \mathcal{D}'(\mathbb{R}^1)$ and let T be a distribution so that

$$\frac{dT}{\mathrm{d}t} = f, \ \text{in } \mathcal{D}'(\mathbb{R}^1) \Rightarrow (f, \varphi) = \left(\frac{dT}{dt}, \varphi\right) = (T, -\varphi'), \ \forall \varphi \in \mathcal{D}'(\mathbb{R}^1).$$

Therefore T is defined only for test functions from Φ_0. We define the distribution T_0 as follows:

$$(T_0, \varphi) = (T, \varphi_0), \tag{8.3.10}$$

where φ_0 is uniquely determined by the test function φ by means of the decomposition (8.3.6). We can verify without difficulty that T_0 defined in (8.3.10) is a distribution, namely, it is a primitive of the distribution f.

We apply now the distribution T to decomposition (8.3.6) and we obtain

$$\begin{aligned}(T, \varphi) &= \left(T, \varphi_1 \int_{-\infty}^{\infty} \varphi(\tau)\mathrm{d}\tau\right) + (T, \varphi_0) \\ &= (T, \varphi_1) \int_{-\infty}^{\infty} \varphi(\tau)\mathrm{d}\tau) + (T_0, \varphi) = C_1 \int_{-\infty}^{\infty} \varphi(\tau)\mathrm{d}\tau + (T_0, \varphi) \\ &= (C_1, \varphi) + (T_0, \varphi) = (C_1 + T_0, \varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R}^1).\end{aligned}$$

Thus, $T = C_1 + T_0$ in $\mathcal{D}'(\mathbb{R}^1)$ and the proof is finished. ■

We conclude the considerations on the primitive of a distribution with some notions regarding the primitives of higher order.

Definition 8.3.2 We say that the distribution $F \in \mathcal{D}'(\mathbb{R}^1)$ is a primitive of order m for the distribution $f \in \mathcal{D}'(\mathbb{R}^1)$ if

$$F^{(m)} = f, \text{ in } \mathcal{D}'(\mathbb{R}^1) \Leftrightarrow \left(F^{(m)}, \varphi\right) = (f, \varphi), \ \forall \varphi \in \mathcal{D}'(\mathbb{R}^1).$$

As far as the primitives of higher order are concerned, we have two results included in the following theorem.

Theorem 8.3.5 *The following statements are true:*
(i) Any distribution $f \in \mathcal{D}'(R^1)$ admits a primitive of any order.
(ii) Two primitives of the same order m differ between them by a polynomial of degree $m-1$, that is, a distribution generated by a polynomial function of degree $m-1$.

Proof (i) The existence of a primitive of higher order will be proven by mathematical induction. For $m = 1$, we already proved a theorem of existence of a primitive. We proved also that any two primitives (of the first order) differ by a constant which can be considered as a polynomial of degree 0. Since F is a distribution, it admits a primitive, according to the same theorem of existence. We can therefore assume that f admits a primitive of order $m - 1$. Denote by G such a primitive and then we have

$$G^{(m-1)} = f, \text{ in } \mathcal{D}'(\mathbb{R}^1) \Longleftrightarrow \left(G^{(m-1)}, \varphi\right) = (f, \varphi), \ \forall \varphi \in \mathcal{D}'(\mathbb{R}^1). \quad (8.3.11)$$

Since G is a distribution, it admits also a primitive which is denoted by F and therefore $F' = G$, in $\mathcal{D}'(\mathbb{R}^1)$. We use now (8.3.11) and then we obtain

$$F^{(m)} = G^{(m-1)} = f,$$

the equalities taking place in $\mathcal{D}'(\mathbb{R}^1)$ and therefore the statement (i) is proven.

To prove the statement (ii) suppose, according to the principle of mathematical induction, that two distributions of order $m - 1$, G_1 and G_2 differ by a polynomial of degree $m - 2$:

$$(G_1 - G_2, \varphi) = \int_{R^1} P_{m-2}(x)\varphi(x)\mathrm{d}x, \ \forall\varphi \in \mathcal{D}'(\mathbb{R}^1). \tag{8.3.12}$$

Denote by F_1 and F_2, respectively, the primitives of the distributions G_1 and G_2, that is,

$$F_1' = G_1, \ F_2' = G_2,$$

the equalities taking place in $\mathcal{D}'(\mathbb{R}^1)$. Then we have

$$\begin{aligned}(G_1 - G_2, \varphi) &= \textstyle\int_{\mathbb{R}^1} [G_1(x) - G_2(x)]\,\varphi(x)\mathrm{d}x \\ &= \textstyle\int_{R^1} \left[F_1'(x) - F_2'(x)\right]\varphi(x)\mathrm{d}x = [F_1(x) - F_2(x)]\ \varphi(x)|_{-\infty}^{\infty}\,\mathrm{d}x \\ -&\textstyle\int_{R^1} [F_1(x) - F_2(x)]\,\varphi'(x)\mathrm{d}x = -\int_{R^1} [F_1(x) - F_2(x)]\,\varphi'(x)\mathrm{d}x.\end{aligned} \tag{8.3.13}$$

On the other hand, integrating by parts in (8.3.12), we obtain

$$\begin{aligned}(G_1 - G_2, \varphi) =\ & P_{m-1}(x)\varphi(x)|_{-\infty}^{\infty} \\ -\int_{-\infty}^{\infty} & P_{m-1}(x)\varphi'(x)\mathrm{d}x = -\int_{-\infty}^{\infty} P_{m-1}(x)\varphi'(x)\mathrm{d}x.\end{aligned} \tag{8.3.14}$$

From (8.3.13) and (8.3.14) we deduce that

$$F_1 - F_2 = P_{m-1},$$

the equality taking place in $\mathcal{D}'(\mathbb{R}^1)$. In fact we have

$$\left(F_1 - F_2, \varphi'\right) = \left(P_{m-1}, \varphi'\right), \ \forall\varphi \in \mathcal{D}(R^1).$$

The complete result is obtained using the theorem of representation of an arbitrary test function with the help the projection on the set Φ_0 (Theorem 8.3.2). ■

8.4 Tensor Product and Product of Convolution

Let Ω be an open set from $\mathbb{R}^n$. All functions $\varphi \in C_0^\infty(\Omega)$ can be prolonged with 0 outside the set Ω, to a function $\tilde{\varphi}$ so that $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$.

Definition 8.4.1 1^o. Given a distribution T on $\mathbb{R}^n$, we define the restriction of T to the set Ω, denoted with $T_{/\Omega}$, by

$$\left(T_{/\Omega}, \varphi\right) = (T, \tilde{\varphi}), \ \forall\varphi \in C_0^\infty(\Omega).$$

2^o. We say that two distributions T_1 and T_2 coincide on Ω if their restrictions to Ω are equal.

3^o. The support of a distribution T, denoted by $supp\ (T)$, is complementary to the largest open set on which its restriction is null.

4^o. We denote by $\mathcal{E}'(\mathbb{R}^n)$ the set of the distributions with compact support.

In the case of the distribution of Dirac we have that $supp\ (\delta_a) = \{a\}$ so that the distribution of Dirac together with all its derivatives are distributions with compact support, $\delta_a \in \mathcal{E}'(\mathbb{R}^n)$.

Theorem 8.4.1 *A distribution T is with compact support if and only if it is defined and continuous on the set $C^\infty(\mathbb{R}^n)$, which is equipped with the uniform convergence on all compact sets from $\mathbb{R}^n$ for functions and their partial derivatives of any order.*

Proof The necessity. Suppose that the distribution T has the compact support $K \subset \mathbb{R}^n$. Let α be a function from $C_0^\infty(\mathbb{R}^n)$ which coincides with 1 on a neighborhood of K. For all functions φ from $C_0^\infty(\mathbb{R}^n)$ we have that the function $\alpha\varphi$ is also from $C_0^\infty(\mathbb{R}^n)$ and coincides with φ on K. Then $(T, \alpha\varphi)$ is well defined and its value is independent of the choice of α (with the condition that it coincides with 1 on a neighborhood of K). If $\{\varphi_\nu\}_\nu$ is a sequence of the functions which is convergent to φ, in the sense of the space $C^\infty(\mathbb{R}^n)$, then the sequence $\{\alpha\varphi\nu\}_\nu$ is convergent to $\alpha\varphi$ on the set $C_0^\infty(\mathbb{R}^n)$ and therefore the sequence $\{(T, \alpha\varphi_\nu)\}_\nu$ is convergent to $(T, \alpha\varphi)$. Then T defines a linear and continuous form on $C^\infty(\mathbb{R}^n)$.

The sufficiency. Let L be a linear and continuous form on the set $C^\infty(\mathbb{R}^n)$. In particular, L is a distribution T, because L is defined on $C_0^\infty(\mathbb{R}^n)$ and is a linear and continuous form with respect to the convergence from this space. Suppose by contradiction that T does not have a compact support. For any ν we can find φ_ν with the support in $\mathbb{R}^n \setminus B(0, \nu)$ so that $(T, \varphi_\nu) = 1$. Determining of φ_ν is allowed because the sequence $\{\varphi_\nu\}_\nu$ is convergent to 0 on $C^\infty(\mathbb{R}^n)$. We obtained thus a contradiction. It remain only to verify that the distribution T can be prolonged to $C^\infty(\mathbb{R}^n)$ and that it coincides with L. We take a sequence $\{\alpha_k\}_k$ of the functions from $C_0^\infty(\mathbb{R}^n)$, which are equal to 1 in the ball $B(0, k)$ and with the support in the ball $B(0, 2k)$. For any function φ from $C_0^\infty(\mathbb{R}^n)$, the sequence $\{\alpha_k\varphi\}_k$ is convergent to φ on the set $C^\infty(\mathbb{R}^n)$ and then the sequence $\{(L, \alpha_k\varphi)\}_k$ is convergent to (L, φ). For k sufficiently big, we have that $(L, \alpha_k\varphi) = (T, \alpha_k\varphi)$, and the number $(T, \alpha_k\varphi)$ is sufficient big, independent of k. ■

Definition 8.4.2 Let us consider the functions $f \in L^1_{loc}(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^m)$. We define the tensor product of the functions f and g by

$$(f \otimes g)(x, y) = f(x)g(y),\ \forall (x, y) \in \mathbb{R}^{n+m}.$$

In the following proposition, we will prove the main properties of the tensor product of the functions.

Proposition 8.4.1 1^o. *If the functions f and g, belong to the sets $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^m)$ respectively, then $f \otimes g \in C^\infty(\mathbb{R}^{n+m})$. In addition, for any multi-indices α and β we have*

$$D^{\alpha+\beta}(f \otimes g) = D^\alpha f \otimes D^\beta g.$$

Here, the derivative $D^{\alpha+\beta}$ *is made of* $|\alpha|$ *times with respect to* x *and of* $|\beta|$ *times with respect to* y*, by taking as in Definition 8.4.1 the function* f *of variable* x *and the function* g *of variable* y*.*

2^o*. The following equality of sets holds true*

$$supp\ (f \otimes g) = supp\ (f) \times supp\ (g),$$

the last set being the Cartesian product of the two supports.

3^o*. If the functions* f *and* g *are so that* $f \in L^1_{loc}(\mathbb{R}^n)$*, and* $g \in L^1_{loc}(\mathbb{R}^m)$*, then* $f \otimes g \in L^1_{loc}(\mathbb{R}^{n+m})$*.*

In addition, the function $f \otimes g$ *generates the distribution* $T_{f\otimes g}$*, defined by*

$$\left(T_{f\otimes g}, \varphi\right) = \int_{\mathbb{R}^{n+m}} (f \otimes g)(x, y)\varphi(x, y)\mathrm{d}x\mathrm{d}y.$$

In particular, if the test function φ *is of the form* $\varphi_1 \otimes \varphi_2$*, then:*

$$\begin{aligned}\left(T_{f\otimes g}, \varphi_1 \otimes \varphi_2\right) &= \int_{\mathbb{R}^{n+m}} (f \otimes g)\,(x, y)\,(\varphi_1 \otimes \varphi_2)\,(x, y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(x)\varphi_1(x)\mathrm{d}x \int_{R^m} g(x)\varphi_2(y)\mathrm{d}y = \left(T_f, \varphi_1\right)\left(T_g, \varphi_2\right).\end{aligned}$$

In the general case, we can write

$$\left(T_{f\otimes g}, \varphi\right) = \left(T_{f(x)}, \left(T_{g(y)}, \varphi(x, y)\right)\right).$$

We can now define the tensor product of two distributions.

Definition 8.4.3 Let us consider the distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ and the distribution $S \in \mathcal{D}'(\mathbb{R}^m)$.

Then their tensor product, denoted by $T \otimes S$, is a distribution from the set $\mathcal{D}'(\mathbb{R}^{n+m})$ which is defined by

$$(T \otimes S, \varphi) = \left(T_x, \left(S_y, \varphi(x, y)\right)\right).$$

It is clear that we have the well definiteness of the distribution $T \otimes S$. Note that if φ is a function with compact support in R^{m+n}, then it is null outside a Cartesian product of compact sets $K_1 \times K_2$. If x does not belong to the compact K_1, then $\varphi(x, y)$ is null and the application $I : (x, y) \longmapsto (S_y, \varphi(x, y))$ is null. Therefore the support of the function I is inside the compact set K_1. We can verify without difficulty that the application I is continuous together with its derivatives of any order and therefore I is a test function on which the functional T acts. An example within the reach of the tensor product of the distributions is obtained with the distribution of Dirac, namely $\delta_a \otimes \delta_b = \delta_{(a,b)}$. Indeed,

$$(\delta_a \otimes \delta_b, \varphi(x, y)) = (\delta_a, (\delta_b, \varphi(x, y))) = (\delta_a, \varphi(x, b)) = \varphi(a, b).$$

On the other hand,

$$\left(\delta_{(a,b)}, \varphi(x, y)\right) = \varphi(a, b).$$

Proposition 8.4.2 1^o. *The following equality of sets holds true:*

$$supp\ (T \otimes S) = supp\ (T) \times supp\ (S).$$

2^o. *If the distributions T and S are tempered distributions so that $T \in \mathcal{E}'(\mathbb{R}^n)$, $S \in \mathcal{E}'(\mathbb{R}^m)$, then $T \otimes S \in \mathcal{E}'(\mathbb{R}^{n+m})$.*

Proof (i) The statement can be immediately deduced based on the definition of the support of a distribution and the definition of the tensor product of distributions.
(ii) It is enough to make this reasoning on compact supports of the distributions T and S, using point (i). ■

Definition 8.4.4 Let us consider the distributions $T, S \in \mathcal{D}'(\mathbb{R}^n)$. We define their product of convolution, denoted by $T * S$, by means of the formula

$$(T * S, \varphi) = \left(T_x \otimes S_y, \varphi(x + y)\right).$$

Observation 8.4.1 *The above product of convolution makes sense because the application $(x, y) \mapsto \varphi(x + y)$ is a function from $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, both in the situation that φ has compact support, and also in the situation that the domain of φ is not compact. In both cases, we have*

$$x \in supp\ (T),\ \ y \in supp\ (S),\ \ x + y \in K = compact,$$

from where we deduce that x and y are bounded, that is, the product of convolution of T with S is well defined.

Example 8.4.1 The product of convolution for the Dirac distribution δ_a with any distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ makes sense

$$(\delta_a * T, \varphi) = \left(T_y, \varphi(y + a)\right).$$

The last operation makes sense because the application $y \longmapsto \varphi(y + a)$ is a function from $C_0^\infty(\mathbb{R}^n)$. In the particular case when $a = 0$ (therefore for $\delta_0 = \delta$), we have

$$(\delta * T, \varphi) = \left(T_y, \varphi(y)\right) \Rightarrow \delta * T = T,\ \text{in}\ \mathcal{D}'(\mathbb{R}^n).$$

Proposition 8.4.3 *(i) If f and g are two functions from $L^1_{loc}(\mathbb{R}^n)$ so that $supp\ (f)$ is bounded, then it makes sense to consider the product of convolution for distributions generated by the functions f and g, that is, $T_f * T_g$.*

In addition, we have $T_f * T_g = T_{f*g}$.
(ii) For any distribution $T \in \mathcal{D}'(\mathbb{R}^1)$, *we have* $\delta_0' * T = T'$.
(iii) If T *is an arbitrary distribution,* $T \in \mathcal{D}'(\mathbb{R}^n)$, *then for any multi-index* α *we have* $D^\alpha \delta_0 * T = D^\alpha T$.
(iv) If it makes sense to consider the product of convolution for the distributions T *and* S, *then we have*

$$D^{\alpha+\beta}(T * S) = D^\alpha T * D^\beta S.$$

(v) If it makes sense to consider the product of convolution $T * S$, *then the application from the set* $\mathcal{D}'(\mathbb{R}^n)$ *into itself, defined by* $(T, S) \longmapsto T * S$, *is continuous with respect to each component.*

Proof (i) Based on the theorem of Fubini, we can write

$$\begin{aligned}\left(T_f * T_g, \varphi\right) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)\varphi(x+y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^n} g(y)\left(\int_{\mathbb{R}^n} f(x)\varphi(x+y)\right)\mathrm{d}y.\end{aligned}$$

Then, with a convenient change of variables, we have

$$\left(T_f * T_g, \varphi\right) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi-\eta)g(\eta)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\eta = \int_{\mathbb{R}^n}(f * g)(\xi)\varphi(\xi)\mathrm{d}\xi.$$

(ii) The statement is motivated by direct calculation

$$\begin{aligned}\left(\delta_0' * T, \varphi\right) &= \left(\delta_{0x}' * T_y, \varphi(x+y)\right) \\ &= \left(T_y, \left(\delta_{0x}', \varphi(x+y)\right)\right) = -\left(T_y, \varphi'(y)\right).\end{aligned}$$

(iii) The proof is analogous to those made at point (ii).
(iv) By direct calculation, we obtain

$$\begin{aligned}&\left(D^{\alpha+\beta}(T * S), \varphi\right) = (-1)^{|\alpha|+|\beta|}\left(T * S, D^{\alpha+\beta}\varphi(x+y)\right) \\ &= (-1)^{|\alpha|+|\beta|}\left(S, \left(T, D^{\alpha+\beta}\varphi(x+y)\right)\right) = (-1)^{|\alpha|}\left(S, \left(D^\beta T, D^\alpha\varphi(x+y)\right)\right) \\ &= \left(D^\alpha S, \left(D^\beta T, \varphi(x+y)\right)\right).\end{aligned}$$

(v) This proof is obvious. ∎

We finish the considerations on the product of convolution for distributions with the following result of regularity.

Theorem 8.4.2 *Let* S *be a distribution with compact support,* $S \in \mathcal{E}'(\mathbb{R}^n)$ *and* f *a function from* $C^\infty(\mathbb{R}^n)$. *Then it makes sense to consider the product of convolution between* S *and the distribution generated by the function* f, *that is,* $S * T_f$. *Moreover, the distribution* $S * T_f$ *can be identified with a function from* $C^\infty(\mathbb{R}^n)$ *so that*

$$\left(S * T_f\right)(x) = \left(S_y, \tau_x \check{f}(y)\right),$$

where we used the notation $\tau_x \check{f}(y) = f(x - y)$.

Proof It can be shown without difficulty that if S is a distribution with compact support, $S \in \mathcal{E}'(\mathbb{R}^n)$, then the application $x \longmapsto (S, \tau_x \check{f})$ is of class $C^\infty(\mathbb{R}^n)$. Then for φ an arbitrary test function, we have

$$\begin{aligned}
\left(S * T_f, \varphi\right) &= \left(S_y, \int_{\mathbb{R}^n} f(x)\varphi(x+y)\mathrm{d}x\right) = \left(S_\eta, \int_{\mathbb{R}^n} f(\xi-\eta)\varphi(\xi)\mathrm{d}\xi\right) \\
&= \left(S_\eta, \int_{\mathbb{R}^n} \tau_\xi \check{f}(\eta)\varphi(\xi)\mathrm{d}\xi\right) = \left(S_\eta, \left(T_{\varphi\xi}, \tau_\xi \check{f}(\eta)\right)\right) = \left(S_y \otimes T_{\varphi x}, \tau_x \check{f}(y)\right) \\
&= \left(T_{\varphi x} \otimes S_y, \tau_x \check{f}(y)\right) = \left(T_{\varphi x}, \left(S_y, \tau_x \check{f}(y)\right)\right) = \int_{\mathbb{R}^n} \varphi(x)\left(S_y, \tau_x \check{f}(y)\right)\mathrm{d}x
\end{aligned}$$

and this ends the proof. ■

8.5 The Fourier Transform in Distributions

We recall that if f is a function from $L^1(\mathbb{R}^n)$, then its Fourier transform can be defined by

$$\mathcal{F}_f(y) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x.y}\mathrm{d}x, \tag{8.5.1}$$

where the product $x.y$ can be defined by $x.y = \sum_{i=1}^{n} x_i y_i$. If we know the Fourier transform of the function f, $\mathcal{F}_f$, then the original function f is calculated by a formula that is analogous with formula (8.5.1), namely

$$\overline{\mathcal{F}_f}(y) = \int_{\mathbb{R}^n} f(x)e^{2\pi i x.y}\mathrm{d}x. \tag{8.5.2}$$

It is clear that the Fourier transform is well defined and bounded, because we have

$$\left|\mathcal{F}_f(y)\right| \leq \int_{\mathbb{R}^n} |f(x)|\mathrm{d}x = \|f\|_{L^1}.$$

Recall, also, in the following proposition, other main properties of the functions.

Proposition 8.5.1 *The main properties of the Fourier transform are*
(i) $\overline{\mathcal{F}_f} = \mathcal{F}_{\check{f}}$;
(ii) $\mathcal{F}_{\overline{f}} = \overline{\mathcal{F}_{\check{f}}}$;

(iii) If the sequence of the functions $\{f_n\}_n$ *is convergent, in* $\mathbb{R}^n$*, to the function* f*, then the sequence* $\{\mathcal{F}_{f_n}\}_n$ *is uniformly convergent on* $\mathbb{R}^n$ *to* $\mathcal{F}_f$*, that is,*

$$f_n \to f, \text{ in } L^1(\mathbb{R}^n) \Rightarrow \mathcal{F}_f, \text{ uniformly in } \mathbb{R}^n;$$

(iv) For a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$*, we will use the notation* $M^\alpha(x) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$*. If for any multi-index* α *so that* $|\alpha| \le k$*, we have that* $M^\alpha f \in L^1(\mathbb{R}^n)$*, then the Fourier transform of* f*,* $\mathcal{F}_f$*, is continuous differentiable on* $\mathbb{R}^n$ *and we have*

$$\mathcal{F}_{M^\alpha f} = \left(\frac{i}{2\pi}\right)^{|\alpha|} D^\alpha \mathcal{F}_f, \ \forall \alpha, \ |\alpha| \le k.$$

(v) If $f \in L^1(\mathbb{R}^n)$ *and if for any multi-index* α*,* $|\alpha| \le k$ *we have that* $D^\alpha f \in L^1(\mathbb{R}^n)$*, then* $M^\alpha \mathcal{F}_f \in L^\infty(\mathbb{R}^n)$*. Moreover*

$$\mathcal{F}_{D^\alpha f} = (2\pi i)^{|\alpha|} M^\alpha \mathcal{F}_f, \ \forall \alpha, \ |\alpha| \le k.$$

(vi) If f *and* g *are functions from* $L^1(\mathbb{R}^n)$*, then their product of convolution exists almost everywhere and* $f * g \in L^1(\mathbb{R}^n)$*. In addition, we have*

$$\mathcal{F}_{f*g} = \mathcal{F}_f . F_g.$$

(vii) If $f, g \in L^1(\mathbb{R}^n)$*, then*

$$\int_{\mathbb{R}^n} \mathcal{F}_f g(x) \mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{F}_g f(x) \mathrm{d}x.$$

Proof Most of these properties are proven in the first part of the book, in the chapter "Operational Calculus". ∎

Definition 8.5.1 We say that the function $f \in C^\infty(\mathbb{R}^n)$ is quickly decreasing if for any multi-indices α and β we have

$$\lim_{|x| \to +\infty} \left| M^\alpha(x) D^\beta f(x) \right| = 0.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the space of the quickly decreasing functions. It is easy to verify the following strict inclusions:

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n).$$

The function $f(x) = e^{-|x|^2}$ is quickly decreasing but does not have a compact support, and the function $g(x) = 1$ is infinitely differentiable but is not quickly decreasing, which proves that the inclusions are strict.

Definition 8.5.2 We say that a sequence $\{f_\nu\}_\nu$ of the functions from $\mathcal{S}(\mathbb{R}^n)$ is convergent to $f \in \mathcal{S}(\mathbb{R}^n)$, if the sequence $\{M^\alpha D^\beta f_\nu\}_\nu$ is convergent to $M^\alpha D^\beta f$, uniformly on $\mathbb{R}^n$, for any multi-indices α and β.

The convergence on the space of quickly decreasing functions is obtained by endowing the space $\mathcal{S}(\mathbb{R}^n)$ with the following family of semi-norms:

$$p_m(f) = \sum_{|\alpha|,|\beta|\leq m} \sup_{\mathbb{R}^n} \left|M^\alpha D^\beta f\right|, \ f \in \mathcal{S}(\mathbb{R}^n).$$

The main properties of the convergence introduced in Definition 8.5.2 are contained in the following Proposition.

Proposition 8.5.2 *The following statements hold true:*
(i) The application $f \longmapsto D^\alpha f$ is continuous. Also, it is defined on $\mathcal{S}(\mathbb{R}^n)$ with values in $\mathcal{S}(\mathbb{R}^n)$.
(ii) We say that the function g has a slow growth if $g \in C^\infty(\mathbb{R}^n)$ and satisfies the property

$$\forall\alpha, \ \exists p(\alpha) > 0 \text{ so that } \lim_{|x|\to+\infty} \|x\|^{-p(\alpha)} D^\alpha g(x) = 0.$$

The set of the functions with slow growth is denoted by $\mathcal{O}(\mathbb{R}^n)$. If the function $g \in \mathcal{O}(\mathbb{R}^n)$, then the application $f \longmapsto gf$ is a continuous function from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.
(iii) The following inclusion holds true:

$$\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n), \ \forall p \geq 1.$$

(iv) The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$, in the sense of the topology given by the convergence of sequences.
(v) The application $(f, g) \longmapsto f * g$ is a continuous function defined on the product space $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ with values in the space $\mathcal{S}(\mathbb{R}^n)$.
(vi) For any function $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\mathcal{F}_{M^\alpha f} = \left(\frac{i}{2\pi}\right)^{|\alpha|} D^\alpha \mathcal{F}_f, \ \mathcal{F}_{D^\alpha f} = (2\pi i)^{|\alpha|} M^\alpha \mathcal{F}_f.$$

(vii) $\mathcal{F}$ is an algebraic isomorphism and also a topological isomorphism

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n),$$

whose converse isomorphism is $\overline{\mathcal{F}}$ (the inverse Fourier transform).
(viii) The space $\mathcal{S}(\mathbb{R}^n)$ endowed with the product of convolution "" is an algebra and we have the relations*

$$\mathcal{F}_{f*g} = \mathcal{F}_f.\mathcal{F}_g, \ \mathcal{F}_{fg} = \mathcal{F}_f * \mathcal{F}_g.$$

Proof The proofs for these properties can be found in any book of Fourier analysis and are, in general, accessible. For instance, for (iii) we have that if $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$|f(x)|^p \leq C \prod_{j=1}^{n} \frac{1}{1+x_j^2}$$

and therefore

$$\|f\|_{L^p(\mathbb{R}^n)}^p \leq C \prod_{j=1}^{n} \int_{\mathbb{R}} \frac{1}{1+x_j^2} \mathrm{d}x = C\pi^n < \infty.$$

Regarding the automorphism from point (vii) we can show, first, that $\mathcal{F}_f \in \mathcal{S}(\mathbb{R}^n)$, using the formulas from point (vi). Then we can show that

$$\overline{\mathcal{F}_f}(y) = \mathcal{F}_f(-y) = \mathcal{F}_{f(-x)}(y).$$

Definition 8.5.3 A temperate distribution is any linear and continuous functional defined on the space of quickly decreasing functions $\mathcal{S}(\mathbb{R}^n)$.

The space of temperate distributions on the space $\mathbb{R}^n$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

A simple example of temperate distribution is given by the distribution T_f generated by a function f, of class $C^\infty(\mathbb{R}^n)$ and for which there is $k \geq 0$ so that

$$\left(1+|.|^2\right)^k f \in L^1(\mathbb{R}^n).$$

In particular, if $f \in L^1(\mathbb{R}^n)$, then $T_f \in \mathcal{S}'(\mathbb{R}^n)$.

If T is a temperate distribution then all its derivatives are also temperate distributions.

If T is a temperate distribution and $g \in \mathcal{O}^m(\mathbb{R}^n)$, then the product gT is also a temperate distribution. Moreover, the application $T \longmapsto gT$ is an application of the space $\mathcal{S}'(\mathbb{R}^n)$ with values in itself. It can be verified without difficulty that if the distribution T has compact support then it is a temperate distribution.

We believe that the reader can prove, without difficulty, the properties of temperate distributions included in the following proposition.

Proposition 8.5.3 *In the set of temperate distributions the following properties are satisfied:*
(i) If S is a temperate distribution on $\mathbb{R}^n$, and T is a temperate distribution on R^m, then the tensor product of the two distributions is a temperate distribution from R^{n+m}.
(ii) If the distribution T has compact support and the function f belongs to the space $\mathcal{S}(\mathbb{R}^n)$, then the product of convolution between f and T, $f * T$, can be identified with a function from the space $\mathcal{S}(\mathbb{R}^n)$ and the application $f \longmapsto f * T$ is continuous from the space $\mathcal{S}(\mathbb{R}^n)$ into itself.

Definition 8.5.4 Let T be a temperate distribution. Then its Fourier transform is defined by

$$(\mathcal{F}_T, \varphi) = \left(T, \mathcal{F}_\varphi\right), \ \forall \varphi \in C_0^\infty(\mathbb{R}^n), \tag{8.5.3}$$

formula (8.5.3) remaining valid also in the situation in which $\varphi \in \mathcal{S}(\mathbb{R}^n)$.
The inverse Fourier transform of a temperate distribution can be defined by

$$\left(\overline{\mathcal{F}_T}, \varphi\right) = \left(T, \overline{F_\varphi}\right), \ \forall \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{8.5.4}$$

We should note the fact that the values (T, F_φ) and $(T, \overline{F_\varphi})$ from (8.5.3) and from (8.5.4), respectively, are well defined because for any function φ from the space $C_0^\infty(\mathbb{R}^n)$ or from $\mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}_\varphi$ is a function from the space $\mathcal{S}(\mathbb{R}^n)$ and T is a temperate distribution.

Example 8.5.1 Let h be a function from $L^1(\mathbb{R}^n)$. Then the distribution T_h, generated by the function h, is a temperate distribution. By direct calculations, we obtain that the Fourier transform of the temperate distribution T_h is equal to the distribution generated by the Fourier transform of the function h. Indeed, we have

$$\begin{aligned}\left(\mathcal{F}_{T_h}, \varphi\right) = \left(T_h, \mathcal{F}_\varphi\right) &= \int_{\mathbb{R}^n} h(x)\mathcal{F}_\varphi(x)\mathrm{d}x \\ = \int_{\mathbb{R}^n} h(x)\int_{\mathbb{R}^n} \varphi(y)e^{-2i\pi x\cdot y}\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^n} \varphi(y)\int_{\mathbb{R}^n} h(x)e^{-2i\pi x\cdot y}\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \varphi(y)\mathcal{F}_h(y)\mathrm{d}y = \left(T_{\langle}, \varphi\right),\end{aligned}$$

that is

$$\mathcal{F}_{T_h} = T_{\mathcal{F}_h}, \ \forall h \in L^1(\mathbb{R}^n),$$

the equality taking place, obviously, in the sense of the distributions.

The main results regarding the Fourier transform for temperate distributions are contained in the following theorem.

Theorem 8.5.1 *The following statements hold true:*
(i) The Fourier transform performs an algebraic isomorphism and a topological isomorphism of the space of temperate distributions $\mathcal{S}'(\mathbb{R}^n)$ *in itself and whose converse isomorphism is the inverse Fourier transform* $\overline{\mathcal{F}}$.
(ii) For all multi-indices α *and for any temperate distribution* T, *the relations are satisfied*

$$\begin{aligned}\mathcal{F}_{M^\alpha T} &= \left(\frac{i}{2\pi}\right)^{|\alpha|} D^\alpha \mathcal{F}_T, \\ -\mathcal{F}_{D^\alpha T} &= (2i\pi)^{|\alpha|} M^\alpha \mathcal{F}_T.\end{aligned}$$

(iii) If the distribution T is from the space $\mathcal{E}'(\mathbb{R}^n)$, then its Fourier transform belongs to the space $\mathcal{O}^m(\mathbb{R}^n)$ and can be identified with the following function:

$$x \longmapsto \mathcal{F}_T(x) = T_y\left(e^{-2i\pi x.y}\right).$$

(iv) If T is a temperate distribution and the distribution S is from the space $\mathcal{E}'(\mathbb{R}^n)$ then it makes sense to consider the product of convolution $T * S$. Moreover, the Fourier transform of the product of convolution $T * S$ is equal to the usual product of the Fourier transforms, that is

$$\mathcal{F}_{T*S} = \mathcal{F}_T.\mathcal{F}_S \in \mathcal{S}'(\mathbb{R}^n).$$

Proof Points (i) and (ii) can be easily proven based on the properties of the Fourier transform for functions.

For detailed demonstrations of the points (iii) and (iv), the reader can use the book *Theories des distributions*, Hermann Paris, 1973, due to L. Schwartz [35]. ■

If we compute the Fourier transform of the distribution δ_a of Dirac, we find that the distribution which is obtained, $\mathcal{F}_{\delta_a}$, can be identified with the following function:

$$x \longmapsto \delta_{ay}\left(e^{-2i\pi x.y}\right) = e^{-2i\pi x.a}.$$

In particular, the Fourier transform of the distribution $\delta_0 = \delta$, $\mathcal{F}_\delta$, can be identified with the constant function 1.

If we consider the temperate distribution T_1 on $\mathbb{R}$, associated with the constant function equal to 1, which obviously is from $L^1_{loc}(\mathbb{R})$, and we apply the formulas of differentiation, from (ii) we obtain

$$\mathcal{F}_{T_1'}(x) = \mathcal{F}_0(x) = 0 = (2i\pi)x\mathcal{F}_{T_1}(x).$$

In this way, it can be shown that $\mathcal{F}_{T_1}$ is proportional with δ_0. If we apply the transform $\mathcal{F}_{T_1}$ to the function $x \longmapsto e^{-\pi x^2}$ we are led to the conclusion that $\mathcal{F}_{T_1} = \delta_0$.

The last result regarding the temperate distributions show that the Fourier transform is an automorphism of the space $L^2(\mathbb{R}^n)$.

Theorem 8.5.2 (Fourier–Plancherel) *The Fourier transform $\mathcal{F}$ and the other transform $\overline{\mathcal{F}}$ are two isometrics of the space $L^2(\mathbb{R}^n)$ into itself.*

Proof We have seen that if the function f is from the space $\mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}_f$ is also from the space $\mathcal{S}(\mathbb{R}^n)$. Moreover

$$\int_{\mathbb{R}^n} f(x)\overline{f(x)}\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\tilde{f}(-x)\mathrm{d}x = (f * \tilde{f})(0),$$

where we used the notation $\tilde{f}(x) = \overline{f(-x)}$.

Using the inverse Fourier transform, we obtain

$$(f * \tilde{f})(0) = \int_{\mathbb{R}^n} \mathcal{F}_{f*\tilde{f}}(y)\mathrm{d}y = \int_{\mathbb{R}^n} \mathcal{F}_f(y)\mathcal{F}_{\overline{f}}(y)\mathrm{d}y = \int_{\mathbb{R}^n} \mathcal{F}_f(y)\overline{\mathcal{F}_f}(y)\mathrm{d}y.$$

Also, for $f \in L^2(\mathbb{R}^n)$ we have $\mathcal{F}_f \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\begin{aligned}\left|(\mathcal{F}_f, \varphi)\right| &= \left|(f, \mathcal{F}_\varphi)\right| = \left|(u, \mathcal{F}_\varphi)_{L^2}\right| \\ &\leq \|f\|_{L^2}\|\mathcal{F}_\varphi\|_{L^2} = \|f\|_{L^2}\|\varphi\|_{L^2}, \ \varphi \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$

This inequality can be extended through density to all functions $\varphi \in L^2(\mathbb{R}^n)$. Thus, we deduce that $\mathcal{F}_f$ defines a linear and continuous functional on the Hilbert space $L^2(\mathbb{R}^n)$. Therefore, we have

$$\mathcal{F}_f \in L^2(\mathbb{R}^n) \text{ and } \|\mathcal{F}_f\|_{L^2} \leq \|f\|_{L^2}.$$

By replacing f with $\overline{\mathcal{F}_f}$ the contrary inequality is obtained, so we are led to the equality

$$\|f\|_{L^2} = \|\mathcal{F}_f\|_{L^2}, \ f \in L^2(\mathbb{R}^n),$$

which proves that $\mathcal{F}$ is an isometrics on the space $L^2(\mathbb{R}^n)$.

The fact that $\overline{\mathcal{F}}$ is an isometrics can be immediately obtained. ■

Chapter 9
Integral Formulas

9.1 Differential Operators

Definition 9.1.1 A partial differential equation of order k is any relation of the form

$$F\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \dots, \frac{\partial^k u}{\partial x_1^k}, \dots, \frac{\partial^k u}{\partial x_n^k}\right) = 0, \tag{9.1.1}$$

that is, a relationship between the n-dimensional variable $x = (x_1, x_2, \dots, x_n)$, $x \in \mathbb{R}^n$, the function $u = u(x)$ and the partial derivatives of the function u of order less than or equal to k.

Relation (9.1.1) is valid for $x \in \Omega$, where Ω is an open set from the $n-$dimensional space $\mathbb{R}^n$.

The partial derivative of order k of the function $u = u(x)$ can be written in the most general case, in the form

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} u(x), \tag{9.1.2}$$

where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$.

We say that $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multi-index of length k and denote by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ the length of the multi-index (in the present case the length is k). For the partial derivative from (9.1.2), when there is no danger of confusion, shorthand notations are also used

$$\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} u(x) \quad \text{or} \quad D^{\alpha} u(x).$$

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_9

Definition 9.1.2 A classical solution of the partial differential equation (9.1.1) on the set Ω is any function u which admits partial derivatives that are continuous up to and including order k, which when is replaced in the equation, transforms it into an identity on the open set Ω.

Observation 9.1.1 *In the following study, we will consider only differential equations with linear or quasilinear partial derivatives of the first order $k = 1$ and of second order $k = 2$, respectively. This means that F is assumed to depend linearly on the partial derivatives of the function u, of first order and of second order, respectively. In the case of quasilinear equations, both in the case of the equations of first order and of second order, the coefficients by which the function F can be expressed can depend on the unknown function u.*

We introduce now the main vector differential operators.

Definition 9.1.3 (i) For the scalar function $u : \Omega \to \mathbb{R}$, the gradient operator (denoted grad u) can be defined by means of the formula

$$\text{grad}\, u(x) = \left(\frac{\partial u}{\partial x_1}(x), \frac{\partial u}{\partial x_2}(x), \ldots, \frac{\partial u}{\partial x_n}(x) \right).$$

(ii) For the vector function $\vec{u} : \Omega \to \mathbb{R}^n$, we define the divergence operator (denoted by $div\ \vec{u}$), by means of the formula

$$\text{div}\ \vec{u}(x) = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}(x).$$

(iii) In the case $\Omega \subset \mathbb{R}^2$ and the vector function $u : \Omega \to \mathbb{R}$ has two components, the curl operator (denoted by $curl\ u$) can be defined by means of the formula

$$\text{curl}\, u = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1} \right).$$

For the case in which $\Omega \subset \mathbb{R}^3$, the vector function $\vec{u} : \Omega \to \mathbb{R}^3$ has three components u_1, u_2, u_3 and then the curl operator can be defined by means of the 3×3 dimensional formal determinant

$$\text{curl}\, \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

(iv) For the scalar function $u : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^n$, we define the Laplace operator (denoted by $\Delta\ u$) by the formula

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x).$$

Observation 9.1.2 1^o. *It can immediately be verified that the divergence operator is the trace of the Jacobean matrix defined by*

$$\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \dots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}.$$

2^o. *For the uniformity of the notation in the case of the gradient, divergence, and curl operators, for $n = 3$, we use the Hamilton operator $\bigtriangledown$ called also the nabla operator and defined by*

$$\bigtriangledown = \frac{\partial}{\partial x_1}\vec{i} + \frac{\partial}{\partial x_2}\vec{j} + \frac{\partial}{\partial x_3}\vec{k}.$$

Thus, instead of grad u *we will write* $\bigtriangledown u(x)$, *instead of* $div\ u$ *we will write* $\bigtriangledown .u(x)$, *and instead of* $curl\ u$ *we will write* $\bigtriangledown \times u(x)$.

3^o. *Clearly, the gradient operator is applied to a scalar field and the result is a vector field, while the divergence operator is applied to a vector field and the result is a scalar field. The only differential operator which preserves the nature of the field is the curl operator which is applied to a vector field, and the result is also a vector field.*

With the help of the differential operators introduced in Definition 9.1.3, we can define the differential operators of second order. This is achieved by combining two of the three differential operators introduced above. It is clear that the single possible combinations are the following:

$$\text{div}\,(\text{grad}\ u)\,,\ \ \text{grad}\,(\text{div}\ \vec{u})\,,\ \ \text{div}\,(\text{curl}\ \vec{u})\,,\ \ \text{curl}\,(\text{grad}\ u)\,,\ \ \text{curl}\,(\text{curl}\ \vec{u})\,.$$

The values of these differential operators of second order are given in the proposition which follows.

Proposition 9.1.1 *For the differential operators of second order, the following formulas are satisfied:*

(i) div (grad u) $= \delta u$, *if* $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$;
(ii) div (curl u) $= 0$, *if* $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^2$;
(iii) div (curl $\vec{u}$) $= 0$, *if* $\vec{u} \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^3$;
(iv) curl (grad u) $= 0$, *if* $u \in C^2(\Omega)$;
(v) grad (div $\vec{u}$) $=$ curl (curl $\vec{u}$) $+ \Delta\vec{u}$;
(vi) curl (curl $\vec{u}$) $=$ grad (div $\vec{u}$) $- \Delta\vec{u}$.

Proof For points (i)–(v), the proofs are obtained without any difficulty, using Definition 9.1.3. The point (vi) is obtained from (v). ■

9.2 Classical Integral Formulas

Definition 9.2.1 A hypersurface in $\mathbb{R}^n$ is, by definition, the set S of all points $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ which verify the relation $G(x_1, x_2, \ldots, x_n) = 0$, where G is a continuously differentiable function defined on $\mathbb{R}^n$ and which verifies the conditions required for the application of the theorem of the implicit functions with respect to one of its variables.

A hypersurface S can be represented locally in the parametric form as follows:

$$S = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ : \ (x_1, x_2, \ldots, x_n) = g(s_1, s_2, \ldots, s_{n-1})\right\},$$

where g is a given continuously differentiable function $g : D \to \mathbb{R}^n$. Here, D is an open set from $\mathbb{R}^{n-1}$.

Theorem 9.2.1 *Let S be a regular and compact hypersurface from the space $\mathbb{R}^n$, with $n \geq 2$, having the boundary ∂S. For any vector function $\vec{u}$ which is continuously differentiable on an open set that contains S, the following equality holds true:*

$$\int_{\partial S} \sum_{i=1}^{n} u_i \mathrm{d}x_i = \int_S \sum_{k<j} \left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\right) \mathrm{d}x_k \wedge \mathrm{d}x_j, \tag{9.2.1}$$

where with $\mathrm{d}x_k \wedge \mathrm{d}x_j$ we denote the exterior product of the two differential forms $\mathrm{d}x_k$ and $\mathrm{d}x_j$. In the particular case when $n = 2$, formula (9.2.1) acquires the following simpler form:

$$\int_{\partial S} [P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y] = \int_S \left(\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right) \mathrm{d}x\mathrm{d}y.$$

If $n = 3$, formula (9.2.1) becomes

$$\int_{\partial S} \sum_{i=1}^{n} u_i \mathrm{d}x_i = \int_S curl\vec{u}(x) d\vec{\sigma(x)},$$

where $d\vec{\sigma(x)}$ represents the oriented area element from the space $\mathbb{R}^3$.

Proof Essentially, the proof is based on integration of differentiable forms. In any book of mathematical analysis, a more general form of Stokes formula can be found

$$\int_{\partial S} \omega = \int_S d\omega,$$

where ω is a $n-1$-dimensional differentiable form of class C^1 on the hypersurface S. If ω has the form

$$\omega = \sum_{i=1}^{n} u_i \mathrm{d}x_i,$$

then

$$d\omega = \sum_{k<j} \left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} \right) \mathrm{d}x_k \wedge \mathrm{d}x_j,$$

from where the desired result is obtained. ■

Theorem 9.2.2 (Ostrogradsky formula) *Let K be a regular compact set from $\mathbb{R}^3$ having the boundary ∂K. Then for any vector function $\vec{u}$ which is continuously differentiable on an open set Ω which contains the compact set K, we have*

$$\int_{\partial K} \vec{u}(x) d\vec{\sigma}(x) = \int_K div\, \vec{u}(x) \mathrm{d}x,$$

where $d\vec{\sigma}(x)$ represents the oriented area element on the boundary ∂K.

Proof As in the case of the formula of Stokes, the proof is based on integration of differentiable forms. For a differentiable $n-1$-dimensional form ω of class C^1 on K, we have

$$\omega = \sum_{i=1}^{n} u_i d\sigma(x)_i \Rightarrow d\omega = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$$
$$= \mathrm{div}\, \vec{u}\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3,$$

so that we can immediately deduce the formula from the statement. ■

Observation 9.2.1 *The Ostrogradsky formula is also called the flux-divergence formula.*

Theorem 9.2.3 (Green's formulas) *Let K be a compact set of class C^1 from $\mathbb{R}^3$ having the boundary ∂K.*

(i) If φ is a continuously differentiable function defined on the open set Ω that contains the compact set K, and ψ is a function of class C^2 on Ω, then we have the following identity:

$$\int_{\partial K} \varphi(x) \frac{\partial \psi}{\partial \nu}(x) d\sigma(x) = \int_{\partial K} \mathrm{grad}\ \varphi(x).\mathrm{grad}\ \psi(x) \mathrm{d}x + \int_K \varphi(x) \delta\psi(x) \mathrm{d}x. \tag{9.2.2}$$

(ii) If the functions φ and ψ are of class C^2 on Ω, then we have the following identity:

$$\int_{\partial K} \varphi(x) \frac{\partial \psi}{\partial \nu}(x) d\sigma(x) - \int_{\partial K} \psi(x) \frac{\partial \varphi}{\partial \nu}(x) d\sigma(x)$$
$$= \int_K [\varphi(x) \Delta\psi(x) - \psi(x)\Delta\varphi(x)]\, \mathrm{d}x, \tag{9.2.3}$$

where we denote by $\frac{\partial \psi}{\partial \nu}$ *the derivative in the direction of the unit normal (called also the normal derivative) of the function* ψ *and which can be defined by*

$$\frac{\partial \psi}{\partial \nu}(x) = \text{grad}\, \psi(x).\vec{\nu}.$$

With $\vec{\nu}$ *we denote the normal to the surface* ∂K*, oriented, by convention, outside* K*.*

Proof For point (i), it is enough to consider the vector function $\vec{u}$ given by $\vec{u} = \varphi.\text{grad}\, \psi$ and apply the Ostrogradsky formula. The functions φ and ψ are scalar functions. To prove (ii), we write, close to the formula (9.2.2), a similar formula in which we change the role of the functions φ and ψ. The obtained relation is subtracted, member by member, from (9.2.2) and so we obtain the desired formula (9.2.3). ■

Observation 9.2.2 *Formula (9.2.2) will be called the first Green formula, and formula (9.2.3) will be called the second Green formula.*

At the end of this chapter, we recall some common notations.

Let Ω be an open set from $\mathbb{R}^n$. Then, we will use the following notations

- $C^0(\Omega)$: the set of continuous functions on $\mathbb{R}^n$;
- $C^0(\overline{\Omega})$: the set of restrictions to $\overline{\Omega}$ of the continuous functions on $\mathbb{R}^n$;
- $C^k(\Omega)(k \geq 1)$: the set of the functions which admit derivatives up to order k and the derivatives of order k are continuous on Ω;
- $C^k(\overline{\Omega})(k \geq 1)$: the set of restrictions to Ω of the functions which admit derivatives up to order k and the derivatives of order k are continuous on the whole space $\mathbb{R}^n$;
- $C_0^k(\Omega)(k \geq 1)$: the set of the functions from $C^k(\Omega)$ which in addition have compact support included in Ω;
- $C^\infty(\Omega)$: the set of the functions which are infinitely differentiable on Ω;
- $C_0^\infty(\Omega)$: the set of the infinitely differentiable functions on Ω which in addition have compact support included in Ω.
  The set $C_0^\infty(\Omega)$ endowed with the topology of the strict inductive limit (introduced in Chap. 1) is also denoted by $\mathcal{D}(\Omega)$.

Chapter 10
Partial Differential Equations of the First Order

10.1 The Cauchy Problem

In this chapter, we consider partial differential equations of the form

$$\sum_{j=1}^{n} a_j(x, u(x)) \frac{\partial u}{\partial x_j}(x) = b(x, u(x)), \ \forall x \in \Omega \subset \mathbb{R}^n, \tag{10.1.1}$$

called *quasilinear partial differential equations of the first order*, where the coefficients $a_j, \ j = \overline{1, n}$ and the right-hand side b are given functions which can depend on the unknown function $u = u(x)$. An important particular case of the equation (10.1.1) is *the linear equation with partial derivatives of the first order*, whose general form is

$$\sum_{j=1}^{n} a_j(x) \frac{\partial u}{\partial x_j}(x) + a_0(x)u(x) = b(x) \, , \ \forall x \in \Omega \subset \mathbb{R}^n, \tag{10.1.2}$$

in which the coefficients a_j, a_0 and the right-hand side b are given functions which do not depend on the unknown function.

Definition 10.1.1 We say that we have a Cauchy problem associated with the quasilinear partial differential equation (10.1.1) if we provide a given and regular hypersurface (that is, continuously differentiable) S, that contains all points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for which $G(x_1, x_2, \ldots, x_n) = 0$ and prescribe a function Φ defined on S so that the unknown function coincides with Φ on S.

Observation 10.1.1 *We say that the hypersurface S is regular, that is, continuously differentiable, if the function G which defines the hypersurface S,*

$$S = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ : \ G(x_1, x_2, \ldots, x_n) = 0 \right\},$$

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is continuously differentiable on $\mathbb{R}^n$ *and verifies the conditions of the theorem of the implicit functions with respect to one of the variables.*

Let us consider two simple examples.
1^o. Let us consider the Cauchy problem

$$\frac{\partial u}{\partial x}(x) = 0, \text{ in } \Omega_1,$$
$$u(x_1, 0) = \Phi(x_1), \ \forall x_1 \in \mathbb{R},$$

where we used the notation $\Omega_1 = \{x = (x_1, x_2) : x_2 > 0\}$.

For this problem, the "initial" condition, which is also called the Cauchy condition, that is, $u(x_1, 0) = \Phi(x_1)$, together with the partial differential equation imply

$$\Phi'(x_1) = 0 \text{ if and only if } \Phi = \text{constant}.$$

2^o. Now consider the problem

$$\frac{\partial u}{\partial x_1}(x) = 0, \text{ in } \Omega_2,$$
$$u(x_1, x_1) = \Phi(x_1), \ \forall x_1 \in \mathbb{R},$$

where we used the notation $\Omega_2 = \{x = (x_1, x_2) : x_2 > x_1\}$. It can be easily shown that for this problem, the initial condition together with the equation with partial derivatives lead to the conclusion that the solution of the Cauchy problem is the function u given by $u(x_1, x_2) = \Phi(x_2)$.

From these two examples, we deduce that the presence of the hypersurface S is essential in the existence of a solution of the Cauchy problem.

In the 2-dimensional case, the solution u of a partial differential equation of the first order can be associated with a surface from $\mathbb{R}^3$ by an equation of the form $x_3 = u(x_1, x_2)$.

In the following proposition, we obtain a condition for a surface to be tangent for which we have a geometric interpretation.

Proposition 10.1.1 *The integral surface of the Eq. (10.1.1) is tangent in each point to the characteristic direction* (a_1, a_2, b)*, defined by the coefficients of the quasilinear equation with partial derivatives of the first order.*

Proof The result is immediately obtained if the quasilinear equation is interpreted as a scalar product of two vectors from $\mathbb{R}^3$. Then, the tangent vectors to the integral surface are

$$\left(1, 0, \frac{\partial u}{\partial x_1}\right) \text{ and } \left(0, 1, \frac{\partial u}{\partial x_2}\right).$$

It is obvious then that the normal vector to this surface is

$$\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, -1\right),$$

and this ends the proof. ■

Definition 10.1.2 A characteristic curve associated with the characteristic direction (a_1, a_2, b), is any curve which is tangent in any of its point to the characteristic direction.

According to this definition we deduce that a characteristic curve is obtained by solving the following system of partial differential equations

$$\frac{\partial x_1}{\partial t} = a_1, \quad \frac{\partial x_2}{\partial t} = a_2, \quad \frac{\partial x_3}{\partial t} = b.$$

Proposition 10.1.2 *For any triplet* $\left(x_1^0, x_2^0, x_3^0\right) \in \mathbb{R}^3$, *there is only one characteristic curve which starts from this triplet.*

Proof Obviously, we have here a result of existence of the solutions which is obtained from the general theory of ordinary differential equations. ■

A final result regarding to the characteristic curves is included in the following proposition.

Proposition 10.1.3 *Let* $\left(x_1^0, x_2^0, x_3^0\right)$ *be an arbitrary point on a characteristic surface. The characteristic curve which passes through this point is fully contained in the characteristic surface.*

Proof Let us consider the system of relations

$$x_1 = x_1(t),$$
$$x_2 = x_2(t),$$
$$x_3 = x_3(t),$$

that is, the parametric representation of the characteristic curve that passes through the point of coordinates $\left(x_1^0, x_2^0, x_3^0\right)$.

We introduce the function $U(t)$ by

$$U(t) = x_3(t) - u\left(x_1(t), x_2(t)\right). \tag{10.1.3}$$

It is clear that U is null for $t = 0$. We compute the derivative of the function U starting from the relation of definition (10.1.3)

$$\begin{aligned} U'(t) &= \frac{\partial x_3}{\partial t} - \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t} - \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t} \\ &= b - a_1\frac{\partial u}{\partial x_1} - a_2\frac{\partial u}{\partial x_2} = 0, \end{aligned} \tag{10.1.4}$$

in which we used the fact that u is the solution of the quasilinear equation (10.1.1) in the particular case $n = 2$.

From (10.1.4) we deduce that U is a constant and because for $t = 0$ we have that $U = 0$, we deduce that U is identically null and this proves that the characteristic curve is fully contained in the integral surface. ■

10.2 Existence and Uniqueness of the Solution

As a corollary of Proposition 10.1.3 we deduce that the integral surface, associated with the solution u of the quasilinear equation, is a reunion of characteristic curves that pass through the points located on this surface. On the other hand, the above results regarding the characteristic curves suggest a solving method for quasilinear equations. We will prove, in the following theorem, a result of existence in a more general case.

Theorem 10.2.1 *Let S be a hypersurface of class C^1 (that is, the function g that defines the hypersurface is continuously differentiable). Suppose that the coefficients a_j, $j = \overline{1, n}$ and the the right-hand side b of the quasilinear equation are given and continuously differentiable functions. We also suppose that the vector with the following components*

$$(a_1, (x, \Phi(x)), a_2(x, \Phi(x)), \ldots, a_n(x, \Phi(x)))\,,$$

is not tangent to the hypersurface S in any point x from S. Then there is only one solution of the Cauchy problem defined in a neighborhood of S.

Proof We consider that the hypersurface S is defined in a parametric form by means of the function $g : \mathbb{R}^{n-1} \to \mathbb{R}^n$,

$$g : \begin{pmatrix} \mathbb{R}^{n-1} & \longrightarrow & \mathbb{R}^n \\ s = (s_1, s_2, \ldots, s_{n-1}) & \longmapsto & (g_1(s), g_2(s), \ldots, g_n(s)) \end{pmatrix}. \tag{10.2.1}$$

The condition that the vector $(a_1, a_2, \ldots, a_n)$ is not tangent to the hypersurface S may be reformulated by imposing to the following determinant

$$\begin{vmatrix} \frac{\partial g_1}{\partial s_1} & \frac{\partial g_1}{\partial s_2} & \ldots & \frac{\partial g_1}{\partial s_{n-1}} & a_1(g(s), \Phi(g(s))) \\ \frac{\partial g_2}{\partial s_1} & \frac{\partial g_2}{\partial s_2} & \ldots & \frac{\partial g_2}{\partial s_{n-1}} & a_2(g(s), \Phi(g(s))) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial g_n}{\partial s_1} & \frac{\partial g_n}{\partial s_2} & \ldots & \frac{\partial g_n}{\partial s_{n-1}} & a_n(g(s), \Phi(g(s))) \end{vmatrix}$$

to be nonzero for all those values of s for which $g(s)$ belongs to the hypersurface S. To prove the existence and the uniqueness of the solution of the Cauchy problem, considered above, we will consider a new Cauchy problem, namely

$$\begin{cases} \frac{\partial x_j}{\partial t}(s,t) = a_j(x_1(s,t), x_2(s,t), \dots, x_n(s,t), y(s,t)) \,,\ j = 1, 2, \dots, n \\ \frac{\partial y}{\partial t}(s,t) = b(x_1(s,t), x_2(s,t), \dots, x_n(s,t), y(s,t)) \\ x_j(s,0) = g_j(s) \,,\ j = 1, 2, \dots, n \\ y(s,0) = \Phi(g(s)) \end{cases} \tag{10.2.2}$$

We have here a system of partial differential equations with respect to t with initial conditions for $t = 0$ in which s is a parameter. If the data that define the Cauchy problems of the form (10.1.2) are continuously differentiable, then we ensured the existence and the uniqueness of a solution (x, y) for such kind of Cauchy problem on an interval $[0, t_0]$. Taking into account the dependence of class C^1 of the solution of a differential system with respect to a parameter, we deduce in addition that the solution of this Cauchy problem is continuously differentiable with respect to the parameter s.

Consequently, we can consider the application

$$\begin{pmatrix} \mathbb{R}^{n-1} \times [0, t_0] & \rightarrow & \mathbb{R}^n \times \mathbb{R} \\ (s,t) & \longmapsto & (x(s,t), y(s,t)) \end{pmatrix},$$

which associates to the pair (s, t), the pair $(x(s, t), y(s, t))$.

Obvious this application is locally invertible in a neighborhood of the point $(s, 0)$, because the application satisfies the conditions of the local inversion theorem.

In fact, the determinant

$$\begin{vmatrix} \frac{\partial x_1}{\partial s_1}(s,0) & \frac{\partial x_1}{\partial s_2}(s,0) & \dots \frac{\partial g_1}{\partial s_{n-1}}(s,0) & \frac{\partial x_1}{\partial t}(s,0) \\ \frac{\partial x_2}{\partial s_1}(s,0) & \frac{\partial x_2}{\partial s_2}(s,0) & \dots \frac{\partial x_2}{\partial s_{n-1}}(s,0) & \frac{\partial x_2}{\partial t}(s,0) \\ \dots & \dots & \dots & \dots \\ \frac{\partial x_n}{\partial s_1}(s,0) & \frac{\partial x_n}{\partial s_2}(s,0) & \dots \frac{\partial x_n}{\partial s_{n-1}}(s,0) & \frac{\partial x_n}{\partial t}(s,0) \end{vmatrix}$$

is equal to the determinant

$$\begin{vmatrix} \frac{\partial g_1}{\partial s_1} & \frac{\partial g_1}{\partial s_2} & \dots \frac{\partial g_1}{\partial s_{n-1}} & a_1(g(s), \Phi(g(s))) \\ \frac{\partial g_2}{\partial s_1} & \frac{\partial g_2}{\partial s_2} & \dots \frac{\partial g_2}{\partial s_{n-1}} & a_2(g(s), \Phi(g(s))) \\ \dots & \dots & \dots & \dots \\ \frac{\partial g_n}{\partial s_1} & \frac{\partial g_n}{\partial s_2} & \dots \frac{\partial g_n}{\partial s_{n-1}} & a_n(g(s), \Phi(g(s))) \end{vmatrix}$$

and this determinant, by the hypothesis, is nonzero.

Thus, we can consider the application $x \mapsto (s(t), t(s))$ and with its help we define the function $u(x) = y(s(x), t(s))$. It remains now to verify that the function u, defined in this way, satisfies the initial condition and quasilinear partial differential equation. We have

$$\forall x \in S \; : \; u(g(s)) = y(s(g(s)), t(g(s))) = y(s, 0) = \Phi(g(s)),$$

where we take into account the uniqueness result from the local inversion theorem.

On the other hand, we have

$$\sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}(s) = \sum_{j=1}^{n} a_j(x)\sum_{k=1}^{n-1}\left\{\sum_{k=1}^{n-1}\frac{\partial y}{\partial s_k}\frac{\partial s_k}{\partial x_j} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial x_j}\right\}$$
$$= \sum_{j=1}^{n}\frac{\partial y}{\partial s_k}\left\{\sum_{j=1}^{n} a_j(x)\sum_{k=1}^{n-1}\frac{\partial s_k}{\partial x_j}\right\} + \frac{\partial y}{\partial t}\sum_{j=1}^{n} a_j(x)\frac{\partial t}{\partial x_j}$$
$$= \sum_{k=1}^{n-1}\frac{\partial y}{\partial s_k}\left\{\sum_{j=1}^{n}\frac{\partial x_j}{\partial t}\frac{\partial s_k}{\partial x_j}\right\} + \frac{\partial y}{\partial t}\sum_{j=1}^{n}\frac{\partial x_j}{\partial t}\frac{\partial t}{\partial x_j},$$

in which we take into account the definition of the differential system for which x_j and y are solutions. In addition, we have

$$\sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}(x) = 1$$
$$= \sum_{k=1}^{n-1}\frac{\partial y}{\partial s_k}\frac{\partial s_k}{\partial t} + \frac{\partial y}{\partial t}.1 = \frac{\partial y}{\partial t} = b(x, u(x)).$$

Thus, we proved the existence of a solution of the Cauchy problem in a neighborhood of the hypersurface S, with the hypothesis that S can be parametrized in the form (10.1.1). But, this parametrization is possible by application of the theorem of the implicit functions to function G which defines the hypersurface S. The theorem will be fully demonstrated if we prove the uniqueness of the solution of the Cauchy problem.

To this end we will consider the particular solutions u of the Cauchy problems defined in a neighborhood of each point from S. To demonstrate the local uniqueness it is enough to prove that the hypersurface defined in $\mathbb{R}^{n+1}$ by the solution u of the problem, is the reunion of the curves obtained by solving the following differential system

$$\begin{cases} \frac{\partial x_j}{\partial t} = a_j(x, y), \ \ j = 1, 2, .., n \\ \frac{\partial y}{\partial t} = b(x, y). \end{cases} \tag{10.2.3}$$

The reunion of the families with $n-1$ parameters of the integral curves of a differential system of the form (10.1.3), defines a solution of the quasilinear partial differential equation of the first order. Conversely, let u be a solution of the quasilinear partial differential equation. We can use the decomposition

$$\begin{cases} \frac{\partial x_j}{\partial t} = a_j(x, u(x)), \ \ j = 1, 2, .., n \\ x_j(0) = x_j^0, \ \ x^0 = \left(x_j^0\right) \in \mathbb{R}^n. \end{cases} \tag{10.2.4}$$

Now, we define the function y by $y(t) = u(x, t)$, so we get the result

$$\frac{\partial y}{\partial t} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial t} = \sum_{j=1}^{n} a_j(x, u) \frac{\partial u}{\partial x_j} = b(x, u).$$

This proves that if the graph of the function u intersects an integral curve of the differential system (10.2.4) in a point $\left(x^0, u(x^0)\right)$ then the graph contains the whole integral curve. The result of uniqueness which refers to the solutions of the differential systems thus prove the uniqueness of the solution of the Cauchy problem for the quasilinear partial differential equation. ■

Observation 10.2.1 *From the proof of the Theorem 10.2.1 we can deduce the following conclusions:*

1^o. *If the hypersurface S is defined in the parametric form starting from a function $g : \mathbb{R}^{n-1} \to \mathbb{R}^n$, then the tangent plane to the hypersurface S in a point s is generated by $n - 1$ dimensional vectors of the form*

$$\left(\left(\frac{\partial g_1}{\partial s_1}, \frac{\partial g_2}{\partial s_1}, \ldots, \frac{\partial g_n}{\partial s_1} \right), \left(\frac{\partial g_1}{\partial s_2}, \frac{\partial g_2}{\partial s_2}, \ldots, \frac{\partial g_n}{\partial s_2} \right), \ldots, \left(\frac{\partial g_1}{\partial s_{n-1}}, \frac{\partial g_2}{\partial s_{n-1}}, \ldots, \frac{\partial g_n}{\partial s_{n-1}} \right) \right).$$

The condition from the statement of the theorem (and also the form of the condition from the proof) has the following significance: the vector of components

$$\left(a_1\left(g(s), \Phi\left(g(s)\right)\right), a_2\left(g(s), \Phi\left(g(s)\right)\right), \ldots, a_n\left(g(s), \Phi\left(g(s)\right)\right)\right)$$

does not belong to the tangent plane, mentioned above.

2^o. *The proof of Theorem 10.2.1 suggests a method of solving the Cauchy problem associated with a quasilinear equation with partial derivatives of the first order.*

We want to now give an example of the Cauchy problem associated with a quasilinear partial differential equation.

Example 10.2.1 Let us solve the following Cauchy problem

$$\begin{aligned} & x_1 \frac{\partial u}{\partial x_1}(x_1, x_2, x_3) + 2x_2 \frac{\partial u}{\partial x_2}(x_1, x_2, x_3) \\ & \quad + \frac{\partial u}{\partial x_3}(x_1, x_2, x_3) = 3u(x_1, x_2, x_3), \ \forall (x_1, x_2, x_3) \in \mathbb{R}^3, \\ & u(x_1, x_2, 0) = \Phi(x_1, x_2), \ \forall (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

In the present case, the hypersurface S is the plane

$$\{x, \ x = (x_1, x_2, x_3) \ : \ x_3 = 0\}.$$

The hypersurface S can be parametrized by means of the function g

$$\begin{pmatrix} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (s_1, s_2) & \longmapsto & (s_1, s_2, 0) \,. \end{pmatrix}$$

The condition of no tangency from the proof of Theorem 10.2.1 is reduced to the study of the determinant

$$\begin{vmatrix} 1 & 0 & s_1 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{vmatrix},$$

which, obviously, is nonzero. We introduce a differential system, accompanied by initial conditions, of the form

$$\begin{cases} \frac{\partial x_1}{\partial t}(s_1, s_2, t) = x_1(s_1, s_2, t), \\ \frac{\partial x_2}{\partial t}(s_1, s_2, t) = 2x_2(s_1, s_2, t), \\ \frac{\partial x_3}{\partial t}(s_1, s_2, t) = 1, \\ \frac{\partial y}{\partial t}(s_1, s_2, t) = 3y(s_1, s_2, t), \\ x_1(s_1, s_2, 0) = s_1, \\ x_2(s_1, s_2, 0) = s_2, \\ x_3(s_1, s_2, 0) = 0, \\ y(s_1, s_2, 0) = \Phi(s_1, s_2). \end{cases}$$

By solving this differential system and by taking into account the initial conditions, we obtain the solution

$$\begin{cases} x_1(s_1, s_2, t) = s_1 e^t, \\ x_2(s_1, s_2, t) = s_2 e^{2t}, \\ x_3(s_1, s_2, t) = t, \\ y(s_1, s_2, t) = \Phi(s_1, s_2) e^{3t}. \end{cases}$$

We can now express the values of $s_1,\ s_2$ and t as functions of $x_1,\ x_2$ and x_3, and this corresponds to the step of inversion of the application $(t, s) \mapsto\ x$. So, we have

$$\begin{aligned} t &= x_1, \\ s_1 &= x_1 e^{-x_3}, \\ s_2 &= x_2 e^{-2x_3}, \end{aligned}$$

so that, finally, we find the solution

$$u\,(x_1, x_2, x_3) = \Phi\left(x_1 e^{-x_3}, x_2 e^{-2x_3}\right) e^{3x_3}.$$

At the end of the paragraph, we will solve a Cauchy problem attached to a nonlinear partial differential equation of the first order.

Example 10.2.2 Consider the Cauchy problem

$$F(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) = 0,$$
$$x_i = \varphi_i,\ i = \overline{1, n-1},\ x_n = 1,\ u = e^{\varphi_1},\ \text{on } (S), \tag{10.2.5}$$

where

$$F(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} x_i p_i + f(p_1, p_2, \ldots, p_n).$$

Also, we used the notations of Monge

$$p_i = \frac{\partial u}{\partial x_i},\ i = \overline{1, n}.$$

Equation (10.2.5)$_1$ is called *the generalized equation of Clairaut*. We will expand the notations of Monge

$$\frac{\partial F}{\partial x_i} = X_i,\ \frac{\partial F}{\partial p_i} = P_i,\ i = \overline{1, n},\ \frac{\partial F}{\partial u} = U$$

and then we obtain the characteristic system

$$\frac{\mathrm{d}x_i}{\mathrm{d}s} = P_i,\ \frac{\mathrm{d}p_i}{\mathrm{d}s} = -(X_i + p_i U),\ i = \overline{1, n},\ \frac{du}{\mathrm{d}s} = \sum_{i=1}^{n} p_i P_i,$$

so if we take into account the Cauchy conditions

$$x_i(0) = x_i^0,\ p_i(0) = p_i^0,\ i = \overline{1, n},\ u(0) = u_0,$$

we obtain the solution

$$x_i = (x_i^0 + a_i^0)e^s - a_i^0,\ p_i = p_i^0\ , i = \overline{1, n},\ u = (u_0 + b_0)e^s - b_0, \tag{10.2.6}$$

where

$$a_i^0 = \frac{\partial f}{\partial p_i}(p_1^0, p_2^0, \ldots, p_n^0),\ i = \overline{1, n},$$
$$b_0 = \sum_{i=1}^{n} p_i^0 a_i^0 - f(p_1^0, p_2^0, \ldots, p_n^0).$$

The values $x_i^0,\ p_i^0,\ i = \overline{1, n}$ and u_0 are determined by the relations

$$\begin{aligned}
&x_i^0 = \alpha_i(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}),\ i = \overline{1,n},\ u_0 = \alpha(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}),\\
&F(x_1^0, x_2^0, \ldots, x_n^0, u_0, p_1^0, p_2^0, \ldots, p_n^0) = 0, \qquad (10.2.7)\\
&\sum_{i=1}^{n} p_i^0 \frac{\partial \alpha_i}{\partial \varphi_j}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}) = \frac{\partial \alpha}{\partial \varphi_j}(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}),\ j = \overline{1, n-1},
\end{aligned}$$

where the functions $\alpha_1, \alpha_2, \ldots, \alpha_n$ and α define the hypersurface (S). In our case the relations (10.2.7) become

$$\begin{aligned}
&x_i^0 = \varphi_i,\ i = \overline{1, n-1},\ x_n = 1,\\
&u_0 = e^{\varphi_1},\ p_1^0 = e^{\varphi_1},\ p_j^0 = 0,\ j = \overline{2, n-1}, \qquad (10.2.8)\\
&e^{\varphi_1} = \varphi_1 e^{\varphi_1} + p_n^0 + f(e^{\varphi_1}, 0, \ldots, 0, p_n^0).
\end{aligned}$$

Suppose that the function f satisfies the conditions of the theorem of implicit functions that allows us the solve the Eq. $(10.2.8)_3$ with respect to p_n^0. We replace the expression of p_n^0, of $x_i^0,\ i = \overline{1, n-1}$ from $(10.2.8)_1$ and of $p_i^0,\ i = \overline{1, n-1}$ from $(10.2.8)_2$ in the solution (10.2.6) of the characteristic system and we will obtain the desired hypersurface.

Chapter 11
Linear Partial Differential Equations of Second Order

11.1 The Cauchy Problem

Let us consider the differential operator

$$L(u)(x)=\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x)+\sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}(x)+a_0(x)u(x), \quad (11.1.1)$$

in which the coefficients $a_{ij}\ (i, j = \overline{1, n}),\ a_j\ (j = \overline{1, n})$ and a_0 are regular functions which depend only on one variable x and which are defined on an open set Ω from R^n, where Ω is not necessarily bounded.

Observation 11.1.1 *If the function u is regular (twice continuously differentiable), according to the classic Schwartz criterion, we have the relations*

$$\forall x \in \Omega,\ \forall\, i, j = \overline{1, n}\ :\ \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \frac{\partial^2 u}{\partial x_j \partial x_i}(x).$$

We will write the coefficients $a_{ij}(x)$ in the form

$$a_{ij}(x) = \frac{1}{2}\left[a_{ij}(x) + a_{ji}(x)\right] + \frac{1}{2}\left[a_{ij}(x) - a_{ji}(x)\right],$$

that is, we highlight here the the symmetric part and the antisymmetric part, respectively, of the coefficients $a_{ij}(x)$

$$a^s_{ij}(x) = \frac{1}{2}\left[a_{ij}(x) + a_{ji}(x)\right],\ a^a_{ij}(x) = \frac{1}{2}\left[a_{ij}(x) - a_{ji}(x)\right],$$

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_11

so we can write

$$a_{ij}(x) = a^s_{ij}(x) + a^a_{ij}(x).$$

Then, by direct calculation, we see that

$$\sum_{i,j=1}^{n} a^a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \sum_{i<j} a^a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + 0 + \sum_{i>j}^{n} a^a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0.$$

In conclusion, the differential operator (11.1.1) becomes

$$\mathcal{L}(u)(x) = \sum_{i,j=1}^{n} a^s_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}(x) + a_0(x)u(x),$$

from where we deduce that we can use the differential operator $\mathcal{L}(u)$ in the form (11.1.1) with the hypothesis that all the coefficients $a_{ij}(x)$ of the partial derivatives of second order are symmetrical.

In the 1-dimensional case, the partial derivatives become ordinary derivatives (of second order) such that the partial differential equation of second order becomes a differential equation of second order and the Cauchy problem, in this case, can be formulated as follows

$$\begin{cases} \mathcal{L}(u)(x) = a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \forall x \in (a, b), \\ u(x_0) = u_0, \\ u'(x_0) = u_1. \end{cases} \tag{11.1.2}$$

Analyzing problem (11.1.2), we note that we need to fix a point x_0 on the interval (a, b) and two real numbers u_0 and u_1 corresponding to the initial values of u and of u', respectively in the fixed point x_0.

To define a Cauchy problem in the n-dimensional case, we need a hypersurface S, defined with the help of a function G supposed to be continuously differentiable and which verifies the condition

$$\forall x \in S \ : \ |\nabla G(x)| \neq 0.$$

This condition will allow the application of the theorem of implicit functions. Also, in view of the formulation of the Cauchy problem, we need a field of vectors l which must verify the condition

$$\forall x \in S \ : \ |l(x)| \neq 0.$$

Then, the Cauchy problem consists in the introduction of two functions u_0 and u_1, defined on the hypersurface S, so that

$$\begin{aligned}
&\mathcal{L}(u)(x) = f(x),\ \forall x \in \Omega,\\
&u(x) = u_0(x),\ \forall x \in S \cap U,\\
&\frac{\partial u}{\partial t}(x) = \frac{l(x).\nabla u(x)}{|l(x)|} = u_1(x),\ \forall x \in S \cap U,
\end{aligned} \tag{11.1.3}$$

where we denote by U a neighborhood of a fixed point $x_0 \in S$.

For an elementary understanding of the notions, we will consider the following simple examples.

Example 11.1.1 We define the hypersurface S by

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2 \ :\ x_2 = 0\right\}.$$

In the present case the function G, which defines in the theory above the hypersurface S, is thus, $G(x_1, x_2) = x_2$.

1^o. We will take the vector field l in the form $l(x) = (0, 1)$. Then the Cauchy problem can be written in the form

$$\begin{aligned}
\mathcal{L}(u)(x) &= f(x),\ \forall x \in \Omega \subset \mathbb{R}^2,\\
u(x_1, 0) &= u_0(x_1),\ \forall x_1 \in \mathbb{R},\\
\frac{\partial u}{\partial x_2}(x_1, 0) &= u_1(x_1),\ \forall x_1 \in \mathbb{R}.
\end{aligned}$$

Here, we have no a priori relationship between u_0 and u_1.

2^o. Consider the vector field l of the form $l(x) = (1, 0)$. Then the Cauchy problem can be written in the form

$$\begin{aligned}
\mathcal{L}(u)(x) &= f(x),\ \forall x \in \Omega \subset \mathbb{R}^2,\\
u(x_1, 0) &= u_0(x_1),\ \forall x_1 \in \mathbb{R},\\
\frac{\partial u}{\partial x_1}(x_1, 0) &= u_1(x_1),\ \forall x_1 \in \mathbb{R}.
\end{aligned}$$

It is clear that in this example we have

$$\frac{\partial u}{\partial x_1}(x_1, 0) = u_0'(x_1),$$

and this involves a relation between u_0 and u_1.

We will see that the hypersurface S will play an important role in the proof of the existence of a solution of the Cauchy problem.

Definition 11.1.1 Let $\mathcal{L}$ be an arbitrary differential operator with partial derivatives

$$\mathcal{L}(u)(x) = \sum_{|\alpha| \le k} a_\alpha(x) \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(x).$$

(i) A vector field $\xi \in \mathbb{R}^n$ is called a characteristic vector in the point x for the operator $\mathcal{L}$, if it satisfies the equation

$$\sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha = 0.$$

We will denote with $car_{\mathcal{L}}(x)$ the set of all characteristic vectors in the point x for the differential operator $\mathcal{L}$.

(ii) A hypersurface S is called a characteristic hypersurface for the operator $\mathcal{L}$ in the point x if the normal vector $\vec{\nu(x)}$ to the surface S in the point x is a vector field from $car_{\mathcal{L}}(x)$.

(iii) A hypersurface S is called an uncharacteristic hypersurface if and only if it is not a characteristic hypersurface at any point x, that is, the normal vector to the hypersurface S does not belong to the set $car_{\mathcal{L}}(x)$ for any point x from S.

For a better understanding of the notions, introduced in Definition 11.2.1, we will consider some simple examples.

Example 11.1.2 (i) For a differential operator with partial derivatives of the first order, we find again the notions introduced in Theorem 11.1.1 from this chapter. Concretely, the notion of uncharacteristic hypersurface is expressed in Theorem 11.1.1 in the following form:

The vector of components $(a_1, a_2, \ldots, a_n)$ *is not tangent to the surface* S *in any point from* S.

In Definition 11.2.1, the assertion above is equivalent to the fact that the normal vector to the hypersurface S is not a characteristic vector, that is, it does not verify the equality

$$\sum_{j=1}^{n} a_j(g(s), u(g(s)))\nu_j(g(s)) = 0. \tag{11.1.4}$$

Equality (11.1.4) can be interpreted as a scalar product between the vector of components $(a_1, a_2, \ldots, a_n)$ and the normal vector $\vec{\nu}$. Of course, according to (11.1.4) this scalar product is null. Thus we deduce that the vector of components $(a_1, a_2, \ldots, a_n)$ is not orthogonal with the normal vector $\vec{\nu}(x)$ to the surface S in the point $x \in S$.

(ii) For a differential operator of second order, the condition "the vector ξ is uncharacteristic for $\mathcal{L}$ in the point x", can be reformulated, equivalently, in the form

$$\langle A(x).\xi, \xi\rangle \neq 0,$$

where we denoted by $A(x)$ the symmetrical matrix of the coefficients of partial derivatives of second order.

(iii) In the simple case

$$\mathcal{L} = \frac{\partial}{\partial x_1}$$

we have $\text{car}_{\mathcal{L}}(x) = \{\xi \ : \ \xi_1 = 0\}$, that is, the set $car_{\mathcal{L}}(x)$ is a hyperplane.
(iv) If $\mathcal{L} = \Delta$ (the laplacian), then $car_{\mathcal{L}}(x) = \{0\}$.
(v) If

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

then

$$\text{car}_{\mathcal{L}}(x) = \{(\xi_0, \xi_1, \ldots, \xi_n) \ : \ \xi_1 = \xi_2 = \cdots = \xi_n\},$$

that is, the set $car_{\mathcal{L}}(x)$ is a straight line.
(vi) If

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

then

$$\text{car}_{\mathcal{L}}(x) = \left\{(\xi_0, \xi_1, \ldots, \xi_n) \ : \ \xi_0^2 = \xi_1^2 + \cdots + \xi_n^2\right\}.$$

The method used to solve the Cauchy problem associated with the linear differential operator $\mathcal{L}$ of second order is suggested in the following theorem, by S. Kovalewskaia.

Theorem 11.1.1 *Suppose that the data which characterizes the Cauchy problem associated with the differential operator $\mathcal{L}$, that is, a_{ij}, a_j, a_0, f, u_0, u_0, and u_1, are analytical functions, and S is not a characteristic surface for $\mathcal{L}$, in a neighborhood of a fixed point x^0. Then, there is a unique solution which is an analytic function of the Cauchy problem in a neighborhood of x^0.*

Proof First, we want to mention that here an analytic function is a function which admits a series representation in the whole neighborhood of each point x^0 and this series is of the form

$$\sum_{|\alpha| \leq k} A_\alpha (x_1 - x_1^0)^{\alpha_1} (x_2 - x_2^0)^{\alpha_2} \ldots (x_n - x_n^0)^{\alpha_n}.$$

The idea of proof consists in the evaluation of the Cauchy problem above with a differential system of the first order, whose coefficients and the initial conditions are just the data which characterize the Cauchy problem, formulated above. ■

11.2 Classification of Partial Differential Equations of Second Order

Consider again the differential linear operator of second order

$$\mathcal{L}(u)(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}(x) + a_0(x)u(x).$$

We can make the convention that the matrix A of the coefficients $a_{ij}(x)$ is assumed to be a symmetric matrix, based on the considerations from the beginning of Sect. 11.1.

Because A is a symmetric matrix, it can be brought into its diagonal form and then we can compute its eigenvalues. Let us introduce the following notations

– $n_+(x)$ the number of strictly positive eigenvalues;
– $n_-(x)$ the number of strictly negative eigenvalues;
– $n_0(x)$ the number of null eigenvalues.

Then, the following equality is obvious

$$n(x) = n_+(x) + n_-(x) + n_0(x),$$

where $n(x)$ is number of all eigenvalues.

Definition 11.2.1 We have the following types of differential operators of second order:

– (i) The operator $\mathcal{L}$ is called elliptic in the point x_0 if $n_+(x_0) = n(x_0)$. If $n_-(x_0) = n(x_0)$, then the operator $\mathcal{L}$ is also elliptic.

– (ii) The operator $\mathcal{L}$ is called elliptic on an open set Ω if $\mathcal{L}$ is elliptic in any point x_0 from Ω.

– (iii) The operator $\mathcal{L}$ is called hyperbolic in x_0 if $n_+(x_0) = n(x_0) - 1$ and $n_-(x_0) = 1$. If $n_-(x_0) = n(x_0) - 1$ and $n_+(x_0) = 1$, then the operator $\mathcal{L}$ is also hyperbolic.

– (iv) The operator $\mathcal{L}$ is called hyperbolic on an open set Ω if it is hyperbolic in any point x_0 from Ω.

– (v) The operator $\mathcal{L}$ is called parabolic in x_0 if $n_0(x_0) \neq 0$.

– (vi) The operator $\mathcal{L}$ is called parabolic on an open set Ω if it is parabolic in any point x_0 from Ω.

In the spirit of this definition, we will classify some differential operators of second order.

Example 11.2.1 (i) The Laplace's operator Δ is elliptic in $\mathbb{R}^n$, because the matrix A of the coefficients is identically equal to the unit matrix and then all its eigenvalues are equal to 1.

(ii) The operator of waves, defined by

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

is hyperbolic on $\mathbb{R}^n$ because the matrix of the coefficients, which has the dimension $(n+1) \times (n+1)$), has on the main diagonal only only elements of -1, in position (1,1) and 1 otherwise. The other elements of the matrix are null.

(iii) The operator of heat, defined by

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

is parabolic in the space $\mathbb{R}^n$ because the matrix of the coefficients (of dimension $(n+1) \times (n+1)$) has on the main diagonal only elements of 0, in position (1, 1) and 1 otherwise. The other elements of the matrix are null.

(iv) If we consider the operator

$$\mathcal{L} = \frac{\partial^2}{\partial x_1^2} + x_1 \frac{\partial^2}{\partial x_2^2},$$

then we have an example of a differential operator which has different types in different areas from $\mathbb{R}^2$.

In all that follows, we will meet some elliptic operators (for which the Laplacian is the most important representative), some parabolic operators (for which the most important representative is the operator of heat) and, finally, some hyperbolic operators (for which the operator of waves is the most important representative).

11.3 Linear Elliptic Operators

In this paragraph, we will consider differential operators of the form

$$\mathcal{L}(u)(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{j=1}^{n} a_j(x) \frac{\partial u}{\partial x_j}(x) + a_0(x) u(x), \quad (11.3.1)$$

in which the matrix of the coefficients $a_{ij}(x)$ is assumed to be symmetric and satisfies the condition

$$\forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n \ : \ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j > 0, \quad (11.3.2)$$

and this is reduced to considering only the case in which the symmetrical matrix of coefficients a_{ij} has only strictly negative eigenvalues.

Definition 11.3.1 An operator $\mathcal{L}$ of the form (11.3.1) is called coercive and continuous on the open set Ω if there exist two real, strictly positive constants c and C so that

$$\forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n \ :\ c\|\xi\|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq C\|\xi\|^2. \tag{11.3.3}$$

It is clear that the Laplacian operator is continuous and coercive on any open set from $\mathbb{R}^n$, because

$$\forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n \ :\ \|\xi\|^2 = \sum_{i,j=1}^{n} \delta_{ij}\xi_i\xi_j, \tag{11.3.4}$$

that is, the condition (11.3.3) from the definition 3.1 holds true with $c = C = 1$ and $a_{ij}(x) = \delta_{ij}$, where δ_{ij} represents the Kronecker's symbol, that is, $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

We now consider some types of boundary value problems associated with elliptic operators.

Definition 11.3.2 Let Ω be a arbitrary open set, $\Omega \subset \mathbb{R}^n$.

Assume that Ω is bounded and denote by $\partial\Omega$ its boundary.

(i) We call the Dirichlet problem associated with the differential operator $\mathcal{L}$, the boundary value problem defined by

$$\begin{aligned} \mathcal{L}(u)(x) &= f(x),\ \forall x \in \Omega, \\ u(x) &= g(x)\ ,\ \forall x \in \partial\Omega, \end{aligned}$$

in which functions f and g are predetermined. If the function g is identical null on $\partial\Omega$, then we say that we have the homogeneous Dirichlet problem. On the contrary, the Dirichlet problem is nonhomogeneous.

(ii) We call the Neumann problem associated with the differential operator $\mathcal{L}$ the boundary value problem defined by

$$\begin{aligned} \mathcal{L}(u)(x) &= f(x),\ \forall x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) &= g(x),\ \forall x \in \partial\Omega, \end{aligned}$$

in which the functions f and g are predetermined and ν is the normal oriented outside the surface $\partial\Omega$. If the function g is identically null on $\partial\Omega$ then we say that we have a Neumann homogeneous problem.

On the contrary, we say that the Neumann problem is nonhomogeneous.

(iii) We call the mixed boundary value problem associated with the differential operator $\mathcal{L}$, the boundary value problem in which the boundary condition involve both the unknown function u and also its derivative in the direction of the normal

$$\mathcal{L}(u)(x) = f(x), \ \forall x \in \Omega,$$
$$\frac{\partial u}{\partial \nu}(x) + \alpha(x)u(x) = g(x), \ \forall x \in \partial\Omega,$$

in which the functions $f(x)$, $\alpha(x)$, and $g(x)$ are predetermined.

Observation 11.3.1 *Normally, in the case of the mixed boundary value problem, the boundary $\partial\Omega$ is partitioned in two subsets Γ_1 and Γ_2 so that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_1} \cap \Gamma_2 = \partial\Omega$ and the boundary condition is indicated in the form:*

$$u(x) = g_1(x), \ \forall x \in \overline{\Gamma_1},$$
$$\frac{\partial u}{\partial \nu}(x) = g_2(x), \ \forall x \in \Gamma_2.$$

An edifying example of the boundary value problem is *the problem of Poincaré*, from which by convenient particularizations we obtain other types of the boundary value problem.

Example 11.3.1 Let us consider the real functions $p_i(x), \ i = \overline{1, n}, \ q(x)$ and $r(x)$, defined on the boundary $\partial\Omega$ of the domain Ω. The problem of Poincaré consists of

$$\mathcal{L}u(x) = f(x), \ \forall x \in \Omega,$$
$$\sum_{i=1}^{n} p_i(x)\frac{\partial u(x)}{\partial x_i} + q(x)u(x) = r(x), \ \forall x \in \partial\Omega, \tag{11.3.5}$$

where by the values of the functions $\partial u(x)/\partial x_i$ and $u(x)$ in the points $x \in \partial\Omega$, we understand the limits of these functions computed by points from inside the domain Ω, which tend to points of the boundary $\partial\Omega$.

In the particular case, in which

$$p_i(x) = 0, \ i = \overline{1, n} \text{ and } q(x) \neq 0, \ \forall x \in \partial\Omega,$$

the boundary condition $(11.3.5)_2$ becomes

$$u(x) = g(x), \ \forall x \in \partial\Omega, \tag{11.3.6}$$

where

$$g(x) = \frac{r(x)}{q(x)}, \ \forall x \in \partial\Omega.$$

The boundary condition (11.3.6) together with Eq. (11.3.5)$_1$ constitute the Dirichlet boundary value problem, which is also known under the name of *the first boundary value problem*.

If

$$q(x) = 0, \ \forall x \in \partial\Omega,$$

then the boundary condition (11.3.5)$_2$ becomes

$$\sum_{i=1}^{n} p_i(x) \frac{\partial u(x)}{\partial x_i} = r(x), \ \forall x \in \partial\Omega. \tag{11.3.7}$$

Then the problem of Poincaré (11.3.5) is reduced to the problem consisting of equation (11.3.5)$_1$ and the boundary condition (11.3.7) which is known as the problem with the boundary condition for the oblique derivative.

If the functions $p_i(x)$ have the values

$$p_i(x) = \cos \widehat{(\nu, x_i)}, \ i = \overline{1, n}, \tag{11.3.8}$$

then the boundary condition (11.3.7) is the boundary condition of Neumann. Hence equation (11.3.5)$_1$ together with the boundary condition (11.3.7), in which the functions $p_i(x)$ have the expressions (11.3.8), is the Neumann boundary value problem. The problem of Neumann is is also known as *the second boundary value problem*.

If it satisfies the condition

$$q(x) \not\equiv 0, \ \text{on } \partial\Omega,$$

and the functions $p_i(x)$ have the expressions from (11.3.8), the problem of Poincaré is reduced to the mixed boundary value problem, which is also known as *the third boundary value problem*.

For the boundary value problems considered above, we now introduce different types of solutions.

Definition 11.3.3 A classical solution of the boundary value problem associated with the differential operator $\mathcal{L}$ with one of the boundary conditions indicated in Definition 3.2, is any function u of class $C^2(\Omega) \cap C^0(\overline{\Omega})$ or $C^2(\Omega) \cap C^1(\overline{\Omega})$ which verifies the equation $\mathcal{L}(u)(x) = f(x)$ on Ω and a boundary condition on $\partial\Omega$.

Observation 11.3.2 1^o. *If u is a classical solution of a boundary value problem associated with the operator $\mathcal{L}$, then f is continuous on Ω.*

2^o. *If u is a classical solution of a boundary value problem associated with the operator $\mathcal{L}$, then for any function $\varphi \in C_0^2(\Omega)$ (therefore, φ is twice continuously differentiable and supp φ is a compact set included in Ω), we have*

$$\int_\Omega \mathcal{L}(u)(x)\varphi(x)\mathrm{d}x = \int_\Omega f(x)\varphi(x)\mathrm{d}x = \int_\Omega \mathcal{L}'(\varphi)(x)u(x)\mathrm{d}x,$$

where with $\mathcal{L}'$ we denote the following linear differential operator

$$\mathcal{L}'(u)(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) - \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j}(x) + a_0(x)u(x).$$

Other types of the solutions of the boundary value problems associated with the operator $\mathcal{L}$ are contained in the next definition.

Definition 11.3.4 (i) A strong solution of the equation $\mathcal{L}(u) = f$, in a topological space X to whom the function f belongs, is any function u which belongs to X for which $\mathcal{L}(u)$ belongs to X and is equal to f.

(ii) A weak solution of the equation $\mathcal{L}(u) = f$ in a topological space X to whom the function f belongs, is any function u which belongs to X and for which

$$(\mathcal{L}(u), \varphi) = (f, \varphi), \forall \varphi \in Y \subseteq X',$$

where with X' we denote the topological dual of the space X, and with Y we denote a vector subspace of X'.

Example 11.3.2 1^o. Let us consider the space $X = C^0(\overline{\Omega})$. Then a strong solution of the equation $\mathcal{L}(u) = f$ in $C^0(\overline{\Omega})$ is any function u that is continuous on $\overline{\Omega}$, so that $\mathcal{L}(u)$ is continuous on $\overline{\Omega}$ and $\mathcal{L}(u) = f$ on Ω. In this way, we deduce that any classical solution is also a strong solution.

2^o. Let us consider the space of functions $X = L^2(\Omega)$. Then a weak solution of the equation $\mathcal{L}(u) = f$ in $L^2(\Omega)$ is any function $u \in L^2(\Omega)$ for which $\mathcal{L}(u) \in L^2(\Omega)$ and $\mathcal{L}(u)(x) = f(x)$, almost for all $x \in \Omega$.

From definitions and examples above, we deduce that that any strong solution is also a weak solution. But not any strong solution is also a classical solution. It is sufficient to consider a counterexample. If we take

$$X = C^0(\bar{(}\Omega)) \text{ si } \mathcal{L}(u) = \frac{\partial^2 u}{\partial x \partial y} \text{ pe } \mathbb{R}^2,$$

then we see that a strong solution cannot be a classical solution.

A function u of the form $u(x, y) = f(x) + g(y)$, where the functions f and g are continuously differentiable on $\mathbb{R}^2$, is a strong solution of equation $\mathcal{L}(u) = 0$. But it is clear that u is not necessarily a twice continuously differentiable function on $\mathbb{R}^2$.

In the following, in the study of elliptic equations of second order, we pay attention to the following three aspects:

– (i) What conditions should be satisfied in order to prove a theorem of existence of a weak solution. In most cases, the existence will be prove with the help of a variational formulation of the considered boundary value problem.

– (ii) What conditions should be satisfied in order to prove a theorem of uniqueness of the weak solution for the considered boundary value problem.

– (iii) Which is the "degree" of regularity of the weak solution.

We will analyze the hypotheses of regularity that must be imposed on the data that define the boundary value problem, the coefficients of the differential operator and the right-hand side function, for which the weak solution can be a strong solution and also just a classical solution.

The theorem of existence of a weak solution of a boundary value problem associated with a continuous and coercive elliptic operator will be based on the well-known Lax–Milgram theorem, which will be approached later.

Chapter 12
Harmonic Functions

12.1 Definitions and Properties

Definition 12.1.1 We call a harmonic function on the open set $\Omega \subset \mathbb{R}^n$, any function u which is twice continuously differentiable on Ω and which verifies the equation $\Delta u(x) = 0, \forall x \in \Omega$, where Δ is the operator of Laplace

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}.$$

The main properties of harmonic functions are given in the following theorem.

Theorem 12.1.1 *If the function u is harmonic on the open set Ω whose boundary $\partial\Omega$ is regular (that is, it is defined with the help of some continuously differentiable functions), then we have the identity*

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x) d\sigma(x) = 0,$$

where ν represents the outside normal to the surface $\partial\Omega$.

(i). Let u be a harmonic function on the open set Ω, x_0 a fixed point in Ω, and $\overline{B(x_0, r)}$ a closed ball having the center in x_0, with the radius r and included in Ω.

Then, we have the representation

$$u(x_0) = \frac{1}{r^{n-1}\omega_n} \int_{S(x_0,r)} u(y) d\sigma(y) = \frac{1}{\omega_n} \int_{S(0,1)} u(x_0 + ry) d\sigma(y),$$

where with ω_n we denoted the area of the sphere centered in 0 of radius 1, $S(0, 1) \subset \mathbb{R}$, that is, as it is known

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$$\omega_n = \frac{(2\pi)^{n/2}}{\Gamma(n/2)}.$$

We denote by Γ*, as usual, the function of Euler of second species, namely*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t.$$

(ii). In the same hypotheses as above, we have the representation

$$u(x_0) = \frac{n}{r^n \omega_n} \int_{B(x_0,r)} u(y)\mathrm{d}y = \frac{n}{\omega_n} \int_{B(0,1)} u(x_0 + ry)\mathrm{d}y.$$

Proof (i). This statement can be immediately proven by applying the second Green formula for the functions u and 1.
(ii). Consider the function v_0 defined by

$$v_0(y) = \begin{cases} \|x_0 - y\|^{2-n} & , \text{ if } n > 2, \\ -\ln(\|x_0 - y\|) & , \text{ if } n = 2, \end{cases}$$

and the corona $C_{\varepsilon,r} = B(x_0, r) \setminus \overline{B(x_0, \varepsilon)}$ defined for $\varepsilon > 0$, arbitrarily small. Obviously, $C_{\varepsilon,r} \subset \Omega$. By direct calculations we can easily verify that the function v_0 is harmonic on $C_{\varepsilon,r}$ and

$$\frac{\partial v_0}{\partial \nu}(y) = \begin{cases} (2-n)/r^{n-1} & , \forall y \in S(x_0, r), \\ -(2-n)/\varepsilon^{n-1} & , \forall y \in S(x_0, \varepsilon). \end{cases} \tag{12.1}$$

On the other hand, using the second formula of Green, we have

$$\begin{aligned} 0 = &\int_{S(x_0,r)} \left[v_0(y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial v_0}{\partial \nu}(y) \right] d\sigma(y) \\ &+ \int_{S(x_0,\varepsilon)} \left[v_0(y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial v_0}{\partial \nu}(y) \right] d\sigma(y), \end{aligned} \tag{12.2}$$

such that by taking into account (12.1), formula (12.2) becomes

$$\begin{aligned} 0 =& r^{2-n} \int_{S(x_0,r)} \frac{\partial u}{\partial \nu}(y) d\sigma(y) + \varepsilon^{2-n} \int_{S(x_0,r)} \frac{\partial u}{\partial \nu}(y) d\sigma(y) \\ &+ (2-n) r^{1-n} \int_{S(x_0,r)} u(y) d\sigma(y) - (2-n)\varepsilon^{1-n} \int_{S(x_0,r)} u(y) d\sigma(y). \end{aligned} \tag{12.3}$$

The first two integrals from the right-hand side of the relation (12.3) are null because the function u is harmonic in the ball $B(x_0, \varepsilon)$ and also in the ball $B(x_0, r)$. We deduce then that

$$r^{1-n}\int_{S(x_0,r)} u(y)d\sigma(y) = \varepsilon^{1-n}\int_{S(x_0,r)} u(y)d\sigma(y),$$

and the equality is equivalent to the equality

$$\frac{1}{r^{n-1}\omega_n}\int_{S(x_0,r)} u(y)d\sigma(y) = \frac{1}{\varepsilon^{n-1}\omega_n}\int_{S(x_0,\varepsilon)} u(y)d\sigma(y). \tag{12.4}$$

Because the equality (12.4) holds true for $\varepsilon > 0$ arbitrarily small, we can pass here to the limit with $\varepsilon \to 0$ and based on the conditions of regularity of the function u in x_0 we deduce that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{n-1}\omega_n}\int_{S(x_0,\varepsilon)} u(y)d\sigma(y) = u(x_0),$$

and then the first equality from (ii) is proved. To obtain the second equality from (ii) we will make an elementary change of variables.
(iii). Based on the equality from (ii) we deduce that for any $\rho \le r$ we have

$$u(x_0) = \frac{1}{\rho^{n-1}\omega_n}\int_{S(x_0,\rho)} u(y)d\sigma(y),$$

and then

$$\int_0^r \rho^{n-1}u(x_0)d\rho = \frac{1}{\omega_n}\int_0^r\int_{S(x_0,\rho)} u(y)d\sigma(y)d\rho.$$

From this equality, using the theorem of Fubini, we are led to

$$\begin{aligned}\frac{r^n}{n}u(x_0) &= \frac{1}{\omega_n}\int_0^r\int_{S(x_0,\rho)} u(y)d\sigma(y)d\rho\\ &= \frac{1}{\omega_n}\int_{B(x_0,r)} u(y)\mathrm{d}y. \end{aligned}\tag{12.5}$$

Hence the first equality from (iii) is proven. In order to prove the second equality from (iii) we must make an elementary change of variables in the equality (12.5). ■

Observation 12.1.1 *The property stated in point (ii) of Theorem 12.1.1 is called the mean value property for the surface integrals, and the property stated at point (iii) of Theorem 12.1.1 is known under the name of the mean value property for volume integrals.*

In the following theorem, we will prove a reciprocal result for the mean value property in the case of the integral of the surface.

Theorem 12.1.2 *Let u be a continuous function on the open set Ω and which, in addition, has the property that for any $x \in \Omega$ and for any $r > 0$ for which the closed* *ball $\overline{B(x,r)}$ is included in Ω, we have the representation*

$$u(x) = \frac{1}{\omega_n} \int_{S(0,1)} u(x + ry) d\sigma(y).$$

Then, the function u is infinitely differentiable and harmonic on Ω.

Proof Consider Φ a function with radial symmetry. Hence, it is of the form $\Phi(x) = \psi(\|x\|)$. Suppose, in addition, that Φ is infinitely differentiable, has compact support included in the ball $B(0, 1)$ and satisfies the equality

$$\int_{B(0,1)} \Phi(x)\mathrm{d}x = 1.$$

Denote by Φ_ε the function

$$\Phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \Phi\left(\frac{x}{\varepsilon}\right).$$

We can immediately deduce that $\Phi_\varepsilon \in C_0^\infty(B(0, \varepsilon))$ and

$$\int_{B(0,\varepsilon)} \Phi_\varepsilon(x)\mathrm{d}x = 1.$$

Denote by Ω_ε the set

$$\Omega_\varepsilon = \{x \in \Omega : \overline{B(x, \varepsilon)} \subset \Omega\}.$$

For any $x \in \Omega_\varepsilon$, the application which associates to y the number $\Phi_\varepsilon(x - y)$ is a function with compact support included in Ω.

On the other hand, we have the equality

$$\begin{aligned}\int_{R^n} u(y)\Phi_\varepsilon(x - y)\mathrm{d}y &= \int_{R^n} u(x - y)\Phi_\varepsilon(y)\mathrm{d}y \\ &= (u * \Phi_\varepsilon)(x) = \frac{1}{\varepsilon^n} \int_{B(0,\varepsilon)} u(x - y)\Phi\left(\frac{y}{\varepsilon}\right)\mathrm{d}y.\end{aligned}$$

If we make the change of variable $y = \varepsilon z$, the equality above becomes

$$\begin{aligned}\int_{R^n} u(y)\Phi_\varepsilon(x - y)\mathrm{d}y &= \int_{B(0,1)} u(x - \varepsilon z)\Phi(z)dz \\ &= \int_0^1 \int_{S(0,r)} u(x - \varepsilon z)\Phi(z)d\sigma(z)dz.\end{aligned}$$

If in this last integral we make the change of variable $z = rw$, we obtain

$$
\begin{aligned}
(u * \Phi_\varepsilon)(x) &= \int_0^1 r^{n-1} \int_{S(0,1)} u(x - \varepsilon r w)\Phi(rw) d\sigma(w) \mathrm{d}r \\
&= \int_0^1 r^{n-1}\psi(r) \left(\int_{S(0,1)} u(x - \varepsilon r w)\sigma(w) \right) \mathrm{d}r \\
&= \omega_n u(x) \int_{S(0,1)} r^{n-1}\psi(r)\mathrm{d}r = u(x).
\end{aligned}
$$

The last equality can be verified by direct calculations

$$
1 = \int_{B(0,1)} \Phi(x)\mathrm{d}x = \int_0^1 \int_{S(0,r)} \Phi(z)d\sigma(z)\mathrm{d}r = \omega_n \int_0^1 r^{n-1}\Psi(r)\mathrm{d}r.
$$

Thus, we proved that the function u coincides with the infinitely differentiable function $u * \Phi_\varepsilon$ on Ω_ε (it is clear that the function $u * \Phi_\varepsilon$ is infinitely differentiable because it is the product of convolution between u and an infinitely differentiable function). Therefore, we deduce that the function u is infinitely differentiable on the set Ω_ε, for any $\varepsilon > 0$, from where we deduce that u is infinitely differentiable on Ω. The proof of the theorem will be complete if we show that the function u is harmonic on Ω. Based on the mean value property for the integral of the surface, we immediately obtain the mean value property for the integral of volume

$$
\forall x \in \Omega \ : \ u(x) = \frac{n}{\omega_n} \int_{B(0,1)} u(x + rz)dz.
$$

Let us denote by $M(x, r)$ the quantity

$$
M(x, r) = \frac{n}{\omega_n} \int_{B(0,1)} u(x + rz).
$$

Based on the hypotheses of the theorem, we deduce that, in fact, $M(x, r) = u(x)$. If we take into account the fact that u is infinitely differentiable on Ω, by direct calculation we obtain

$$
\Delta_x M(x, r) = \frac{n}{\omega_n} \int_{B(0,1)} \Delta u(x + rz)dz,
$$

and if we apply Green's formula, we are led to

$$
\Delta_x M(x, r) = \frac{n}{\omega_n} \int_{S(0,1)} \nabla u(x + rz)\ z\ d\sigma(z). \tag{12.6}
$$

On the other hand, we can compute the following partial derivative

$$\begin{aligned}\frac{\partial M}{\partial r}(x,r) &= \frac{n}{\omega_n}\int_{B(0,1)} \nabla u(x+rz)zdz \\ &= \frac{n}{\omega_n}\frac{1}{r^n}\int_{B(x,r)} \nabla u(y)\frac{y-x}{r}\mathrm{d}y \\ &= \frac{n}{\omega_n}\frac{1}{r^{n+1}}\int_0^r\int_{S(x,\rho)} \nabla u(y)(y-x)d\sigma(y)\mathrm{d}y \\ &= \frac{n}{\omega_n}\frac{1}{r^{n+1}}\int_0^r \rho^n\int_{S(0,1)} \nabla u(y)(x+\rho z)zd\sigma(z)dz.\end{aligned} \tag{12.7}$$

From the evaluations (12.6) and (12.7), we obtain the equation

$$\Delta_x M(x,r) = \frac{1}{r^{n+1}}\frac{\partial}{\partial r}\left(r^{n+1}\frac{\partial M}{\partial r}(x,r)\right),$$

and this equation holds true on the set

$$\{(x,r)\ :\ x\in\Omega,\ d(x,\partial\Omega)>r\}.$$

But $M(x,r)$ is in fact, just $u(x)$ which is independently of r. Hence, we can deduce that $\Delta_x M$ is null, which is equivalent to the fact that the function u is harmonic in Ω. ■

Observation 12.1.2 *1^o. From the proof of Theorem 12.1.2 we deduce that if the function u is harmonic on the set Ω then u is infinitely differentiable on Ω.*

2^o. Let $\{u_k\}_k$ be a sequence of harmonic functions which is uniformly convergent to the function u on any compact set from Ω. Then u is a harmonic function on Ω. This statement can be proven using the mean value property for the integrals of volume.

12.2 The Maximum Principle

A first result on the maximum principle, which has a great theoretical importance, is shown in the following theorem.

Theorem 12.2.1 *Any harmonic function defined on $\mathbb{R}^n$ which is bounded from above or from below is constant on $\mathbb{R}^n$.*

Proof We pay attention to the case when the function u is bounded from below because if u is a harmonic function and is bounded from above then $-u$ is also harmonic and bounded from below. Using, eventually, a constant translation, we can assume that the function u is bounded from below by 0, that is, the values of u are positive or null. Let x_1 and x_2 be two distinct points from $\mathbb{R}^n$. It is clear that we can choose two numbers r_1 and r_2 so that

$$r_1 \geq r_2 + \|x_1 - x_2\| > r_2 > 0,$$

and this ensures that the ball $B(x_1, r_1)$ contains the ball $B(x_2, r_2)$. We can choose, for instance,

$$r_1 = r_2 + \|x_1 - x_2\|.$$

Because the function u is harmonic and has positive or null values on $\mathbb{R}^n$, we have

$$\begin{aligned} u(x) &= \frac{n}{r_2^n \omega_n} \int_{B(x_2,r_2)} u(y)\mathrm{d}y \leq \frac{n}{r_2^n \omega_n} \int_{B(x_1,r_1)} u(y)\mathrm{d}y \\ &\leq \frac{r_1^n}{r_2^n} \frac{n}{r_1^n \omega_n} \int_{B(x_1,r_1)} u(y)\mathrm{d}y = \frac{r_1^n}{r_2^n} u(x_1). \end{aligned}$$

If in this inequality we pass to the limit with $r_2 \to \infty$, then we can immediately deduce that $u(x_2) \leq u(x_1)$. The contrary inequality, $u(x_1) \leq u(x_2)$, is obtained based on the fact that in all previous considerations , x_1 and x_2 play symmetrical roles. ■

The maximum principle result which will be proven in the following theorem is due to Hopf.

Theorem 12.2.2 (Maximum principle). *Consider Ω an open and connected set, $\Omega \subseteq \mathbb{R}^n$ and u a harmonic and continuous function on $\overline{\Omega}$. If there exists a point $x_0 \in \Omega$ so that*

$$\forall x \in \Omega \ : \ u(x) \leq u(x_0),$$

then u is a constant function on Ω.

Proof Let $\overline{B(x_0, r_0)}$ be a closed ball with the center in the point x_0 and the radius r_0 so that $\overline{B(x_0, r_0)} \subseteq \Omega$. Because u is a harmonic function on Ω, we deduce that u is harmonic on B(x_0, r_0), and then we have the representation

$$u(x_0) = \frac{n}{r_0^n \omega_n} \int_{B(x_0,r_0)} u(y)\mathrm{d}y \leq u(x_0).$$

We deduce then that

$$\int_{B(x_0,r_0)} [u(y) - u(x_0)]\mathrm{d}y = 0,$$

and, therefore, $u(x) = u(x_0)$, for almost all $x \in \overline{B(x_0, r_0)}$. But the function u was assumed to be continuous and therefore

$$u(x) = u(x_0), \forall x \in \overline{B(x_0, r_0)}.$$

Let us consider the set Ω_0 defined by

$$\Omega_0 = \left\{ x_0 \in \Omega \ : \ u(x) \leq u\left(x_0\right) = \max_{x \in \Omega} u(x), \ \forall x \in \Omega \right\}.$$

Based on the hypotheses of the theorem, we deduce that Ω_0 is nonempty. Ω_0 is the closure of the set of those $x \in \Omega$ for which $u(x) = \max\limits_{x \in \Omega} u(x)$.

The result above proves that Ω_0 is an open subset from Ω which was assumed to be a connected set. Thus, we deduce that $\Omega_0 = \Omega$ and, therefore, the function u is constant on Ω. ■

Observation 12.2.1 *1°. From the proof of Theorem 12.3.2, we deduce that if Ω is an open and connected set from $\mathbb{R}^n$, and the function u is harmonic and nonconstant on Ω, then u can reach neither the maximum value nor the minimum value in Ω.*

2°. If Ω is a bounded set, and the function u is harmonic on Ω and $u \in C^0(\Omega)$ then we have the estimate

$$\forall x \in \Omega \ : \ \min_{x \in \partial\Omega} u(x) \leq u(x) \leq \max_{x \in \partial\Omega} u(x).$$

If in addition, the set Ω is connected (since it is bounded) and u is nonconstant on Ω, then the estimate above becomes

$$\forall x \in \Omega \ : \ \min_{x \in \partial\Omega} u(x) < u(x) < \max_{x \in \partial\Omega} u(x).$$

12.3 Representation of the Harmonic Functions

Definition 12.3.1 A Newtonian potential (or of a single layer potential) in $\mathbb{R}^n$ $(n \geq 2)$, is the function Γ defined for $x \neq 0$ by

$$\Gamma(x) = \begin{cases} \frac{1}{(2-n)\omega_n} \|x\|^{2-n} & , \text{if } n \geq 3, \\ \frac{1}{2\pi} \ln(\|x\|) & , \text{if } n = 2. \end{cases}$$

Observation 12.3.1 *It is a simple exercise to prove that the Newtonian potential Γ is a harmonic function in $\mathbb{R}^n \setminus 0$.*

Other two important properties of the Newtonian potential are formulated in the following proposition. Its proof is immediate and for this reason, we leave it to the reader.

Proposition 12.3.1 *For the potential of a single layer, we have the following estimates:*

1°.

$$\forall x \neq 0 \ : \ \left| \frac{\partial \Gamma}{\partial x_i}(x) \right| \leq \frac{1}{\omega_n} \|x\|^{-n}, \ i = 1, 2, ..., n;$$

2°.

$$\forall x \neq 0 \ : \ \left| \frac{\partial^2 \Gamma}{\partial x_i \partial x_j}(x) \right| \leq \frac{1}{\omega_n} \|x\|^{-n}, \ i, j = 1, 2, ..., n.$$

The theorem of representation which follows is known under the name of *the theorem of the three potentials*.

Theorem 12.3.1 *Let* Ω *be an open set from* $\mathbb{R}^n (n > 2)$ *having the regular border* $\partial\Omega$. *Then any function* $f \in C^2(\overline{\Omega})$ *admits the representation*

$$f(x) = \int_\Omega \Gamma(x-\xi)\delta f(\xi) \mathrm{d}\xi + \int_{\partial\Omega} f(\xi) \frac{\partial \Gamma}{\partial \nu}(x-\xi) d\sigma(\xi)$$
$$- \int_{\partial\Omega} \frac{\partial f}{\partial \nu} \Gamma(x-\xi) d\sigma(\xi), \ \forall x \in \Omega.$$

Proof Because Ω is an open set we have that for $x \in \Omega$ and for a sufficiently small ε, the ball centered in x and with radius ε, $B(x, \varepsilon)$, is fully included in Ω. Consider the set Ω_ε defined by $\Omega_\varepsilon = \Omega \setminus \overline{B(x, \varepsilon)}$. If we take into account the fact that $\Delta\Gamma(x - \xi)$ is equal to 0 for any x and ξ distinct, by applying the second Green formula, we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} \Delta f(\xi)\Gamma(x-\xi)\mathrm{d}\xi &= \int_{\partial\Omega} \Gamma(x-\xi) \frac{\partial f}{\partial \nu}(\xi) d\sigma(\xi) \\ - \int_{\partial\Omega} f(\xi) \frac{\partial \Gamma}{\partial \nu}(x-\xi) d\sigma(\xi) &+ \int_{S(x,\varepsilon)} \Gamma(x-\xi) \frac{\partial f}{\partial \nu_\varepsilon}(\xi) d\sigma(\xi) \\ &- \int_{S(x,\varepsilon)} f(\xi) \frac{\partial \Gamma}{\partial \nu_\varepsilon}(x-\xi) d\sigma(\xi), \end{aligned} \tag{12.8}$$

where ν_ε represents the normal to the sphere $S(x, \varepsilon)$ oriented to the inside of the sphere $S(x, \varepsilon)$. We will estimate the last two integrals from the right-hand side of the relation (12.8), assuming that $n \geq 3$. Thus

$$\begin{aligned} &\left| \int_{S(x,\varepsilon)} \Gamma(x-\xi) \frac{\partial f}{\partial \nu_\varepsilon}(\xi) d\sigma(\xi) \right| \\ \leq &\frac{c_1}{(n-2)\omega_n} \frac{1}{\varepsilon^{n-2}} \int_{S(x,\varepsilon)} d\sigma(\xi) \leq \frac{c_1}{(n-2)\omega_n} \frac{1}{\varepsilon^{n-2}} \varepsilon^{n-1} \omega_n = \frac{c_1}{n-2} \varepsilon, \end{aligned} \tag{12.9}$$

where the constant c_1 was chosen so that

$$\forall \xi \in S(x, \varepsilon) \ : \ |f(\xi)| \leq C_1,$$

and this is possible since $f \in C^2(\overline{\Omega})$.

From (12.9), by passing to the limit with $\varepsilon \to 0$, we deduce that

$$\lim_{\varepsilon\to 0}\left|\int_{S(x,\varepsilon)} \Gamma(x-\xi)\frac{\partial f}{\partial \nu_\varepsilon}(\xi)d\sigma(\xi)\right| = 0. \tag{12.10}$$

On the other hand, by taking into account the last integral from (12.8), we have

$$\begin{aligned}\int_{S(x,\varepsilon)} f(\xi)\frac{\partial \Gamma}{\partial \nu_\varepsilon}(x-\xi)d\sigma(\xi) &= f(x)\int_{S(x,\varepsilon)} \frac{\partial \Gamma}{\partial \nu_\varepsilon}(x-\xi)d\sigma(\xi)\\ &\quad+\int_{S(x,\varepsilon)} \frac{\partial \Gamma}{\partial \nu_\varepsilon}(x-\xi)[f(\xi)-f(x)]d\sigma(\xi)\\ &= -f(x)+\int_{S(x,\varepsilon)} [f(\xi)-f(x)]\frac{\partial \Gamma}{\partial \nu_\varepsilon}(x-\xi)d\sigma(\xi).\end{aligned} \tag{12.11}$$

Because f is continuously differentiable on $\overline{\Omega}$ we deduce that the function f is Lipschitz function. Therefore, there is a constant c_2 so that

$$\forall \xi \in S(x,\varepsilon) : |f(x)-f(\xi)| \le c_2|x-\xi| \le c_2\varepsilon,$$

and then we obtain

$$\left|\int_{S(x,\varepsilon)} [f(\xi)-f(x)]\frac{\partial \Gamma}{\partial \nu_\varepsilon}(x-\xi)d\sigma(\xi)\right| \le C\varepsilon.$$

Therefore, by passing to the limit with $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_{S(x,\varepsilon)} f(\xi)\frac{\partial \Gamma}{\partial \nu_\varepsilon}(x-\xi)d\sigma(\xi) = f(x).$$

If we pass to the limit in (12.8) with $\varepsilon \to 0$ and we take into account the estimate above, we obtain the formula of representation from the statement of the theorem. ■

Observation 12.3.2 *From Theorem 12.3.1 we deduce that, in particular*

$$\forall f \in C_0^\infty(\Omega),\ \forall x \in \Omega\ :\ f(x) = \int_\Omega \Gamma(x-\xi)\delta f(\xi)\mathrm{d}\xi,$$

because both f and its normal derivative are null on the boundary of Ω. In this way, the next distributions make sense

$$\Delta\Gamma(x-\xi) = \delta_x(\xi).$$

As we have already stated, the formula of representation of f given in Theorem 12.3.1 is called the "Theorem of the three potentials". We now define the three potentials.

Definition 12.3.2 1^o. A potential of volume with density ρ_0, where the function ρ_0 is of class $C^0(\overline{\Omega})$, is the function defined by

$$u(x_0) = \int_{\Omega} \Gamma(x - \xi)\rho_0(\xi)\mathrm{d}\xi.$$

2^o. A potential of the surface of single layer with density ρ_1, where the function ρ_1 is of class $C^0(\partial\Omega)$, is the function u_1, defined by

$$u_1(x) = \int_{\partial\Omega} \rho_1(\xi)\frac{\partial\Gamma}{\partial\nu}(x - \xi)d\sigma(\xi).$$

3^o. A potential of the surface of double layer with the density ρ_2, where the function ρ_2 is of class $C^0(\partial\Omega)$, is the function u_2 defined by

$$u_2(x) = \int_{\partial\Omega} \rho_2(\xi)\sigma(x - \xi)d\sigma(\xi).$$

The main properties of these three potentials will be proven in the following theorem.

Theorem 12.3.2 *(i). The potential of the surface of a single layer and the potential of the surface of the double layer are harmonic functions in* Ω.

(ii). If the function ρ_0 *is of class* $C^1(\overline{\Omega})$ *then the potential of volume* u_0 *with the density* ρ_0 *is of class* $C^2(\Omega) \cap C^1(\overline{\Omega})$ *and verifies the equation*

$$\forall x \in \Omega \ : \ \delta u_0(x) = \rho_0(x).$$

Proof (i). We apply the theorem that allows the differentiation under the integral sign. Let x_0 be a point from Ω and denote by d_0 the distance from x_0 to the boundary $\partial\Omega$. We can verify immediately that for any multi-indices α there is a constant C_α so that for any x from the ball $B(x_0, d0/2)$ and for any $\xi \in \partial\Omega$ we have

$$\left|\frac{\partial^{|\alpha|}\Gamma}{\partial x^\alpha}(x - \xi)\right| \le C_\alpha; \ \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}|\left(\frac{\partial T}{\partial\nu}\right)(x - \xi)\right| \le C_\alpha.$$

Because the functions ρ_1 and ρ_2 are integrable on $\partial\Omega$, we can differentiate under the integral sign and obtain

$$\frac{\partial^{|\alpha|}u_1}{\partial x^\alpha}(x) = \int_{\partial\Omega} \rho_1(\xi)\frac{\partial^{|\alpha|}}{\partial x^\alpha}\left(\frac{\partial\Gamma}{\partial\nu}\right)(x - \xi)d\sigma(\xi),$$
$$\frac{\partial^{|\alpha|}u_2}{\partial x^\alpha}(x) = \int_{\partial\Omega} \rho_2(\xi)\frac{\partial^{|\alpha|}\Gamma}{\partial x^\alpha}(x - \xi)d\sigma(\xi).$$

So, we proven that the two potentials of the surface (u_1 of single layer and u_2 of double layer) are infinitely differentiable functions on Ω. From the previous computations we deduce, in particular, that

$$\delta u_1(x) = \int_{\partial\Omega} \rho_1(\xi)\delta\left(\frac{\partial\Gamma}{\partial\nu}\right)(x-\xi)d\sigma(\xi) = 0,$$
$$\delta u_2(x) = \int_{\partial\Omega} \rho_2(\xi)\delta\Gamma(x-\xi)d\sigma(\xi) = 0,$$

because $\Delta\Gamma(x-\xi) = 0$, for any distinct x and ξ, and in the last result we reversed the order of differentiation under the integral sign.

(ii). We apply again the theorem of differentiation under the integral sign (and this is allowed based on the hypotheses) and we obtain

$$\frac{\partial u_0}{\partial x_i}(x) = \int_{\Omega} \rho_0(\xi)\frac{\partial\Gamma}{\partial x_i}(x-\xi)\mathrm{d}\xi.$$

Thus, we deduce that $u_0 \in C^1(\overline{\Omega})$. On the other hand, by taking into account that

$$\frac{\partial\Gamma}{\partial x_i}(x-\xi) = -\frac{\partial\Gamma}{\partial \xi_i}(x-\xi),$$

we can immediately deduce that

$$\frac{\partial u_0}{\partial x_i}(x) = -\int_{\Omega} \rho_0(\xi)\frac{\partial\Gamma}{\partial x_i}(x-\xi)\mathrm{d}\xi$$
$$= \int_{\Omega} \Gamma(x-\xi)\frac{\partial\rho_0}{\partial x_i}(\xi)\mathrm{d}\xi - \int_{\partial\Omega} \rho_0(\xi)\Gamma(x-\xi)\nu_i(\xi)d\sigma(\xi), \tag{12.12}$$

in which we applied the formula of Ostrogradski. The first integral from the right-hand side of the formula (12.12) is a potential of volume with density $\partial\rho_0/\partial x_i$. The last integral from (12.12) is a potential of the surface of a single layer with the density $\rho_0\nu_i$, where ν_i are the cosine directors of the normal.

Because $\partial\rho_0/\partial x_i \in C^0(\overline{\Omega})$, we can show without difficulty that the potential of volume with density $\partial f_0/\partial x_i$ is a function of class $C^1(\overline{\Omega})$. Based on the point (i) of the theorem we have that the potential of the surface of a single layer with density $\rho_0\nu_i$ is an infinitely differentiable function on Ω.

Let ψ_0 be a function, $\psi \in C_0^2(\Omega)$. Based on Theorem 3.1 we have that for any $x \in \Omega$, the following representation formula holds true

$$\psi(x) = \int_{\Omega} \Gamma(x-\xi)\Delta\psi(\xi)\mathrm{d}\xi$$
$$+ \int_{\partial\Omega} \psi(\xi)\frac{\partial\Gamma}{\partial x_i}(x-\xi)d\sigma(\xi) - \int_{\partial\Omega} \frac{\partial\psi}{\partial x_i}(\xi)\Gamma(x-\xi)d\sigma(\xi). \tag{12.13}$$

By taking into account that the function ψ together with its gradient become null on the boundary $\partial\Omega$, from (12.13) we deduce that

$$\psi(x) = \int_\Omega \Gamma(x-\xi)\Delta\psi(\xi)\mathrm{d}\xi,$$

and then

$$\int_\Omega \psi(x)\Delta u_0(x)\mathrm{d}x = \int_\Omega u_0(x)\Delta\psi(x)\mathrm{d}x + 0$$
$$= \int_\Omega \Delta\psi(x)\left(\int_\Omega \Gamma(x-\xi)\rho_0(\xi)\mathrm{d}\xi\right)\mathrm{d}x.$$

In this formula, we apply the theorem of Fubini and we obtain

$$\int_\Omega \psi(x)\Delta u_0(x)\mathrm{d}x = \int_\Omega \rho_0(\xi)\int_\Omega \Gamma(x-\xi)\Delta\psi(x)\mathrm{d}x\mathrm{d}\xi$$
$$= \int_\Omega \rho_0(\xi)\psi(\xi)\mathrm{d}\xi,$$

and from this equality we are led to

$$\forall\psi \in C_0^2(\Omega) \;:\; \int_\Omega \psi(x)\left[\Delta u_0(x) - \rho_0(\xi)\right]\mathrm{d}x = 0.$$

If we assume that the density ρ_0, which is of class $C_0^2(\Omega)$, is also a function from $L^2(\Omega)$, we obtain the equality $\Delta u_0 = \rho_0$, almost everywhere on Ω. But, by taking into account that the functions u_0 and ρ_0 are continuous we deduce that in fact, the equality

$$\Delta u_0(x) = \rho(x),$$

holds true for any $x \in \Omega$, and this ends the proof. ■

In the following, we will approach the results regarding to classic solutions of the boundary value problems introduced in Chapter IV.

Theorem 12.3.3 *Let Ω be an open set from $\mathbb{R}^n (n \geq 2)$ having regular boundary and the function $f \in L^1(\mathbb{R}^n)$ which for $n = 2$ satisfies the condition*

$$\left|\int_{\|y\|>1} f(y)\ln(\|y\|)\mathrm{d}y\right| < \infty.$$

Then the function u_f defined by

$$u_f(x) = (f * \Gamma)(x) = \int_{\mathbb{R}^n} f(y)\Gamma(x-y)\mathrm{d}y, \tag{12.14}$$

is integrable on any compact set from $\mathbb{R}^n$ and satisfies the condition

$$\forall \Phi \in \mathbb{C}_0^\infty(\mathbb{R}^n) \ : \ \int_{\mathbb{R}^n} u_f(x)\Delta\Phi(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\Phi(x)\mathrm{d}x.$$

Proof In the proof, we will use the classical results approached in the following lemma (we will formulate it without proof).

Lemma 12.3.1 *Let us consider the function Φ having the support in the unit ball $B(0, 1) \subset \mathbb{R}^n$.*
Assume in addition that $\Phi \in L^1(\mathbb{R}^n)$, and that its norm, in the sense of $L^1(\mathbb{R}^n)$, is equal to 1. Denote by Φ_ε the function defined by

$$\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi\left(\frac{x}{\varepsilon}\right).$$

Then, we have

1^o. If the function u is from $L^\infty(\mathbb{R}^n)$ and is uniformly continuous on $\mathbb{R}^n$, then the function $(\Phi_\varepsilon * u)_\varepsilon$ is uniformly convergent to u on $\mathbb{R}^n$.

2^o. If the function u is continuous on $\mathbb{R}^n$, then the function $(\Phi_\varepsilon * u)_\varepsilon$ is uniformly convergent to u, on any compact set from $\mathbb{R}^n$.

3^o. If the function u is from $L^P(\mathbb{R}^n)$, $(1 \le p < \infty)$, then the function $(\Phi_\varepsilon * u)_\varepsilon$ is uniformly convergent to u, in $L^P(\mathbb{R}^n)$.

We will come back to the proof of Theorem 3.3. Consider the functions Γ_0 and Γ_∞ defined on $\mathbb{R}^n$ by

$$\Gamma_0(x) = \begin{cases} \Gamma(x) \ , \text{ if } \|x\| \le 1, \\ \ \ 0 \quad \ , \text{ if } \|x\| > 1, \end{cases}$$

$$\Gamma_\infty(x) = \Gamma(x) - \Gamma_0(x).$$

The function Γ_0 is from $L^1(\mathbb{R}^n)$ and then the product of convolution between f and Γ_0 is defined almost everywhere. In addition $f * \Gamma_0 \in L^1(\mathbb{R}^n)$. If $n \ge 3$ then the function Γ_∞ is from $L^\infty(\mathbb{R}^n)$ and consequently the product of convolution between f and Γ_∞ exists. Moreover, $f * \Gamma_\infty \in L^\infty(\mathbb{R}^n)$.

In the case $n = 2$, we have

$$\lim_{\|y\| \to \infty} (\ln \|x - y\| - \ln \|y\|) = 0.$$

Based on the additional assumption imposed on the function f, in the case $n = 2$, we deduce that the product of convolution $f * \Gamma_\infty$ is well defined and is bounded on any compact set from $\mathbb{R}^n$. Then the product of convolution between f and Γ (denoted in the theorem with u_f) exists and is a function from $L^1_{loc}(\mathbb{R}^n)$. To prove the second statement of the theorem, we define the functions f_k and u_k by

$$f_k(x) = \begin{cases} f(x) \ , \text{ if } \|x\| \le k, \\ 0 \quad , \text{ if } \|x\| > k \end{cases}$$

$$u_k = f_k * \Gamma.$$

It is clear that u_k is an integrable function on any compact set from $\mathbb{R}^n$. Suppose that the function $\Phi \in C_0^\infty(\mathbb{R}^n)$. We have

$$\int_{\mathbb{R}^n} u_n(x)\Delta\Phi(x)\mathrm{d}x = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f_n(x-y)\Gamma(y)\mathrm{d}y \right) \Delta\Phi(x)\mathrm{d}x$$

$$= \int_{\mathbb{R}^n} \Gamma(y) \left(\int_{\mathbb{R}^n} f_n(x-y)\Delta\Phi(x)\mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}^n} \Gamma(y) \left(\tilde{f}_n * \Delta\Phi \right)(y)\mathrm{d}y$$

$$= \int_{\mathbb{R}^n} \Gamma(y)\Delta \left(\tilde{f}_n * \Phi \right)(y)\mathrm{d}y = (f_n * \Phi)(0),$$

where we used the notation $\tilde{f}_n(y) = f_n(-y)$. We deduce then that

$$\int_{\mathbb{R}^n} u_n(x)\Delta\Phi(x)\mathrm{d}x = \int_{\mathbb{R}^n} \tilde{f}_n(-y)\Phi(y)\mathrm{d}y = \int_{\mathbb{R}^n} f_n(y)\Phi(y)\mathrm{d}y.$$

Now, it is sufficient to pass to the limit with $n \to \infty$ and then based on a theorem of convergence, which is due to Lebesque, we have

$$\int_{\mathbb{R}^n} u_f(x)\Delta\Phi(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(y)\Phi(y)\mathrm{d}y, \forall \Phi \in C_0^\infty(\mathbb{R}^n),$$

and this ends the proof the theorem. ∎

Observation 12.3.3 *We can easily see that Theorem 3.3 proves that u_f, defined in (12.14), is a solution of the Dirichlet's problem on the space $\mathbb{R}^n$, in the sense of the distributions.*

We intend now to find the sufficient conditions that must be imposed on the function f so that the function u_f defined in (12.14) is a classical solution of the Dirichlet's problem.

To this end, we recall in the beginning the definition of a Hölder function. Let α be a real fixed number, $\alpha \in (0, 1)$. We say that u is a Hölder function (in other words, it satisfies the property of Hölder) of exponent α on the set $\overline{\Omega}$ if it satisfies the equality: there exists a constant C so that

$$\forall x, y \in \overline{\Omega} \ : \ |u(x) - u(y)| \le C|x - y|^\alpha,$$

where the constant C depends only of the set $\overline{\Omega}$. Denote by $C^\alpha(\overline{\Omega})$ the set of all Hölder functions on $\overline{\Omega}$, of exponent α.

Observation 12.3.4 1^o. *It is easy to verify that if the function $f \in C^\alpha(\overline{\Omega})$, then f is continuous on $\overline{\Omega}$.*

2°. If the function f is of class $C^1(\overline{\Omega})$, then $f \in C^\alpha(\overline{\Omega})$, a result which is obtained without difficulty by applying the theorem of finite increases of Lagrange.

Theorem 12.3.4 *Assume that the function $f \in C^\alpha(\overline{\Omega})$. Then*

$$u_f = f * \Gamma \in C^0(\overline{\Omega}) \cap C^2(\Omega)$$

and it verifies the equation

$$\Delta u_f(x) = f(x), \forall x \in \Omega.$$

Proof It is easy to prove the first part of the theorem. To prove that the function u_f is a classical solution of the Dirichlet's problem, we will consider the extension $\overline{f}$ of the function f to the whole space $\mathbb{R}^n$, by taking $\overline{f}(x) = 0$ if u is outside Ω. Then $\overline{f} \in Ł^1(\mathbb{R}^n)$ and for u_f we have

$$\int_{\mathbb{R}^n} u_f(x)\Delta\Phi(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\Phi(x)\mathrm{d}x, \ \forall\Phi \in C_0^\infty(\mathbb{R}^n),$$

if and only if

$$\int_{\mathbb{R}^n} \Delta u_f(x)\Phi(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\Phi(x)\mathrm{d}x, \ \forall\Phi \in C_0^\infty(\mathbb{R}^n),$$

and the proof of the theorem is complete. ■

Definition 12.3.3 The function of Green attached to the set Ω, is a function G, $G : \Omega \times \overline{\Omega} \to \mathbb{R}$, which verifies the following properties:

1°. $\forall x \in \Omega \ : \ y \longmapsto G(x, y) - \Gamma(x - y)$ is a harmonic function on Ω and continuous on $\overline{\Omega}$;

2°. $\forall x \in \Omega, \ \forall y \in \partial\Omega : G(x, y) = 0.$

In the following proposition, we prove a result of symmetry regarding the function of Green.

Proposition 12.3.2 *For $\forall x, y \in \Omega$, we have $G(x, y) = G(y, x)$.*

Proof Let x, y, z be arbitrary points in Ω. On the set

$$\Omega_s = \Omega \setminus \overline{(B(x,r) \cup B(y,r))}, \ r > 0,$$

we define the functions u and v by $u(y) = G(x, y), \ v(y) = G(z, y)$. It is clear that the functions u and v are harmonic in Ω_r because the functions $u(y) - \Gamma(x - y)$ and $v(y) - \Gamma(z - y)$ are harmonic in Ω_r and the applications $y \mapsto \Gamma(x - y)$ and $y \mapsto \Gamma(z - y)$ are harmonic on the set Ω_r. Then we can write

$$
\begin{aligned}
0 = \int_{\Omega_r} (u\Delta v - v\Delta u)\,\mathrm{d}y = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma(y)\\
\int_{S(x,r)} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma(y) - \int_{S(y,r)} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma(y),
\end{aligned}
$$

where, by convention, the normal ν is assumed to be oriented to the outside of the respective domain.

On the other hand, we have

$$
\begin{aligned}
&\int_{S(x,r)} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma(y)\\
&= \int_{S(x,r)} \left[(u-\Gamma(x-y))\frac{\partial v}{\partial \nu} - (v-\Gamma(z-y))\frac{\partial u}{\partial \nu}\right] d\sigma(y)\\
&\quad + \int_{S(x,r)} \left[\Gamma(x-y)\frac{\partial v}{\partial \nu} - \Gamma(z-y)\frac{\partial u}{\partial \nu}\right] d\sigma(y)\\
&= \int_{S(x,r)} \begin{pmatrix} (u-\Gamma(x-y))\frac{\partial v-\Gamma(z-y))}{\partial \nu} \\ -(v-\Gamma(z-y))\frac{\partial(u-\Gamma(x-y))}{\partial \nu}\end{pmatrix} d\sigma(y)\\
&\quad + \int_{S(x,r)} \begin{pmatrix} (u-\Gamma(x-y))\frac{\partial \Gamma(z-y)}{\partial \nu} \\ -(v-\Gamma(z-y))\frac{\partial \Gamma(x-y)}{\partial \nu}\end{pmatrix} d\sigma(y)\\
&\quad + \int_{S(x,r)} \left[\Gamma(x-y)\frac{\partial v}{\partial \nu} - \Gamma(z-y)\frac{\partial u}{\partial \nu}\right] d\sigma(y).
\end{aligned}
$$

If we pass to the limit with $r \to 0$, then the first integral tends to zero because the functions under the integral sign are at least of class C^1 on Ω. In the second integral, the term

$$
\int_{S(x,r)} \left[(u-\Gamma(x-y))\frac{\partial \Gamma(z-y)}{\partial \nu}\right] d\sigma(y),
$$

tends to zero, as $r \to 0$, because the functions which appear here are at least of class C^1 on Ω. Based on the Theorem 12.3.2 of characterization of the potentials of the surface, for the other integrals, we have

$$
\begin{aligned}
-\int_{S(x,r)} \left[(v-\Gamma(z-y))\frac{\partial \Gamma(x-y)}{\partial \nu}\right] d\sigma(y) = -(v(x) - \Gamma(z-x)),\\
\int_{S(x,r)} \left[\Gamma(x-y))\frac{\partial v}{\partial \nu} - \Gamma(z-y)\frac{\partial u}{\partial \nu}\right] d\sigma(y)\\
= \int_{S(x,r)} \left[\Gamma(x-y))\frac{\partial(v-\Gamma(z-y))}{\partial \nu} - \Gamma(z-y)\frac{\partial(u-\Gamma(x-y))}{\partial \nu}\right] d\sigma(y)\\
+ \int_{S(x,r)} \left[\Gamma(x-y))\frac{\partial \Gamma(z-y)}{\partial \nu} - \Gamma(z-y)\frac{\partial \Gamma(x-y)}{\partial \nu}\right] d\sigma(y),
\end{aligned}
$$

and therefore

$$\int_{S(x,r)} \left[\Gamma(x-y)) \frac{\partial v}{\partial \nu} - \Gamma(z-y) \frac{\partial u}{\partial \nu} \right] d\sigma(y) \to 0,$$

as $r \to 0$.

Let us underline the fact that

$$\int_{\partial\Omega} \left[u \frac{\partial u}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right] d\sigma(y) = 0,$$

because u and v are null on the boundary $\partial\Omega$.

In conclusion, we have

$$0 = -v(x) + u(x),$$

and this is equivalent to

$$G(z, x) = G(x, z),$$

and the proof Proposition 12.3.2 is complete. ■

The result from Proposition 12.3.2 allows the extension of the function of Green to the set $\overline{\Omega} \times \overline{\Omega}$, by considering that

$$\forall x \in \partial\Omega, \ \forall y \in \Omega \ : \ G(x, y) = 0.$$

We can now give representations for the classic solutions of Dirichlet's problem both in the case of the homogeneous problem and in the case of the nonhomogeneous problem.

Theorem 12.3.5 *(i). If the function f verifies the hypotheses of the theorem 3.5, then the solution u_f of the Dirichlet's homogeneous problem*

$$\begin{cases} \Delta u_f(x) = f(x), \ \forall x \in \Omega, \\ u_f(x) = 0, \ \forall x \in \partial\Omega, \end{cases}$$

is equal to

$$u_f(x) = \int_\Omega G(x, y) f(y) \mathrm{d}y,$$

where G is the function of Green attached to Laplace operator on the set Ω.

(ii). Let g be a continuous function on $\partial\Omega$, $g \in C^0(\partial\Omega)$. The solution v_g of the Dirichlet's problem

$$\begin{cases} \Delta v_g(x) = 0, \ \forall x \in \Omega, \\ v_g(x) = g(x), \ \forall x \in \partial\Omega \end{cases}$$

is equal to

$$v_g(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x, y) g(y) d\sigma(y),$$

where we denoted by

$$\frac{\partial G}{\partial \nu_y}$$

the so-called Poisson kernel of the Laplacian in Ω.

Proof (i). We can write

$$\begin{aligned} \int_\Omega G(x, y) f(y) \mathrm{d}y &= \int_\Omega \Gamma(x-y) f(y) \mathrm{d}y + \int_\Omega [G(x, y) - \Gamma(x-y)] f(y) \mathrm{d}y \\ &= \left(\overline{f} * \Gamma\right)(x) + \int_\Omega [G(x, y) - \Gamma(x-y)] f(y) \mathrm{d}y, \end{aligned}$$

where $\overline{f}$ is the extension of f to the whole space $\mathbb{R}^n$, giving to f the null value outside Ω.

The Laplacian of the first quantity is equal to $f(x)$, in virtue of the Theorem 3.4. Because the application $y \mapsto G(x, y) - \Gamma(x - y)$ is a harmonic function in Ω and continuous on $\overline{\Omega}$, based on the properties of the function of Green, we deduce that the Laplacian of the second quantity is null.

If $x \in \partial\Omega$, based on the properties of the function of Green we deduce that

$$\int_\Omega G(x, y) f(y) \mathrm{d}y = \int_\Omega 0.f(y) \mathrm{d}y = 0,$$

from where we deduce that the function u_f defined by

$$u_f = \int_\Omega G(x, y) f(y) \mathrm{d}y,$$

is a classical solution of Dirichlet's problem.

The point (ii) is proved analogously. ∎

If we take the ball centered in the origin and having radius R, $B(0, R)$, in the set Ω we obtain the result from the following theorem.

Theorem 12.3.6 *If the function* $f \in C^1(\overline{B(0, R)})$ *and the function* $g \in C^0(S(0, R))$, *then the solution* $u_{f,g}$ *of the nonhomogeneous Dirichlet problem*

$$\begin{cases} \delta u_{f,g}(x) = f(x), \ \forall x \in \Omega, \\ u_{f,g}(x) = g(x), \ \forall x \in \partial\Omega \end{cases}$$

is a function from $C^2(\Omega) \cap C^0(\overline{\Omega})$ *and is given by the formula*

$$u_{f,g}(x) = \int_{S(0,R)} P_R(x,\xi)g(\xi)d\sigma(\xi) + \int_{0,R} G_R(x,\xi)f(\xi)\mathrm{d}\xi,$$

where we used the relations

$$P_R(x,\xi) = \frac{\mathbb{R}^2 - \|x\|^2}{\omega_n R\|x-\xi\|^n},$$

$$G_R(x,\xi) = -\frac{1}{(n-2)\omega_n}\left(\frac{1}{\|x-\xi\|^{n-2}} - \frac{\left(\frac{R}{\|x\|}\right)^{n-2}}{\left\|\left(\frac{R}{\|x\|}\right)^2 x - \xi\right\|^{n-2}}\right),$$

for $n \geq 3$ *and* $x \neq 0$. *If* $n=2$ *and* $x \neq 0$ *then we have*

$$G_R(x,\xi) = \frac{1}{2\pi}\ln\frac{\|x\|\left\|\left(\frac{R}{\|x\|}\right)^2 x - \xi\right\|}{R\|x-\xi\|}.$$

Proof The result is immediately obtained, as a particularization of the results from the previous theorems. ■

Chapter 13
Weak Solutions of Classical Problems

13.1 The Sobolev Spaces $\mathbf{H^1(\Omega)}$ and $\mathbf{H_0^1(\Omega)}$

The Sobolev spaces, which will be defined in the following, are spaces on which weak solutions can be defined (in a sense to be defined later) for classical boundary value problems.

Definition 13.1.1 Let Ω be an open set from $\mathbb{R}^n$ whose boundary $\partial\Omega$ is supposed to be a regular surface (at least of class C^1). We say that the function u belongs to the Sobolev space $H^1(\Omega)$ if and only if (by definition) $u \in L^2(\Omega)$ and $\forall i \in 1, 2, \ldots, n$, $\exists v_i \in L^2(\Omega)$ so that

$$\int_\Omega u(x)\frac{\partial\varphi}{\partial x_i}(x)\mathrm{d}x = -\int_\Omega v_i(x)\varphi(x)\mathrm{d}x,\ \forall\varphi \in C_1^0(\Omega).$$

The function v_i is called the weak derivative of the function u, with regard to the variable x_i.

Example 13.1.1 Let Ω be an open and bounded set from $\mathbb{R}^n$.

1^o. Any function $u \in C^1(\overline{\Omega})$ is from the Sobolev space $H^1(\Omega)$ and its classical derivatives are equal to the weak derivatives. To prove the statement it is sufficient to note that, on the one hand, the function $u \in L^2$, and on the other hand $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$. Then with the first formula of Green, we obtain

$$\int_\Omega u(x)\frac{\partial\varphi}{\partial x_i}(x)\mathrm{d}x = -\int_\Omega \frac{\partial u}{\partial x_i}(x)\varphi(x)\mathrm{d}x + 0,$$

because φ is null on the boundary $\partial\Omega$. We deduce thus that the integral

$$\int_{\partial\Omega} u(x)\varphi(x)\nu_i d\sigma(x)$$

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
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is null. In the last integral, we denoted by ν_i the component of order i of the unit normal ν oriented outside $\partial\Omega$ (called also directional cosine of order i of the unit normal), and $d\sigma(x)$ represents the measure on the boundary $\partial\Omega$. This example allows us to identify the weak derivative with the strong derivative and in the following we will use the notation $v_i = \frac{\partial u}{\partial x_i}$ just if the function u (which belongs to the Sobolev space $H^1(\Omega)$) is not differentiable in the usual sense.

2^o. On the interval $(-1, 1)$, we consider the function $x \longmapsto u(x) = |x|$.

Clearly, we have that $u \in H^1(-1, 1)$. In fact, we can show without difficulty that $u \in L^2(-1, 1)$. In addition, for any function $\varphi \in C_0^1((-1, 1))$, we have

$$\int_{-1}^{1} u(x)\varphi'(x)\mathrm{d}x = -\int_{-1}^{0} x\varphi'(x)\mathrm{d}x + \int_{0}^{1} x\varphi'(x)\mathrm{d}x$$
$$= \varphi(-1) + \int_{-1}^{0} \varphi(x)\mathrm{d}x - \varphi(1) - \int_{0}^{1} \varphi(x)\mathrm{d}x = \int_{-1}^{0} \varphi(x)\mathrm{d}x - \int_{0}^{1} \varphi(x)\mathrm{d}x,$$

because $\varphi(-1) = \varphi(1) = 0$ $(\varphi \in C_0^1(-1, 1))$. Therefore, we deduce that there is the function v defined by

$$v(x) = \begin{cases} -1 \ , \text{ if } x \in (-1, 0), \\ \ \ 1 \ \ , \ \text{ if } x \in (0, 1) \end{cases}$$

which obviously belongs to $L^2(-1, 1)$, and which satisfies the equality

$$\int_{-1}^{1} u(x)\varphi'(x)\mathrm{d}x = -\int_{-1}^{1} v(x)\varphi(x)\mathrm{d}x.$$

Example 2^o allows us to expand the differentiability (in the sense of the weak differentiability) also for functions which are not necessarily differentiable in the classical sense.

The following result will be formulated without demonstration because we consider it as a known result and the proof can be found in any book of the functional analysis.

Theorem 13.1.1 *The function u is in the Sobolev space $H^1(\Omega)$ if and only if there exists a sequence $\{u_k\}_k$ of functions from the space $C_0^\infty(R^n)$, which is convergent in the following sense:*

- *$\{(u_n)_{|\Omega}\}_n$ is convergent to u in $L^2(\Omega)$;*
- *for any index $i \in \{1, 2, \ldots, n\}$, we have*

$$\left\{ \frac{\partial u_n}{\partial x_i}|_\Omega \right\}_n \to \frac{\partial u}{\partial x_i}, \text{ in } L^2(\Omega).$$

In the following theorem, we will prove that the Sobolev space $H^1(\Omega)$ is, in fact, a Hilbert space with respect to a conveniently chosen scalar product.

Theorem 13.1.2 *The Sobolev space $H^1(\Omega)$ is a Hilbert space with respect to the following scalar product:*

$$< u, v >_{H^1(\Omega)}= \int_\Omega u(x)v(x)\mathrm{d}x + \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)\mathrm{d}x.$$

Proof It is easy to verify that the following application

$$(u, v) \longmapsto \int_\Omega u(x)v(x)\mathrm{d}x + \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)\mathrm{d}x,$$

is a scalar product on $H^1(\Omega)$. Now, we want to prove that the space $H^1(\Omega)$ endowed with the norm associated with this scalar product is a complete metric space. Let us consider a sequence $\{u_k\}_k$, which is a Cauchy sequence in the norm

$$\|u\|_{H^1(\Omega)} = \sqrt{< u, u >}.$$

In particular, this sequence is Cauchy with respect to the norm from $L^2(\Omega)$, and the sequences $\{\frac{\partial u_k}{\partial x_i}\}_k$, for each $i = 1, 2, \ldots, n$, are also Cauchy sequences with respect to the norm from $L^2(\Omega)$. Thus, we deduce the existence of a subsequence $\{u_{k'}\}_{k'}$ and of the functions $u, v_1, v_2, \ldots, v_n$ from $L^2(\Omega)$ so that

$$u_{k'} \to u, \text{ for } k' \to \infty,$$
$$\forall i \in 1, 2, \ldots, n, \ \frac{\partial u_{k'}}{\partial x_i} \to v_i, \text{ for } k' \to \infty,$$

the convergences taking place in the strong topology from $L^2(\Omega)$.

Therefore, we deduce that v_i is the weak derivative of u with respect to the variable x_i, by passing to the limit in the equality

$$\int_\Omega u_{k'}(x)\frac{\partial \varphi}{\partial x_i}(x)\mathrm{d}x = -\int_\Omega \frac{\partial u_{k'}}{\partial x_i}(x)\varphi(x)\mathrm{d}x,$$

and this proves the fact that $u \in H^1(\Omega)$. We can verify then, without difficulty, that the sequence $\{u_{k'}\}_{k'}$ is convergent to u in $H^1(\Omega)$, and the proof of the theorem is complete. ■

Observation 13.1.1 1^o. *Using the application*

$$\begin{pmatrix} H^1(\Omega) \to & \left(L^2(\Omega)\right)^{n+1} \\ u \quad \mapsto & \left(u, \frac{\partial u}{\partial x_i}, \frac{u}{x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) \end{pmatrix},$$

we can show that the Sobolev space $H^1(\Omega)$ is a separable Hilbert space, because $\left(L^2(\Omega)\right)^{n+1}$ is a separable Hilbert space.

2^o. *On the Sobolev space $H^1(\Omega)$, we have two notions of convergence:*

(i) - strong convergence: the sequence $\{u_k\}_k$ is convergent to u in the strong topology from $H^1(\Omega)$ if and only if (by definition) we have

$$\int_\Omega |\nabla (u_k - u)|^2 \, \mathrm{d}x + \int_\Omega (u_k - u)^2 \, \mathrm{d}x \to 0, \ \textit{as } n \to \infty.$$

We denote the strong convergence in the form: $u_k \to \infty$, as $k \to \infty$.

(ii) - weak convergence: the sequence $\{u_k\}_k$ is convergent to u in the weak topology from $H^1(\Omega)$ if and only if (by definition) we have

$$\forall v \in H^1(\Omega) \ : \ \int_\Omega \nabla (u_k - u) \nabla v \mathrm{d}x + \int_\Omega (u_k - u) \, v \mathrm{d}x \to 0, \ \textit{as } k \to \infty.$$

We denote the weak convergence in the form: $u_k \rightharpoonup k$, as $k \to \infty$.

The necessity of the introduction of the weak topology is certified from the result of compactness (which is stated without proof) that is valid, in general, for all Hilbert spaces, therefore it is valid also in the case of the Sobolev space $H^1(\Omega)$.

Theorem 13.1.3 *Let H be a Hilbert space, which as we know is a reflexive space. From any sequence $\{x_k\}_k$ which is bounded in H, we can extract a subsequence which is weakly convergent.*

An important subspace of the Sobolev space $H^1(\Omega)$, very useful in the following, is $H_0^1(\Omega)$.

Definition 13.1.2 We say that the function u belongs to the space $H_0^1(\Omega)$ if and only if (by definition), there is a sequence $\{u_k\}_k$ of the functions from $C_0^\infty(\Omega)$ which is convergent to u in the strong topology of $H^1(\Omega)$.

Observation 13.1.2 1^o. *The space $H_0^1(\Omega)$ is known also under the name of Sobolev space.*

2^o. *In the definition of the Sobolev space $H_0^1(\Omega)$, we can substitute the condition that the sequence $\{u_k\}_k$ contains functions from $C_0^\infty(\Omega)$ with the condition that the terms u_k of the sequence are from $C_0^1(\Omega)$.*

3^o. *The Sobolev space $H_0^1(\Omega)$ is effectively included in the Sobolev space $H^1(\Omega)$ because from Theorem 13.1.1 and the Definition 13.1.2 it is certified that in both cases the support of the functions u_k is included in Ω.*

4^o. *We can state that the space $H_0^1(\Omega)$ contains those functions from the space $H^1(\Omega)$ which become null on the boundary $\partial\Omega$.*

5^o. If Ω is the whole space $\mathbb{R}^n$, then $H^1(\Omega)$ coincides with $H_0^1(\Omega)$ because the restrictions regarding the support disappear. It is one of the rare cases when the two Sobolev spaces coincide.

6^o. The space $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ because it is the closure of the space $C_0^\infty(\Omega)$ (or $C_0^1(\Omega)$) in the strong topology from $H^1(\Omega)$. Consequently, we can consider that $H_0^1(\Omega)$ is a Hilbert space with respect to the norm from $H^1(\Omega)$.

In the case in which Ω is an open and bounded set from $\mathbb{R}^n$, we can define an equivalent norm on the space $H_0^1(\Omega)$.

Theorem 13.1.4 (Inequality of Friedrichs). *Let Ω be an open and bounded set from $\mathbb{R}^n$. Then, we have*

(i) There is a constant C, which depends only on Ω, so that

$$\forall u \in H_0^1(\Omega) \ : \ \|u\|_{L^2(\Omega)} \le C\| \nabla u\|_{L^2(\Omega)},$$

where the constant C is called the constant of Poincaré, for the set Ω.

(ii) The norm from $H^1(\Omega)$ is equivalent to the semi-norm

$$u \mapsto \| \nabla u\|_{L^2(\Omega)}$$

on the space $H_0^1(\Omega)$.

Proof (i) The fact that the set Ω is bounded ensures the existence of a real number a so that Ω is included in the cube $[-a, a]^n$. Then for any function $u \in C_0^1(\Omega)$ and for any point $x \in \Omega$, $x = (x_1, x_2, \ldots, x_n)$, we have the representation

$$u(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_{n-1}, -a)$$
$$+ \int_{-a}^{x_n} \frac{\partial u}{\partial x_n}(x_1, x_2, \ldots, x_{n-1}, s)\mathrm{d}s = \int_{-a}^{x_n} \frac{\partial u}{\partial x_n}(x_1, x_2, \ldots, x_{n-1}, s)\mathrm{d}s,$$

because the function u becomes null on the boundary $\partial\Omega$. Then, we deduce that then that

$$|u(x_1, x_2, \ldots, x_n)| \le \int_{-a}^{x_n} \left| \frac{\partial u}{\partial x_n}(x_1, x_2, \ldots, x_{n-1}, s)\right| \mathrm{d}s$$
$$\le \int_{-a}^{a} \left| \frac{\partial u}{\partial x_n} \right. (x_1, x_2, \ldots, x_{n-1}, s)| \,\mathrm{d}s \le \int_{-a}^{a} |\nabla u\, (x_1, x_2, \ldots, x_{n-1}, s)| \,\mathrm{d}s$$
$$\le \sqrt{2a} \left\{ \int_{-a}^{a} |\nabla u\, (x_1, x_2, \ldots, x_{n-1}, s)|^2 \,\mathrm{d}s \right\}^{1/2},$$

where we applied the inequality of Buniakowsky.
We square both members of the previous inequality, then integrate on Ω and we obtain

$$\int_{\Omega} u^2 (x_1, x_2, \ldots, x_n) \mathrm{d}x \leq 2a \int_{\Omega} \int_{-a}^{a} |\nabla u (x_1, x_2, \ldots, x_{n-1}, s)|^2 \, \mathrm{d}s$$
$$\leq 4a^2 \int_{\Omega} |\nabla u (x_1, x_2, \ldots, x_n)|^2 \, \mathrm{d}x,$$

where we take into account that the set Ω is included in the cube $[-a, a]^n$. If we impose on the constant C to satisfy the condition $C \leq 2a$, the proof of the statement (i) is complete.

(ii) It is easy to verify the inequality

$$\|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq \sqrt{1 + C^2} \| \nabla u\|_{L^2(\Omega)},$$

and this proves that the semi-norm $u \mapsto \| \nabla u\|_{L^2(\Omega)}$ is, in fact, a norm on the space $H_0^1(\Omega)$ and this norm defines on $H_0^1(\Omega)$ a topology equivalent to the topology given by the norm on the space $H^1(\Omega)$ and with this the proof of the Theorem 13.1.4 is complete. ■

Observation 13.1.3 *From the proof of the Theorem 13.1.4 it is certified that it is sufficient that Ω be bounded only in one direction to ensure the existence of a constant of Poincaré and the semi-norm $u \mapsto \| \nabla u\|_{L^2(\Omega)}$ be a norm on the space $H_0^1(\Omega)$.*

At the end of this paragraph, we want to make a small comment about the known Helmholtz's equation, which has the form

$$\Delta u + k^2 u = -f(x), \;\; \forall x \in \mathbb{R}^3.$$

In the case $k = 0$, the equation above is reduced to the equation of Poisson. Therefore, a study of the equation of Helmholtz should be made analogously to that made on the Poisson's equation. However, there are some features, in particular with regards to the uniqueness of the solution, which justify a separate study on the equation of Helmholtz. For instance, the weak solution of the Poisson's equation in the space $\mathbb{R}^3$ is unique in the set of the distributions which become null at infinity. In the case of the homogeneous equation, attached to the equation of Helmholtz, we have the nontrivial solution

$$Im\ \mathcal{E}(x) = -\frac{\sin k|x|}{4\pi|x|},$$

by knowing that a fundamental solution of the equation of Helmholtz is the distribution

$$\mathcal{E}(x) = -\frac{e^{ik|x|}}{4\pi|x|}.$$

However, the uniqueness of the solution can be obtained, for instance, in the case of the so-called *condition of radiation* due to Sommerfeld

$$u(x) = O\left(|x|^{-1}\right), \ |x| \to \infty,$$
$$\frac{\partial u(x)}{\partial |x|} + iku(x) = o(|x|^{-1}), \ |x| \to \infty.$$

13.2 The Lax–Milgram Theorem and Applications

When solving boundary value problems of Dirichlet type and also of Neumann type, an important role is played by the following theorem, which is known in the literature under the name of "the Lax–Milgram theorem".

Theorem 13.2.1 (Lax–Milgram). *Let H be a Hilbert subspace and we will denote by a a bilinear form, $a : H \times H \to \mathbb{R}$, having the following two properties:*

- *there is a constant C so that*

$$\forall x, y \in H \ : \ |a(x, y)| \le C\|x\|\|y\|,$$

which will be called "the property of continuity";
- *there is a constant $c > 0$ so that*

$$\forall x \in H \ : \ a(x, x) \ge c\|x^2\|,$$

which will be called "the property of coercivity".
Then for any continuous linear form $L : H \to \mathbb{R}$, there is a point $x_L \in H$ and it is unique so that

$$\forall y \in H \ : \ a(x_L, y) = L(y).$$

Proof Based on the hypotheses of the theorem, we must show that the conditions of the known theorem of representation due to Riesz are satisfied. For any $y \in H$, the application

$$\begin{pmatrix} H \longrightarrow & \mathbb{R}, \\ y \longmapsto & a(x, y) \end{pmatrix}$$

is linear and continuous. The theorem of Riesz ensures then the existence and uniqueness of a element $Ax \in H$ so that

$$a(x, y) =< Ax, y >, \ \forall y \in H.$$

Because of the uniqueness of the element Ax and by using the linearity of the bilinear form a with respect to its first variable, we deduce that we obtained an operator $A : H \to H$. Let us prove that the operator A is surjective. Indeed, let x_0 be an element from the kernel of A. Then for any $y \in H$ we have

$$< Ax_0, y >= 0 = a\,(x_0, y) \Rightarrow a\,(x_0, x_0) = 0.$$

By using the coercivity of the bilinear form, we deduce that $x_0 = 0$.

On the other hand, the operator A is continuous because

$$|< Ax, y >| = |a(x, y)| \leq C\|x\|\|y\|,\ \forall y \in H,$$

from where we deduce that

$$\|A\|_{\mathcal{L}(H)} \leq C.$$

Now, for the linear and continuous form L we apply the theorem of Riesz. Hence, we deduce the existence and uniqueness of the element $l \in H$ so that

$$L(x) =< l, x >,\ \forall x \in H.$$

It remains only to prove the existence of an element x_L in H so that

$$< Ax_L, y >=< l, y >, \forall y \in H,$$

and this is equivalent with $Ax_L = l$.

For this, we consider the operator $T_\rho : H \to H$ defined for a $\rho > 0$, which will be chosen later, by

$$T_\rho(x) = x - \rho\,(Ax - l)\,.$$

Searching for a point x_L so that $Ax_L = l$ is equivalent to searching for a fixed point x in H of the operator T_ρ. Thus, the problem is reduced to prove that the operator T_ρ, for ρ conveniently chosen, is a contraction. Using the linearity and the continuity of the operator A as well as the coercivity of the bilinear form a, we can do the following calculations

$$\begin{aligned}\left\|T_\rho(x) - T_\rho(y)\right\| &= \|x - y\|^2 - 2\rho < A(x - y), x - y > \\ &+ \rho^2\|Ax - Ay\|^2 = \|x - y\|^2 - 2\rho a(x - y, x - y) + \rho^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\rho c\|x - y\|^2 + \rho^2 C^2\|x - y\|^2.\end{aligned}$$

It is sufficient to choose ρ so that

$$1 - 2\rho c + \rho^2 C^2 < 1,$$

and we obtain that the operator T_ρ is a contraction. For instance, $\rho = \frac{c}{2C^2}$ satisfies the above condition. The proof is concluded. ■

A generalization of the Lax–Milgram theorem was given by Stampacchia.

Theorem 13.2.2 (Stampacchia). *Assume that the hypotheses of the Lax–Milgram theorem are satisfied. Suppose also that the set K is included in H. In addition, we assume that K is a convex set, that is,*

$$\forall x, y \in K,\ \forall z \in [0,1]\ :\ (1-t)x + ty \in K.$$

Then for any linear and continuous form $L : H \to \mathbb{R}$, there is an element $x_{L,K} \in K$ and is unique so that we have

$$\forall y \in K\ :\ a\left(x_{L,K}, y - x_{L,K}\right) \geq L\left(y - x_{L,K}\right).$$

Proof We give only a sketch of the proof. Consider the operator A, as in the proof of the Theorem 13.2.1, and we will build the operator T_ρ as follows

$$T_\rho(x) = proi_K\left(x - \rho(Ax - l)\right).$$

Furthermore, the proof can be continued analogously, as in the proof of the Lax–Milgram Theorem. ■

We will apply in the following the results above in order to solve the classical boundary value problems of Dirichlet and Neumann type.

Let Ω be an open set from $\mathbb{R}^n$ with regular boundary $\partial\Omega$ and consider the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = f, \text{ in } \Omega, \\ u = 0, \text{ pe } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$.

We should note, first that if we multiply both members of the equation $-\Delta u = f$ by the test function $\varphi \in C_0^1(\Omega)$ and then integrate by parts the obtained relation on the domain Ω, then we are led to the equality

$$\int_\Omega \nabla u . \nabla \varphi \mathrm{d}x = \int_\Omega f \varphi \mathrm{d}x$$

by taking into account the null boundary conditions on the boundary $\partial\Omega$ (called homogeneous Dirichlet conditions), it is natural to search for the solution u of this problem in the Sobolev space $H_0^1(\Omega)$.

In the following theorem, we prove the existence and the uniqueness of the solution in this Sobolev space.

Theorem 13.2.3 *For any function $f \in L^2(\Omega)$, there is only one function u_f so that $u_f \in H_0^1(\Omega)$ and which verifies the equality*

$$\forall \varphi \in H_0^1(\Omega)\ :\ \int_\Omega \nabla u_f \nabla \varphi \mathrm{d}x = \int_\Omega f \varphi \mathrm{d}x, \tag{13.2.1}$$

that is, the function u_f is a weak solution (unique) of the homogeneous Dirichlet problem.

Proof We must first note that in formula (13.2.1) we have, in fact, the variational formulation of the homogeneous Dirichlet problem.

Let v_0 be an arbitrary function from the Sobolev space $H_0^1(\Omega)$ which is used to multiply formally both members of the Poisson's equation $-\Delta u = f$. We integrate on Ω the resulting relation and after application of the formula of Green, we obtain

$$\int_\Omega \nabla u. \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x.$$

Thus, it is natural to consider the bilinear form $a(u, v)$ and the linear form $L(v)$, defined, respectively, by

$$a(u, v) = \int_\Omega \nabla u. \nabla v \mathrm{d}x,$$
$$L(v) = \int_\Omega f v \mathrm{d}x.$$

It is clear that $a(u, v)$ is a bilinear form defined on $H_0^1(\Omega) \times H_0^1(\Omega)$. By taking into account the estimate

$$|a(u, v)| \le \| \nabla u\|_{L^2(\Omega)} \| \nabla v\|_{L^2(\Omega)},$$

obtained by applying the Cauchy–Buniakovski–Schwartz inequality, we deduce that the bilinear form $a(u, v)$ is continuous.

We use now the bound from above

$$|L(v)| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} C \| \nabla v\|_{L^2(\Omega)},$$

which is obtained by applying the inequality of Friedrichs. Hence,we deduce that the linear form $L(v)$ is continuous.

With these considerations, we can apply the Lax–Milgram theorem which ensures the existence and uniqueness of an element $u_f \in H_0^1(\Omega)$, (therefore u_f is null on the boundary $\partial\Omega$), which verifies the relation

$$\forall v \in H_0^1(\Omega)\ :\ \int_\Omega \nabla u_f \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x.$$

We will come back now to the first formulation of the problem and after an integration by parts in the first integral, we obtain

$$\forall v \in H_0^1(\Omega) \ : \ \int_\Omega \left(-\Delta u_f - f\right) v \mathrm{d}x = 0 \tag{13.2.2}$$

by taking into account the fact that the relation (13.2.2) holds true for $\forall v \in H_0^1(\Omega)$, we deduce that $-\Delta u - f$ has the value 0 on Ω and the proof of the Theorem 13.2.3 is complete. ■

Observation 13.2.1 1^o. *By using the method presented above, we can solve the following problem of Dirichlet type*

$$-\Delta u = f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \ \textit{in}\ \Omega,$$

$$u = 0, \ \textit{on}\ \partial\Omega,$$

where f and f_i are functions from $L^2(\Omega)$, and the derivatives $\frac{\partial f_i}{\partial x_i}$ are computed in the sense of the distributions. Using the idea from the proof of the Theorem 13.2.3, we introduce the linear form L as follows

$$\begin{aligned} L(v) &= \int_\Omega f v \mathrm{d}x - \sum_{i=1}^{n} \int_\Omega \frac{\partial f_i}{\partial x_i} v \mathrm{d}x \\ &= \int_\Omega f v \mathrm{d}x + \int_\Omega f_i \frac{\partial v}{\partial x_i} \mathrm{d}x, \ \forall v \in H_0^1(\Omega). \end{aligned}$$

Then, the solution is obtained by applying the Lax–Milgram theorem.

2^o. *Consider the nonhomogeneous Dirichlet problem*

$$\begin{aligned} -\Delta u &= f, \ \textit{in}\ \Omega, \\ u &= u_0, \ \textit{on}\ \partial\Omega, \end{aligned}$$

where u_0 is the restriction to the boundary $\partial\Omega$ of a function $u_0^* \in H^1(\Omega)$. We define the function v by $v = u - u_0^*$. We can see that this function is a solution of the problem

$$-\Delta v = f - \Delta u_0^* = f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \ \textit{in}\ \Omega,$$

$$v = 0, \ \textit{on}\ \partial\Omega$$

which is of the same type as the one presented at the first point of the observation. Here f_i are elements from $L^2(\Omega)$ because

$$f_i = \frac{\partial u_0^*}{\partial x_i}.$$

3°. If the set Ω is not bounded, by using the Lax–Milgram theorem we can solve the following nonhomogeneous Dirichlet problem

$$\begin{aligned} -\Delta u + u &= f, \ in\ \Omega, \\ u &= 0, \ pe\ \partial\Omega. \end{aligned}$$

To this end, we introduce the bilinear form $a(u, v)$ defined by

$$a(u, v) = \int_\Omega \nabla u \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x,$$

and the linear form $L(v)$ defined by

$$L(v) = \int_\Omega f v \mathrm{d}x + \sum_{i=1}^n \int_\Omega f_i \frac{\partial v}{\partial x_i} \mathrm{d}x, \ \forall v \in H_0^1(\Omega).$$

Furthermore, we can use the Lax–Milgram theorem because

$$a(u, u) = \|u\|^2_{H^1(\Omega)} = \|u\|^2_{H_0^1(\Omega)}.$$

In the following, we will address the homogeneous Neumann problem. For the case in which $f \in L^2(\Omega)$, we consider the problem

$$\begin{aligned} -\Delta u + u &= f, \ \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \ \text{pe } \partial\Omega, \end{aligned}$$

in which $\frac{\partial u}{\partial \nu}$ represents the normal derivative (or, in other words, the derivative in the direction of the normal) of the function u, computed on the surface $\partial\Omega$. As usual, by the normal derivative we will understand the scalar product between the gradient of the function u and the unit normal $\overrightarrow{v}$ oriented outside the boundary $\partial\Omega$.

As a first result, in the following theorem we give a variational formulation of the homogeneous Neumann problem.

Theorem 13.2.4 *For any function $f \in L^2(\Omega)$, there is an element u_f in the Sobolev space $H^1(\Omega)$ and it is unique such that it is a weak solution of the homogeneous Neumann problem, that is, u_f satisfies the relation*

$$\int_\Omega \nabla u_f \nabla v \mathrm{d}x + \int_\Omega u_f v \mathrm{d}x = \int_\Omega f v \mathrm{d}x, \ \forall v \in H^1(\Omega),$$

which is called the variational formulation of the homogeneous Neumann problem.

Proof We write formally the fact that u verifies the equation $-\Delta u + u = f$ then we multiply both members with the function v which is an arbitrary function from $H^1(\Omega)$. We integrate the resulting equality on Ω and, after application of the formula of Green, we obtain

$$\int_\Omega \nabla u \nabla v \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\sigma(x) + \int_\Omega uv\mathrm{d}x = \int_\Omega fv\mathrm{d}x, \ \forall v \in H^1(\Omega).$$

This relation is equivalent to the following one

$$\int_\Omega \nabla u \nabla v \mathrm{d}x + \int_\Omega uv\mathrm{d}x = \int_\Omega fv\mathrm{d}x, \ \forall v \in H^1(\Omega).$$

We now introduce the bilinear form $a(u, v)$ by

$$a(u, v) = \int_\Omega \nabla u. \nabla v \mathrm{d}x + \int_\Omega uv\mathrm{d}x. \tag{13.2.3}$$

We apply twice the Cauchy–Buniakovsky–Schwartz inequality and we obtain

$$|a(u, v)| \le \| \nabla u\|_{L^2(\Omega)} \| \nabla v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \\ \le 2\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

This estimate together with the observation that

$$\|u\|_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)},$$

leads to the conclusion that the bilinear form $a(u, v)$ is continuous.

On the other hand, from (13.2.3) we deduce that

$$a(u, v) = \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega u^2 \mathrm{d}x = \|u\|^2_{H^1(\Omega)},$$

and this allows us to arrive at the conclusion that the bilinear form $a(u, v)$ is coercive.

Using the algorithm from the Lax–Milgram theorem, we introduce the linear form $L(v)$ defined by

$$L(v) = \int_\Omega fv\mathrm{d}x. \tag{13.2.4}$$

By applying in (13.2.4) the Cauchy–Buniakowski–Schwarz inequality, we find the estimate

$$|L(v)| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}.$$

We can take into account this inequality and the observation that

$$\|v\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)},$$

so that we deduce that the linear form $L(v)$ is continuous.

We now can apply the Lax–Milgram theorem and then we deduce that there is a unique function u_f from the Sobolev space $H^1(\Omega)$ so that

$$\int_\Omega \nabla u_f \nabla v + \int_\Omega u_f v \mathrm{d}x = \int_\Omega f v \mathrm{d}x, \ \forall v \in H^1(\Omega).$$

We will come back to the initial equation and we multiply both members with the function v that is arbitrary in the Sobolev space $H_0^1(\Omega)$. The obtained relation will be integrated, member by member, on the set Ω. After that we apply Green's formula, and we obtain the relation

$$\int_\Omega (-\Delta u_f + u_f - f) v \mathrm{d}x = 0.$$

Based on the fact that the function v is arbitrary, from the equality above we deduce that

$$-\Delta u_f + u_f = f \text{ in } \Omega.$$

Now, we multiply again the initial equation in both members by the function v which is arbitrary, this time in the Sobolev space $H^1(\Omega)$. The obtained relation will be integrated and in agreement with the application of the formula of Green, based on the previous result, we deduce that

$$\int_\Omega \frac{\partial u_f}{\partial \nu} v \mathrm{d}x = 0.$$

We will use again the fact that the function v is arbitrary, so that this relation leads to the conclusion that

$$\frac{\partial u_f}{\partial \nu} = 0, \text{ pe } \partial\Omega,$$

and the proof is completed. ■

We now want to give a justification for the formal calculations made in the proofs of Theorems 13.2.3 and 13.2.4.

Theorem 13.2.5 *(i) Consider that the conditions from the Lax–Milgram theorem are satisfied. In addition, suppose that the bilinear form $a(u, v)$ is symmetric, that is, it verifies the equality*

$$a(x, y) = a(y, x), \ \forall x, y \in H.$$

Then the solution x_L of the variational formulation

$$a\,(x_L, y) = L(y), \ \forall y \in H,$$

is a solution of the problem of minimum

$$\min_{x \in H} \left(a(x, x) - 2L(x)\right),$$

and conversely, that is, this problem of minimum admits a solution which verifies the relation from the variational formulation above.

(ii) Suppose that the hypotheses from the theorem of Stampacchia are satisfied. In addition, assume that the bilinear form $a(u, v)$ is symmetric. Then the solution $x_{L,K}$ of the variational formulation

$$a\left(x_L, y - x_{L,K}\right) \geq L\left(y - x_{L,K}\right), \ \forall y \in K,$$

is a solution of the problem of minimum

$$\min_{k \in K} \left(a(x, x) - 2L(x)\right),$$

and conversely, that is, the problem of minimum admits a solution, which verifies the relation

$$a(x_L, y - x_{L,K}) = L(y - x_{L,K}), \ \forall y \in K.$$

Proof (i) By direct calculation, we obtain these successive relations

$$\begin{aligned} a(y, y) - 2L(y) &= a\,(x_L + y - x_L, x_L + y - x_L) \\ &- 2L(x_L + y - x_L) = a\,(x_L, x_L) + 2a\,(x_L, y - x_L) \\ &+ a\,(y - x_L, y - x_L) - 2L(x_L) - 2L(y - x_L) \\ &= a\,(x_L, x_L) - 2L(x_L) + 2a(x_L, y - x_L) \\ &- 2L(y - x_L) + a\,(y - x_L, y - x_L) \\ &= a\,(x_L, x_L) - 2L(x_L) + a\,(y - x_L, y - x_L)\,, \end{aligned} \tag{13.2.5}$$

in which we take into account that

$$a\,(x_L, y - x_L) - L(y - x_L) = 0.$$

If we consider the obvious inequality

$$a\,(y - x_L, y - x_L) \geq 0,$$

then from (13.2.5) we are led to the inequality

$$a(y, y) - 2L(y) \geq a\left(x_L, x_L\right) - 2L(x_L),$$

and this proves that x_L is a solution of the problem of minimum formulated in the statement of the theorem.
We now prove the reciprocal result. Suppose that the mentioned problem of minimum admits the solution x^*. Then for any $t \in (0, 1]$ and for any $x \in H$, we have

$$a\left(x^* + tx, x^* + tx\right) - 2L\left(x^* + tx\right) \geq a\left(x^*, x^*\right) - 2L(x^*),$$

and this inequality is equivalent to the inequality

$$\begin{aligned} &a(x^*, x^*) + 2ta(x^*, x) + t^2 a(x, x) \\ &\quad - 2L(x^*) - 2tL(x) \geq a(x^*, x^*) - 2L(x^*). \end{aligned}$$

We reduce the similar terms and then we divide by the strictly positive number t and obtain

$$2a(x^*, x) + ta(x, x) - 2L(x) \geq 0.$$

In this relation we pass to the limit with $t \to 0^+$ so that we obtain

$$2a(x^*, x) - 2L(x) \geq 0 \Leftrightarrow a(x^*, x) - L(x) \geq 0.$$

We can obtain a relation which is analogous with this last relation if we transform x in $-x$ such that we can deduce that

$$a(x^*, x) = L(x), \ \forall x \in H,$$

that is, x^*, which is a solution of the problem of minimum, is a solution of the variational formulation, and this ends the proof of point (i).

(ii) This point can be proven analogously. ■

We can apply the results from Theorem 13.2.5 in the study of the homogeneous boundary value problems of Dirichlet type and also of Neumann type, formulated above. Thus, in the case of the homogeneous Dirichlet problem we define the bilinear form $a(u, v)$ on the space $H_0^1(\Omega) \times H_0^1(\Omega)$, by

$$a(u, v) = \int_\Omega \nabla u \nabla v \mathrm{d}x.$$

Because, obviously, the bilinear form $a(u, v)$ is symmetric according to Theorem 13.2.5, the solution of the homogeneous Dirichlet problem is equal to the solution of the problem of minimum

$$\min_{v\in H_0^1(\Omega)}\left(\int_\Omega |\nabla v|^2\mathrm{d}x - 2\int_\Omega f v\mathrm{d}x\right).$$

In the case of the homogeneous Neumann problem, we define the bilinear form $a(u, v)$ on the space $H^1(\Omega)\times H^1(\Omega)$, by

$$a(u, v) = \int_\Omega \nabla u \nabla v\mathrm{d}x + \int_\Omega uv\mathrm{d}x.$$

It is clear that this bilinear form is symmetric and then the solution of the homogeneous Neumann problem coincides with the solution of the problem of minimum

$$\min_{v\in H^1(\Omega)}\left(\int_\Omega |\nabla u|^2\mathrm{d}x + \int_\Omega v^2\mathrm{d}x - 2\int_\Omega f v\mathrm{d}x\right).$$

In the next theorem, we will give a new generalization of the Lax–Milgram theorem.

Theorem 13.2.6 *Let H be a separable Hilbert space and consider the functional $I : H \to \mathbb{R}\cup\{+\infty\}$ with the following three properties:*

1. *I is a proper functional (therefore I is not identically equal to $+\infty$);*
2. *I is lower semi-continuous with respect to the weak topology of the space H, that is,*

$$\forall x_k \rightharpoonup x,\ \textit{for } k\to+\infty,\ \Rightarrow \liminf_{k\to+\infty} I(x_k) \geq I(x);$$

3. *I is coercive, that is,*

$$\lim_{\|x\|\to+\infty}\frac{I(x)}{\|x\|} = +\infty.$$

Then the problem of minimum

$$\min_{x\in H} I(x)$$

has at least one solution.

Proof Let $\{x_k\}_{k\geq 1}$ be a sequence that minimizes the functional I, that is

$$\lim_{k\to+\infty} I(x_k) = \inf_{x\in H} I(x).$$

Based on the property of coercivity of the functional I we deduce that the sequence $\{x_k\}_{k\geq 1}$ is bounded on H and then admits a subsequence $\{x_{k'}\}_{k'}$ which is convergent in the weak topology of the space H to an element $x^* \in H$. From the fact that the functional I is lower semi-continuous with respect to the weak topology of H we deduce that

$$\inf_{x \in H} I(x) = \liminf_{k' \to +\infty} I\left(x_{k'}\right) \geq I\left(x^*\right).$$

This proves that the minimum of the functional I on the space H is x^*, and the proof of the theorem is completed. ■

Observation 13.2.2 1^o. *If I is a convex functional, that is,*

$$I((1-t)x + ty) \leq (1-t)I(x) + tI(y), \ \forall x, y \in H, \ \forall t \in [0, 1],$$

then the continuity of I with respect to the strong topology of the space H ensures that the functional I is lower semi-continuous with respect to the weak topology of the space H.

2^o. *If I is a strictly convex functional, that is,*

$$I((1-t)x + ty) < (1-t)I(x) + tI(y), \ \forall x, y \in H, \ \forall t \in (0, 1),$$

then the element for which the minimum of the functional I is attained on the space H is unique.

3^o. *There are several situations which require that the set of values for the functional I is $R \bigcup \infty$. We highlight here a situation. Consider the problem of minimum*

$$\min_{x \in K} \left(a(x, x) - 2L(x)\right),$$

where K is a closed and convex set from H. This problem is equivalent to the problem of minimum

$$\min_{x \in H} I(x),$$

where the functional $I : H \to \mathbb{R} \cup \{+\infty\}$ can be defined by

$$I(x) = \begin{cases} a(x, x) - 2L(x) \ , \ if \ x \in K \subset H, \\ \qquad +\infty \qquad \ , \quad otherwise. \end{cases}$$

Chapter 14
Regularity of the Solutions

14.1 Some Inequalities

We recall for beginners some very helpful inequalities in the following.

Theorem 14.1.1 *Let Ω an open set from the space $\mathbb{R}^n$. We have the following classical inequalities:*

1^o. The inequality of Cauchy–Buniakovski–Schwartz. If the functions f and g are from $L^2(\Omega)$, then the product fg is a function from $L^1(\Omega)$ and we have

$$\left|\int_\Omega f(x)g(x)\mathrm{d}x\right| \leq \left(\int_\Omega (f(x))^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega (g(x))^2\mathrm{d}x\right)^{1/2}.$$

2^o. The inequality of Hőlder. If the function f is from the space $L^P(\Omega)$ where $1 < p < \infty$, and the function g is from the space $L^q(\Omega)$ where q is so that $1 = \frac{1}{p} + \frac{1}{q}$, then the product fg is a function from $L^1(\Omega)$ and we have

$$\left|\int_\Omega f(x)g(x)\mathrm{d}x\right| \leq \left(\int_\Omega (f(x))^p\mathrm{d}x\right)^{1/p}\left(\int_\Omega (g(x))^q\mathrm{d}x\right)^{1/q}.$$

3^o. The inequality of Young. If the function $f \in L^p(\Omega) \cap L^q(\Omega)$, where p and q are so that $1 \leq p < q \leq \infty$, then $f \in L^r(\Omega)$, for any $r \in [p, q]$ and we have

$$\|f\|_{L^r(\Omega)} \leq \|f\|^{\alpha}_{L^p(\Omega)}\|f\|^{1-\alpha}_{L^q(\Omega)},$$

where α is chosen so that

$$\frac{\alpha}{p} + \frac{1-\alpha}{q} = \frac{1}{r}.$$

The proofs of these inequalities can be found in many books, especially those dedicated to functional analysis.

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_14

In the particular case $n = 1$, hence $\mathbb{R}^n = \mathbb{R}$, we have an interesting result (included in the following proposition) which asserts that any function from $H^1(a, b)$ is equal almost everywhere to a continuous function on $[a, b]$.

Proposition 14.1.1 *Let (a, b) be an open interval from $\mathbb{R}$. We have the continuous inclusion*

$$H^1(a, b) \subset C^0([a, b])$$

in the sense that for any function u from the Sobolev space $H^1(a, b)$ there exists a function $\tilde{u} \in C^0([a, b])$ so that $u(x) = \tilde{u}(x)$, for almost all $x \in [a, b]$ and there exists a constant c, independent of the function u, so that we have

$$\|\tilde{u}(x)\|_{C^0([a,b])} \leq c\|u\|_{H^1(a,b)}.$$

In addition, the function $\tilde{u}$ satisfies the relation

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(x)\mathrm{d}x,$$

where we denoted by u' the weak derivative of the function u.

The proof of this proposition can be found in any book of functional analysis. We return to the general case $n \geq 2$ and we take into account the case when the open set Ω is the whole space $\mathbb{R}^n$.

Theorem 14.1.2 (Gagliardo, Sobolev) *We assume $n > 2$. Then, we have the following continuous inclusion*

$$H^1 \subset L^s(\mathbb{R}^n),$$

where $s = 1/2 - 1/n$ and is called the critical exponent of Sobolev.

Proof We will give only a sketch of the proof. First, the following inequality is proven

$$\|u\|_{L^s(\mathbb{R}^n)} \leq c\|u\|_{H^1(\mathbb{R}^n)},$$

by considering the first case in which the function $u \in C_0^1(\mathbb{R}^n)$ (this case has been proven by Gagliardo). Furthermore, the result is extended to the space $H^1(\mathbb{R}^n)$ by using a density argument. ■

Corollary 14.1.1 *(i). If $n > 2$ we have the continuous inclusion*

$$H^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$$

for any $p \in [2, s]$, where we recall that $s = 1/2 - 1/n$.

(ii). If $n = 2$ *we have the continuous inclusion*

$$H^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^n),$$

for any $p \in [2, \infty)$.

Proof (i). Since the function u is from the Sobolev space $H^1(\mathbb{R}^n)$, we deduce that $u \in L^2(\Omega) \cap L^s(\Omega)$. The desired result is obtained by using the inequality of Young.

(ii). The result is easily obtained by using the inequality of Young. ■

We will make now some considerations in the case in which the open set Ω is the semi-space denoted by $\mathbb{R}^{n+}$ and defined by

$$\mathbb{R}^{n+} = \{(x_1, x_2, \ldots, x_n) : x_n > 0\}.$$

Theorem 14.1.3 *For any open set* Ω *with the boundary* $\partial\Omega$ *of class* C^1 *(included in the case* $\Omega = \mathbb{R}^{n+1},\ n \geq 2$*) we have the following continuous inclusions*

$$H^1(\Omega) \subset L^p(\Omega),\ \forall p \in [2, \infty),\ \textit{for } n = 2,$$
$$H^1(\Omega) \subset L^p(\Omega),\ \forall p \in [2, s],\ s = \frac{1}{2} - \frac{1}{n},\ \textit{for } n \geq 3.$$

Proof We will give details of the proof in the more simple case when the open set Ω coincides with the semi-space $\mathbb{R}^{n+}$ considered above.

We introduce the operator of extension

$$\begin{pmatrix} H^1(\mathbb{R}^{n+}) & \rightarrow & H^1(\mathbb{R}^n) \\ u & \longmapsto & \tilde{u} \end{pmatrix}$$

where the function $\tilde{u}$ is defined by

$$\tilde{u}(x) = \begin{cases} u(x_1, x_2, \ldots, x_n),\ \forall x \in \mathbb{R}^{n+}, \\ u(x_1, x_2, \ldots, -x_n),\ \forall x \in \mathbb{R}^{n-}. \end{cases}$$

We can show, without difficulty, that

$$\frac{\partial \tilde{u}}{\partial x_i} = \widetilde{\frac{\partial u}{\partial x_i}},\ \forall i \in 1, 2, \ldots, n-1,$$

and

$$\frac{\partial \tilde{u}}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_n}(x_1, x_2, \ldots, x_n),\ \forall x \in \mathbb{R}^{n+}, \\ -\frac{\partial u}{\partial x_n} u(x_1, x_2, \ldots, -x_n),\ \forall x \in \mathbb{R}^{n-}. \end{cases}$$

Keeping in mind these derivatives as well as the definition of the norm in the Sobolev space $H^1(\Omega)$, we obtain the inequality

$$\|\tilde{u}\|_{H^1(\mathbb{R}^n)} \leq \sqrt{2}\|u\|_{H^1(\mathbb{R}^{n+})}.$$

To establish this inequality, in the general case, the procedure of extension by density is used. We will proceed by direct calculations. First, we suppose that $n = 2$. The extension $\tilde{u}$ of the function u is a function from $H^1(\Omega)$ and then $\tilde{u} \in L^p(\mathbb{R}^n)$ for any $p \in [2, \infty)$. We deduce that $u \in L^p(\mathbb{R}^n)$ for any $p \in [2, \infty)$ and we have, with regard to the norms, the following estimates

$$\|u\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{2^{p/2}}\|\tilde{u}\|_{L^p(\mathbb{R}^n)} \leq C(p)\|\tilde{u}\|_{H^1(\mathbb{R}^n)} \leq$$
$$\leq \sqrt{2}C(p)\|u\|_{H^1(\mathbb{R}^{n+})},$$

in which we denoted by $C(p)$ a constant which depends only on p. Thus, for $n = 2$, the result is proven. In a similar manner the result for $n \geq 3$ is obtained. In the case in which Ω is an arbitrary open set from the space $\mathbb{R}^n$ with regular boundary of class C^1, we will consider that Ω is situated fully in the same part of a part of its boundary.

In this situation, by using a system of local maps we can reduce the considerations to the case in which Ω is the semi-space $\mathbb{R}^{n+}$, by using a partition of unity suggested by Brezis in his book [9]. ■

In the following theorem, we tackle the case of an open and bounded set, with regular boundary of class C^1.

Theorem 14.1.4 (Rellich–Kondrachov) *Let Ω be an open and bounded set from $\mathbb{R}^n$, $n \geq 1$ with regular boundary of class C^1. Then the following inclusions are compact injections*

$$H^1(a, b) \subset C^0([a, b]),\ \textit{for } n = 1;$$
$$H^1(\Omega) \subset L^p(\Omega), \forall p \in [1, \infty),\ \textit{for } n = 2;$$
$$H^1(\Omega) \subset L^p(\Omega), \forall p \in [1, s),\ s = \frac{1}{2} - \frac{1}{n},\ \textit{for } n \geq 3.$$

This can be reformulated, equivalently, as follows: any sequence which is weakly convergent in $H^1(a, b)$ and in $H^1(\Omega)$, respectively, is convergent relative to the strong topology from the space $C^0([a, b])$ and $L^p(\Omega)$, respectively.

Proof In the book [9] Brezis gave a proof based on the theorem of Ascoli. Here, we will approach this proof in a different manner. We take into account an operator of extension from $H^1(\Omega)$ to $H^1(\mathbb{R}^n)$. Then, we choose a function Ψ in $C_0^\infty(\mathbb{R}^n)$ so that $\Psi \equiv 1$ on Ω and we take into account the open and bounded set Ω^1 which has a regular boundary and contains the support of the function Ψ. Consequently, we can then take the linear and continuous operator L, $L : H^1(\Omega) \to H_0^1(\Omega^1)$, obtained by the composition of the operator of extension above with the application: $u \mapsto \Psi u$.

For the moment, we take into account that the function v is from the space $H^1(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} (1 + |y|^2)|\mathcal{F}v(y)|^2 \mathrm{d}y < \infty,$$

where $\mathcal{F}$ is the Fourier transform.

We consider a sequence $\{u_k\}_k$ which is weak convergent to 0 in the space $H^1(\Omega)$. Based on the Fourier–Plancherel theorem, we have the estimate

$$\|Lu_k\|_{L^2(\Omega')} = \int_{\mathbb{R}^n} |\mathcal{F}(Lu_k)(y)|^2 \,\mathrm{d}y$$
$$= \int_{|y|\le r} |\mathcal{F}(Lu_k)(y)|^2 \,\mathrm{d}y + \int_{|y|>r} |\mathcal{F}(Lu_k)(y)|^2 \,\mathrm{d}y. \tag{14.1.1}$$

The last integral can be written in the following form

$$\int_{|y|>r} |\mathcal{F}(Lu_k)(y)|^2 \mathrm{d}y =$$
$$= \int_{|y|>r} (1 + |y|^2)^{-1}(1 + |y|^2)|\mathcal{F}(Lu_k)(y)|^2 \mathrm{d}y,$$

from where we deduce that

$$\int_{|y|>r} |\mathcal{F}(Lu_k)(y)|^2 \mathrm{d}y$$
$$\le (1 + r^2)^{-1} \int_{|y|>r} (1 + |y|^2)|\mathcal{F}(Lu_k)(y)|^2 \mathrm{d}y.$$

This means that for any $r \to \infty$ we obtain

$$\int_{|y|>r} |\mathcal{F}(Lu_k)(y)|^2 \mathrm{d}y \to 0.$$

We make now an evaluation of the first integral from the right-hand side of the relation (14.1.1), namely

$$\int_{|y|\le r} |\mathcal{F}(Lu_k)|^2 \,\mathrm{d}y = \int_{|y|\le r} \left| \int_{\Omega^1} Lu_k e^{-2i\pi x.y} \mathrm{d}x \right|^2 \mathrm{d}y. \tag{14.1.2}$$

The sequence $\{Lu_k\}_k$ is convergent to 0 in the weak topology of the space $H_0^1(\Omega^1)$ and hence the sequence is convergent almost everywhere.

In the last integral from (14.1.2), we can consider a function Ψ which has compact support and which is identically equal to 1 on Ω^1, without changing the result. The conclusion from the theorem is obtained if we use the theorem of convergence due to Lebesgue. ■

At the end of the paragraph, we will characterize the functions from $H^1(\mathbb{R}^n)$ with the help of the Fourier transform.

Proposition 14.1.2 *The function u is from the space $H^1(\mathbb{R}^n)$ if and only if the function $\left(1+|.|^2\right)^{1/2}\mathcal{F}_u$ is from the space $L^2(\mathbb{R}^n)$.*

Proof Let u be a function from the space $H^1(\mathbb{R}^n)$. Hence, we can consider that $u \in L^2(\mathbb{R}^n)$. Then u defines a tempered distribution, which will be denoted also by u, so that $\mathcal{F}u \in L^2(\mathbb{R}^n)$. Because the weak derivative $\frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^n)$, for any index i, we can apply the results from Proposition 8.5.2 Chap. 8, which prove that the application $y \mapsto y_i \mathcal{F}u(y)$ is from $L^2(\mathbb{R}^n)$ for any index i and the proof of the first implication is concluded.

To prove the reciprocal result, we use again the results from Proposition 8.5.2 Chap. 8. Based on these results, and since $\mathcal{F}_u \in L^2(\mathbb{R}^n)$ we have that $u \in L^2(\mathbb{R}^n)$. Also, because the application $y \mapsto y_i \mathcal{F}_u(y)$ is from $L^2(\mathbb{R}^n)$, for any index i, we obtain that the weak derivative $\frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^n)$, for any index i. Thus, we deduce that the function u is from the space $H^1(\mathbb{R}^n)$ and the proof of the proposition is completed. ■

Let Ω be a bounded domain from $\mathbb{R}^n$ that satisfies the following condition, called *the hypothesis of the cone*, which we recall now.

Let $C_0(\Gamma_s, R)$, $s = 1, 2, \ldots, q$, *be cones with the peak in the origin* 0, *in which* Γ_s *are open sets relative to the sphere* $|x| = 1$. *There exists an open coverage of the set* $\overline{\Omega}$, with the open sets $I_1, I_2, \ldots, I_q$, so that for any $x \in \overline{\Omega} \cap I_s$, the cone $C_x(\Gamma_s, R)$ is contained in Ω.

Theorem 14.1.5 *If the bounded domain Ω satisfies the hypothesis of the cone, then the following inequalities hold true:*

- *the first inequality of Korn*

$$\int_\Omega \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \mathrm{d}x \geq c_1 \|u\|_1^2, \ \forall u \in H_0^1(\Omega); \tag{14.1.3}$$

- *the second inequality of Korn*

$$\int_\Omega \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \mathrm{d}x + \int_\Omega |u|^2 \mathrm{d}x \geq c_2 \|u\|_1^2, \ \forall u \in H^1(\Omega). \tag{14.1.4}$$

Proof The first inequality of Korn is obtained relatively easy, by applying the Fourier transform to the functions u_i and then by using the theorem of Parseval.

The second inequality of Korn has a demonstration which is much more complicated. We will give only a sketch of the proof, inspired by the book [10] of Fichera, where we can find the complete proof.

Consider the set $\{J\}$ of all open balls with the properties

- the center x of the ball J is in $\overline{\Omega}$;
- the radius of the ball J does not exceed $\frac{1}{2}R$;
- the ball J is contained in an open set I_s.

Based on the hypothesis of the cone, we can extract an open covering of the set $\overline{\Omega}$, consisting of balls $J_1, J_2, \ldots, J_m$. Consider a partition of unity

$$\sum_{k=1}^{m} \varphi_k^2(x) = 1,\ \varphi_k \in C^\infty,\ supp\ \varphi_k \subset J_k,\ k = 1, 2, \ldots, m,$$

and then we have

$$\begin{aligned}\int_\Omega u_{i,j} u_{i,j} \mathrm{d}x &= \int_\Omega (\varphi_k u_i)_{,j} (\varphi_k u_i)_{,j} \mathrm{d}x \\ &= \int_\Omega \varphi_{k,j} u_i \varphi_{k,j} u_i \mathrm{d}x - 2\int_\Omega \varphi_k, j\varphi_{k,j} u_i u_{i,j} \mathrm{d}x.\end{aligned}$$

Then it is possible to use the fact that, based on the hypothesis of the cone, we have

$$H^1(\Omega) = \overline{C^\infty(\overline{\Omega})},$$

the closure of $C^\infty(\overline{\Omega})$ is taken in the sense of the topology of the Sobolev space $H^1(\Omega)$. ■

We have to mention that the original form of the second inequality of Korn is

$$\int_\Omega u_{i,j} u_{i,j} \mathrm{d}x \le \int_\Omega \left(u_{i,j} + u_{j,i}\right)\left(u_{i,j} + u_{j,i}\right) \mathrm{d}x, \tag{14.1.5}$$

for any $u \in H^1(\Omega)$, so that

$$\int_\Omega \left(u_{i,j} - u_{j,i}\right) \mathrm{d}x = 0. \tag{14.1.6}$$

It is not very difficult to prove the equivalence between the form (14.1.4) of the second inequality of Korn and the form (14.1.5), with the condition (14.1.6).

14.2 Extensions of Sobolev Spaces

By convention, we call the Sobolev spaces $H^1(\Omega),\ H_0^1(\Omega)$, previously introduced as Sobolev spaces of the first order. In this paragraph we introduce other Sobolev spaces, namely some extensions of higher order.

Definition 14.2.1 Let Ω be an open arbitrary set from the space $\mathbb{R}^n$. For any integer number $m \geq 1$, we define the Sobolev space of order m, by

$$H^m(\Omega) = \left\{ u \in H^{m-1}(\Omega) : \frac{\partial u}{\partial x_i} \in H^{m-1}(\Omega), \forall i \in \{1, 2, \ldots, n\} \right\}.$$

For $m = 0$ we have $H^0(\Omega) = L^2(\Omega)$.

For any $p \geq 1$, we introduce the Sobolev space $W^{1,p}(\Omega)$ by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), \forall i \in \{1, 2, \ldots, n\} \right\}.$$

Observation 14.2.1 1^o. *We can show without difficulty that the Sobolev space* $W^{1,p}(\Omega)$ *is a reflexive Banach space for* $1 < p < \infty$.

2^o. *The Sobolev space* $H^m(\Omega)$ *is a Hilbert space with the norm*

$$\|u\|^2_{H^m(\Omega)} = \|u\|^2_{H^{m-1}(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{H^{m-1}(\Omega)}. \tag{14.2.1}$$

An analogous result to that of Proposition 14.1.2 is given in the following proposition.

Proposition 14.2.1 *A function* u *is from the Sobolev space* $H^m(\mathbb{R}^m)$ *if and only if the function* $(1 + |.|^2)^{m/2}\mathcal{F}_u$ *is from the space* $L^2(\mathbb{R}^n)$, *where* $\mathcal{F}$ *is the Fourier transform.*

Proof To prove this result we can use analogous ideas to those used in the proof of Proposition 14.1.2. ■

Based on Proposition 14.2.1 we deduce that a norm on the Sobolev space $H^m(\Omega)$, equivalent to the basic norm (14.2.1), is defined by

$$\|u\|_{H^m(\Omega)} = \left\| \left(1 + |.|^2\right)^{m/2} \mathcal{F}_u \right\|_{L^2(\Omega)}.$$

Consequently, we can introduce a new Sobolev space.

Definition 14.2.2 For any real and positive number s, we introduce the Sobolev fractional space $H^s(\mathbb{R}^n)$ by

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : |1 + |.|^2|^{s/2}\mathcal{F}_u \in L^2(\mathbb{R}^n) \right\}$$

equipped with the norm

$$\|u\|_{H^s(\Omega)} = \left\| \left(1 + |.|^2\right)^{s/2} \mathcal{F}_u \right\|_{L^2(\Omega)}. \tag{14.2.2}$$

As far as the fractional Sobolev space $H^s(\mathbb{R}^n)$ is concerned, it is natural to approach similar results to those proven in Propositions 14.1.2 and 14.2.1.

Proposition 14.2.2 *We assume that $0 < s < 1/2$. Then the function u is from the space $H^s(\mathbb{R}^n)$ if and only if u is from the space $L^2(\mathbb{R}^n)$ and verifies the following condition*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \mathrm{d}x\mathrm{d}y < \infty.$$

A norm on the fractional Sobolev space $H^s(\mathbb{R}^n)$, which is equivalent to the basic norm (14.2.2), is given by

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

Proof The proof is too technical, but it is given in detail by Lions and Magenes in [13]. ■

In the case in which the open set Ω from $\mathbb{R}^n$ is bounded and with the boundary which is sufficient regular, we introduce another Sobolev space $H^s(\Omega)$ defined by

$$H^s(\Omega) = \left\{ u \in H^{[s]}(\Omega) : \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2(s-[s])}} \mathrm{d}x\mathrm{d}y < \infty \right\}$$

$\forall \alpha, |\alpha| = [s]$, in which $[s]$ is the integer part of the real number s.

This space is endowed with the norm

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{H^{[s]}(\Omega)} + \sum_{|\alpha| \le [s]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2(s-[s])}} \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

An analogous result with that from Theorems 14.3.1 and 14.2.2 is given in the next theorem.

Theorem 14.2.1 *Let Ω be an open set from $\mathbb{R}^n$.*

1^o. If $2m < n$, then the inclusion $H^m \subset L^{m^*}(\Omega)$ is continuous, where $m^* = 2n/(n - 2m)$.

2^o. If $2m > n$, then the inclusion $H^m \subset C^0(\overline{\Omega})$ is continuous.

3^o. More generally, if $m - n/2 \notin Z$, Z is the set of integer numbers, then the inclusion $H^m \subset C^k(\overline{\Omega})$ is continuous, where $k = [m - n/2]$.

Proof The results are obtained by a very technical proof, which can be found in detail in the paper of Dautry and Lions [7]. ■

Now, we introduce the notion of *the trace* of a Sobolev space.

For beginners, will consider the case of the trace on the hyperplane

$$x \in \mathbb{R}^n : x = (x_1, x_2, \dots, x_n), x_n > 0 \subset \mathbb{R}^n.$$

Proposition 14.2.3 *We suppose that the open set Ω coincides with the semi-space*

$$\mathbb{R}^{n+} = \{x \in \mathbb{R}^n : x = (x_1, x_2, \ldots, x_n), x_n > 0\},$$

or Ω is an open and bounded set from $\mathbb{R}^n$ with regular boundary (of class C^1). Then the space $C_0^\infty(\mathbb{R}^n)$ is dense in $H^m(\Omega)$.

Proof (Sketch). In the case $\Omega = \mathbb{R}^{n+}$, we consider the canonical inclusion and we make its convolution with a regularizing sequence. In the case in which Ω is an open and bounded set, we use the procedure with the system of local maps (see also the proof of the Theorem 14.1.3).

Theorem 14.2.2 *The following statements hold true:*

(i). Let $\gamma_0 : C_0^\infty(\mathbb{R}^n)|_{\mathbb{R}^{n+}} \to C_0^\infty(\mathbb{R}^{n-1})$ be the application which associates to a function $u \in C_0^\infty(\mathbb{R}^n)$ the element $u(x', 0)$, where we used the notation $x' = (x_1, x_2, \ldots, x_{n-1})$. Then γ_0 is extended to a linear and continuous application from the space $H^1(\mathbb{R}^{n+})$ to the space $H^{1/2}(\mathbb{R}^{n-1})$.
(ii). The application γ_0 is surjective, and its kernel is $H_0^1(\mathbb{R}^{n+})$.

Proof We choose the function u from $C_0{}^\infty(\mathbb{R}^n)|_{\mathbb{R}^{n+}}$ and write

$$\begin{aligned}
\|u\|^2_{H^1(\mathbb{R}^{n+})} &= \int_{\mathbb{R}^{n+}} (1+|y|^2)|\mathcal{F}_u^2|(y)\mathrm{d}y \\
&= \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} (1+|y|^2)|\mathcal{F}_u|^2(y)\mathrm{d}y'\mathrm{d}y_n \\
&= \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} (1+|y'|^2)|\mathcal{F}_u|^2(y)\mathrm{d}y'\mathrm{d}y_n \\
&\quad + \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} |y_n|^2|\mathcal{F}_u|^2(y)\mathrm{d}y'\mathrm{d}y_n \\
&= \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} (1+|y'|^2)|\mathcal{F}_u|^2(y)\mathrm{d}y'\mathrm{d}y_n \\
&\quad + \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} \left|\frac{\partial \mathcal{F}_u}{\partial y_n}\right|^2 (y)\mathrm{d}y'\mathrm{d}y_n.
\end{aligned}$$

Let $v(y_n) = \mathcal{F}u(y', y_n)$. It is not difficult to verify that the function v is continuous and $v \in H^1(\mathbb{R}^+)$. In addition, we want to remark that

$$|v(0)|^2 = -2\int_0^{+\infty} v'(t)v(t)\mathrm{d}t \le 2\|v'\|_{L^2(\mathbb{R}^+)}\|v\|_{L^2(\mathbb{R}^+)}.$$

Then

$$|\mathcal{F}u|^2(y',0) \le 2\left(|\mathcal{F}_u|^2(y',y_n)\mathrm{d}y_n\right)^{1/2}\left(\int_0^{+\infty}\left|\frac{\partial \mathcal{F}u}{\partial y_n}\right|(y',y_n)\mathrm{d}y_n\right)^{1/2}$$

and this implies that

$$\sqrt{1+|y'|^2}|\mathcal{F}_u|^2(y',0) \le \left(1+|y'|^2\right)\int_0^{\infty} |\mathcal{F}_u|^2\,(y',y_n)\mathrm{d}y_n$$
$$+\int_0^{+\infty}\left|\frac{\partial \mathcal{F}_u}{\partial y_n}\right|^2 (y',y_n)\mathrm{d}y_n.$$

Then, we have the inequality

$$\int_{\mathbb{R}^{n-1}}(1+|y'|^2)|\mathcal{F}_u|^2(y',0)\mathrm{d}y' \le (1+|y'|^2)\int_0^{\infty}\int_{\mathbb{R}^{n-1}}|\mathcal{F}_u|^2(y',y_n)\mathrm{d}y'\mathrm{d}y_n$$
$$+\int_0^{\infty}\int_{\mathbb{R}^{n-1}}|\frac{\partial \mathcal{F}_u}{\partial y_n}|^2(y',y_n)\mathrm{d}y'\mathrm{d}y_n.$$

In conclusion, we have

$$\|\gamma_0 u\|^2_{H^1(\mathbb{R}^{n-1})} \le C\|u\|^2_{H^1(\mathbb{R}^{n+})},$$

so that by the application of the usual procedure of the density and by using Proposition 14.2.1, the first part of the proof is concluded.

(ii). To show that the application γ_0 is surjective, we must prove that for any $u \in H^{1/2}(\mathbb{R}^{n-1})$, there exists the function $\tilde{u} \in H^1(\mathbb{R}^{n+})$ so that $\gamma_0\tilde{u} = u$. We choose the function $\Psi \in C_0^{\infty}(R)/_{\mathbb{R}^+}$ so that $\Psi(0) = 1$ and we define the function $\tilde{u}$ as a solution of the equation

$$\mathcal{F}_u\left(y',y_n\right) = \mathcal{F}_u\Psi\left(\left(1+|y'|^2\right)^{1/2}y_n\right).$$

In this last equality we apply the reciprocal Fourier transform and we obtain

$$\tilde{u} = u * v,$$

where

$$\mathcal{F}v\left(y',y_n\right) = \Psi\left(\left(1+|y'|^2\right)^{1/2}y_n\right).$$

Now, we can verify that

$$\mathcal{F}(\gamma_0,\tilde{u}) = \mathcal{F}u,$$

from where we deduce that $\gamma_0\tilde{u} = u$, because $\tilde{u} \in H^1(\mathbb{R}^{n+})$. For the proof of the fact that the kernel of the application γ_0 is $H_0^1(\mathbb{R}^{n+})$ the reader can use the book [7]. ■

We should mention, at the end of this paragraph, that there exist some generalizations of the results from Theorem 14.2.2 to the space $H^m(\mathbb{R}^{n+1})$ and only to the space $H^m(\Omega)$, where Ω is an open set with regular boundary from $\mathbb{R}^n$.

14.3 Regularity of the Weak Solutions

In this paragraph, we will prove that in certain hypotheses of regularity imposed on the open set Ω from $\mathbb{R}^n$, which is assumed anyway to have the regular boundary, as well as in certain hypotheses imposed on the function the right-hand side f, the weak solution of the homogeneous Dirichlet problem

$$\begin{aligned} &-\Delta u(+u) = f, \text{ in } \Omega, \\ &u = 0, \text{ pe } \partial\Omega, \end{aligned} \tag{14.3.1}$$

becomes a classical solution of this problem. We recall that the function u is a weak solution of the problem (14.3.1) if the solution u of the variational formulation

$$\int_\Omega \nabla u \nabla v \mathrm{d}x \left(+ \int_\Omega uv\mathrm{d}x\right) = \int_\Omega fv\mathrm{d}x, \forall v \in H_0^1(\Omega), \tag{14.3.2}$$

is from the space $H_0^1(\Omega)$.

The term $+u$ which appears in the brackets in the equation from problem (14.3.1) or the term $+\int_\Omega uv\mathrm{d}x$ from the variational formulation (14.3.2) allow us to approach the problem in the formulation (14.3.1) and (14.3.2), respectively, in the case in which the open set Ω is not bounded.

The main result of the paragraph is included in the following theorem.

Theorem 14.3.1 *Let k be an integer number and Ω an open and bounded set from $\mathbb{R}^n$, with the boundary of class C^{k+2} or $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^{n_+}$. If the function f from the right-hand side is from the space $H^k(\Omega)$, then the solution u of the homogeneous Dirichlet problem belongs to the space $H^{k+2}(\Omega)$.*

Proof We will proceed recursively with regards to k. We suppose first that $k = 0$ and then $f \in L^2(\Omega)$, in which $\Omega = \mathbb{R}^n$. Let us consider the real and strictly positive number h and the vector $\vec{h}$ from $\mathbb{R}^n$ with the components $\vec{h} = (h, 0, 0, \ldots, 0)$.

Define the function $v_{\vec{h}} \in H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$, by

$$v_{\vec{h}}(x) = -\left(\tau_{-\vec{h}} - I\right)\left(\tau_{\vec{h}} - I\right)u(x),$$

where τ_h is the operator of translation of vector $\vec{h}$, thus defined by

$$\tau_{\vec{h}}u(x) = u\left(x + \vec{h}\right).$$

We can immediate verify that $v_{\vec{h}}$ can be written in the form

$$v_{\vec{h}}(x) = -2u(x) + u(x + \vec{h}) + u(x - \vec{h}).$$

In the variational formulation (14.3.2), for which u is a solution, we take as test function $v = v_{\vec{h}}$ and we obtain

$$\int_{\mathbb{R}^n} \nabla u(x) \nabla u(x+\vec{h})\mathrm{d}x + \int_{\mathbb{R}^n} \nabla u(x) \nabla u(x-h)\mathrm{d}x$$
$$-2\int_{\mathbb{R}^n} |\nabla u(x)|^2\mathrm{d}x + \int_{\mathbb{R}^n} u(x)u(x+h)\mathrm{d}x + \int_{\mathbb{R}^n} u(x)u(x-h)\mathrm{d}x$$
$$-2\int_{\mathbb{R}^n} u^2(x)\mathrm{d}x = \int_{\Omega} f v_{\vec{h}}\mathrm{d}x,$$

and after application of the inequality of Cauchy–Buniakowski–Schwartz, we deduce that

$$\int_{\mathbb{R}^n} |\nabla(\tau_{\vec{h}} - I)u|^2(x)\mathrm{d}x + \int_{\mathbb{R}^n} |(\tau_{\vec{h}} - I)u|^2(x)\mathrm{d}x$$
$$= \int_{\Omega} f v_{\vec{h}}\mathrm{d}x \leq \|f\|_{L^2(\mathbb{R}^n)}\|v_{\vec{h}}\|_{L^2(\mathbb{R}^n)}. \tag{14.3.3}$$

Let us assume, for the moment, that the inequality

$$\left\|(\tau_{\vec{h}} - I)v\right\|_{L^2(\mathbb{R}^n)} \leq h\|v\|_{H^1(\mathbb{R}^n)}, \tag{14.3.4}$$

is true for any function $v \in H^1(\mathbb{R}^n)$. Then from the inequality (14.3.3), in which we recall that

$$v_{\vec{h}}(x) = -(\tau_{-\vec{h}} - I)(\tau_{\vec{h}} - I)u(x),$$

we are led to the inequality

$$\int_{\mathbb{R}^n} |\nabla(\tau_{\vec{h}} - I)u|^2(x)\mathrm{d}x + \int_{\mathbb{R}^n} |(\tau_{\vec{h}} - I)u|^2(x)\mathrm{d}x$$
$$\leq h\|f\|_{L^2(\mathbb{R}^n)}\|(\tau_{\vec{h}} - I)u\|_{H^1(\mathbb{R}^n)}.$$

This inequality proves that the sequence

$$\left\{\frac{1}{h}\left(\tau_{\vec{h}} - I\right)u\right\}_h$$

is bounded in the space $H^1(\mathbb{R}^n)$ and, consequently, admits a subsequence which is convergent to an element $u_0 \in H^1(\mathbb{R}^n)$, which will be determined in the following.

If we suppose that the function $\varphi \in C_0^1(\mathbb{R}^n)$, we have

$$\frac{1}{h}\int_{\mathbb{R}^n} (\tau_{\vec{h}} - I)u(x)\varphi(x)\mathrm{d}x = \frac{1}{h}\int_{\mathbb{R}^n} u(x)(\tau_{-\vec{h}} - I)\varphi(x)\mathrm{d}x \to$$

$$\to -\int_{\mathbb{R}^n} u(x)\frac{\partial \varphi}{\partial x_1}(x)\mathrm{d}x, \text{ for } h \to 0^+.$$

Therefore, we have the equality

$$\int_{\mathbb{R}^n} u_0(x)\varphi(x)\mathrm{d}x = -\int_{\mathbb{R}^n} u(x)\frac{\partial \varphi}{\partial x_1}(x)\mathrm{d}x,$$

which proves that $u_0 = \frac{\partial u}{\partial x_1}$.

Because $u_0 \in H^1(\mathbb{R}^n)$, we obtain that $\frac{\partial u}{\partial x_1} \in H^1(\mathbb{R}^n)$. As far as other partial derivatives are concerned, the calculations are analogous. Hence, we proved that $u \in H^2(\mathbb{R}^n)$ and the statement of the theorem is verified for $k = 0$.

Let us suppose now that $k \geq 1$, and $f \in H^k(\mathbb{R}^n)$. In particular, we deduce that $f \in H^1(\mathbb{R}^n)$ and then $u \in H^2(\mathbb{R}^n)$.

Consider the function $\varphi \in C_0^2(\mathbb{R}^n)$. We take in the variational formulation as test function $v = \varphi_i = \frac{\partial \varphi}{\partial x_i}$. We have the possibility of this choice by taking into account that $\frac{\partial \varphi}{\partial x_i} \in C_0^1(\mathbb{R}^n)$. Then we are led to the equality

$$\int_{\mathbb{R}^n} \nabla u \nabla \varphi_i \mathrm{d}u + \int_{\mathbb{R}^n} u\varphi_i \mathrm{d}x = \int_{\mathbb{R}^n} f\varphi_i \mathrm{d}x,$$

which is equivalent with

$$\int_{\mathbb{R}^n} \nabla(\frac{\partial u}{\partial x_i}) \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i}\varphi \mathrm{d}x = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}\varphi \mathrm{d}x, \tag{14.3.5}$$

and all calculations being justified in the sense of the distributions.

From (14.3.5) we deduce that the function $\frac{\partial u}{\partial x_i}$ is the solution from the space $H^1(\mathbb{R}^n)$ of a variational formulation associated with a homogeneous Dirichlet problem, because we have that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$, and the right-hand side of the problem is $\frac{\partial f}{\partial x_i}$. Based on recurrence, in the case in which the open set Ω is the whole space $\mathbb{R}^n$, the proof is concluded, except for the demonstration of the inequality (14.3.4).

In the case in which the open set Ω is just the semi-space $\mathbb{R}^{n+}$, we can use the reasoning from the first part of the proof, with the vector $\vec{h}$ in which the real constant h is placed in one of the first $n - 1$ components.

So, we deduce that the partial derivatives of the form $\frac{\partial^2 u}{\partial x_\alpha \partial x_i}$ are from the space $L^2(\mathbb{R}^{n_+})$, for $\alpha \in \{1, 2, \ldots, n-1\}$ and $i \in \{1, 2, \ldots, n\}$.

It is sufficient to show that the equation

$$-\Delta u + u = f,$$

leads to

$$\frac{\partial^2 u}{\partial x_n^2} = -\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} + u - f,$$

and from this we deduce that $u \in H^2(\mathbb{R}^{n+})$. We do not write the proof of the case in which Ω is an arbitrary open set. We recall however that, essentially, in the proof we use the argument of the system of local maps. Details can be found in the book of Brezis [9]. It remains to prove the inequality (14.3.4), namely for any function $v \in H^1(\mathbb{R}^n)$ and any vector $\vec{h}$ from $\mathbb{R}^n$, we have

$$\left\| \left(\tau_{\vec{h}} - I \right) v \right\|_{L^2(\mathbb{R}^n)} \leq |\vec{h}| \|v\|_{H^1(\mathbb{R}^n)}. \tag{14.3.6}$$

We take the function v, for beginners, from $C_0^1(\mathbb{R}^n)$. For any real number t we construct the function $g(t)$ by

$$g(t) = v(x + t\vec{h})$$

and then

$$g'(t) = \nabla v(x + t\vec{h}).\vec{h}.$$

By direct calculations we obtain the relation

$$v(x + t\vec{h}) - v(x) = \int_0^1 \nabla v(x + t\vec{h})\vec{h} \mathrm{d}t,$$

from which we deduce that

$$|v(x + t\vec{h}) - v(x)| \leq |\vec{h}| \int_0^1 |\nabla v(x + t\vec{h})| \mathrm{d}t,$$

and this implies that

$$|v(x + t\vec{h}) - v(x)| \leq |\vec{h}| \left(\int_0^1 |\nabla v(x + t\vec{h})|^2 \mathrm{d}t \right)^{1/2}.$$

We use the square in this inequality and after that we integrate on $\mathbb{R}^n$. We obtain the inequality

$$\int_{\mathbb{R}^n} |v(x + t\vec{h}) - v(x)|^2 \leq |\vec{h}|^2 \int_{\mathbb{R}^n} \int_0^1 |\nabla v(x + t\vec{h})|^2 \mathrm{d}t \mathrm{d}x.$$

Then we are led to

$$\int_{\mathbb{R}^n} |v(x + t\vec{h}) - v(x)|^2 \leq \int_0^1 \int_{\mathbb{R}^n} |\nabla v(x + t\vec{h})|^2 \mathrm{d}t \mathrm{d}x = \int_{\mathbb{R}^n} |\nabla v(x)|^2 \mathrm{d}x,$$

so that the inequality (14.3.6) is proven for $v \in C_0^1(\mathbb{R}^n)$.

The result for the case $v \in H^1(\mathbb{R}^n)$ is obtained, as usual, by density. The inequality (14.3.6) is proven, which concludes the proof of the theorem. ■

Corollary 14.3.1 *We suppose that the hypotheses of the Theorem 14.3.1 are satisfied. If $k > n/2$ then the weak solution of the homogeneous Dirichlet problem is equal, almost everywhere, to a twice continuously differentiable function in $\overline{\Omega}$.*

Proof Theorem 14.3.1 shows that if the function $f \in H^k(\Omega)$ then the function u, the weak solution of the homogeneous Dirichlet problem is from $H^{k+2}(\Omega)$. Based on the result of regularity from Theorem 14.2.1, point 3^o, we deduce that $u \in C^{[k+2-n/2]}(\overline{\Omega})$ which concludes the proof. ■

We can also show, without difficulty, that in the hypotheses of the Theorem 14.3.1 the function u is a classical solution of the homogeneous Dirichlet problem.

Observation 14.3.1 *The readers can derive themselves some regularity results, similar to those shown above, but in the case of the homogeneous Neumann problem.*

Chapter 15
Weak Solutions for Parabolic Equations

15.1 Formulation of Problems

In this chapter, we will deal with the study of the following problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = f(t,x), \forall (t,x) \in Q_T = (0,T) \times \Omega, \\
&u(0,x) = u_0(x), \ \forall x \in \Omega, \\
&u(t,x) = 0, \ \forall (t,x) \in (0,T) \times \partial\Omega,
\end{aligned} \tag{15.1.1}$$

where Ω is an open set from R^n whose boundary $\partial\Omega$, if it exists, is assumed to be a regular surface. Here, we denoted by T a strictly positive real number. The problem (15.1.1) is the problem of the heat propagation and is a prototype for parabolic differential equations of second order.

The function u, which is the solution of this problem, represents the temperature of a body which occupies the domain Ω, which is subject to a heat source whose density of volume is represented by the function f.

In the formulation above, we assumed that the initial distribution of the temperature in the body is u_0, and the temperature on the boundary of the body is maintained constant to zero.

In order to study the problem (15.1.1), we will begin with solving the following nonlinear system:

$$u'(t) + F(u(t)) = 0, \ \forall t > 0, \tag{15.1.2}$$

where F is a function $F : X \to X$, and X is a Banach space. We will attach to this system the initial condition

$$u(0) = u_0, \tag{15.1.3}$$

in which $u_0 \in X$ and X is the Banach space above.

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_15

Theorem 15.1.1 *We suppose that the function F is Lipschitz on the space X with Lipschitz constant L, that is,*

$$\forall x, y \in X : |F(x) - F(y)| \leq L|x - y|.$$

Then, for any $u_0 \in X$, there exists a unique solution u of the problem that consists of the system (15.1.2) and the initial condition (15.1.3) and this solution is a function from the space $C^1([0, +\infty); X)$.

Proof We start with proving the existence of the solution for this problem. We want to remark that if the solution u exists, then it verifies the following integral equation:

$$u(t) = u_0 - \int_0^t F(u(s))\mathrm{d}s, \ \forall t \geq 0. \tag{15.1.4}$$

It is not difficult to prove this statement.

Define the operator Φ by

$$\Phi(u)(t) = u_0 - \int_0^t F(u(s))\mathrm{d}s, \ \forall u \in X. \tag{15.1.5}$$

Then, the existence of a solution of the problem (15.1.2), (15.1.3) or, equivalently of Eq. (15.1.4), is reduced to the existence of a fixed point for the operator Φ. Let $k > 0$ be a real number. Define the vector space X_k by

$$X_k = \{u \in C^0([0,); X)) : \sup_{t \geq 0} e^{-kt} \|u(t)\| < +\infty\}.$$

Therefore, we can endow the space X_k with the norm,

$$\|u\|_{X_k} = \sup_{t \geq 0} e^{-kt} \|u(t)\|,$$

in which $\|u\|$ is the basic norm of the Banach space. Let $\{u_m\}_m$ be a Cauchy sequence of elements from the space X_k. We want to demonstrate that the sequence $\{u_n\}_n$ is convergent. For any $t \geq 0$, the sequence $\{u_m(t)\}_m$ is a sequence of elements from the space X, which is a Banach space. Then, there exists an element $u(t) \in X$ so that

$$\forall t \geq 0 : \lim_{m \to \infty} u_m(t) = u(t), \text{ in } X.$$

Since the sequence $\{u_m\}_m$ is a Cauchy sequence of elements from X_k, we can write

$$\forall \varepsilon > 0, \exists m_0, \text{ so that } \forall m \geq m_0, \ \forall t \geq 0 : e^{-kt} \|u_m(t) - u(t)\| \leq \varepsilon.$$

Hence, we deduce that

$$\forall \varepsilon > 0, \exists m_0, \text{ so that } \forall m \geq m_0, : \sup_{t\geq 0} e^{-kt} \|u_m(t) - u(t)\| \leq \varepsilon.$$

Consequently, the function $u(t)$ is continuous, $u : [0, \infty) \to X_k$ and we have

$$\forall \varepsilon > 0, \exists m_0, \text{ so that } \forall m \geq m_0, : \|u_m(t) - u(t)\|_{X_k} \leq \varepsilon,$$

which means that the sequence $\{u_m\}_m$ is convergent to u, in the space X_k. By taking into account (15.1.5), we obtain the following successive inequalities:

$$\begin{aligned} \|\Phi(u)(t)\| &\leq \|u_0\| + \int_0^t \|F(u(s))\| \mathrm{d}s \\ &\leq \|u_0\| + \int_0^t \|F(u(s)) - F(0) + F(0)\| \mathrm{d}s \\ &\leq \|u_0\| + L \int_0^t \|u(s)\| \mathrm{d}s + t\|F(0)\|, \end{aligned} \tag{15.1.6}$$

in which we take into account that F is a Lipschitz function.

On the other hand, because $u \in X_k$, we have

$$\forall t \geq 0 : \|u(t)\| \leq \|u\|_{X_k} - e^{kt},$$

and then, from (15.1.6) we deduce that

$$\|\Phi(u)(t)\| \leq \|u_0\| + \frac{L}{k}\|u\|_{X_k}(e^{kt} - 1) + t\|F(0)\|,$$

so that

$$e^{-kt}\|\Phi(u)(t)\| \leq \|u_0\| + \frac{L}{k}\|u\|_{X_k}(e^{-kt})t\|F(0)\|. \tag{15.1.7}$$

Based on the relation (15.1.7), we deduce that $\Phi(u) \in X_k$. Also, for any two elements $u, v \in X_k$, by taking into account (15.1.5), we find the estimate

$$\begin{aligned} \|\Phi(u)(t) - \Phi(v)(t)\| &\leq \int_0^t \|F(u(s)) - F(v(s))\| \mathrm{d}s \\ &\leq L \int_0^t \|u(s) - v(s)\| \mathrm{d}s, \end{aligned}$$

and this implies that

$$\|\Phi(u(t)) - \Phi(v)(t)\| \leq L\|u - v\|_{X_k} \int_0^t e^{ks} \mathrm{d}s,$$

and, finally,

$$\|\Phi(u(t)) - \Phi(v)(t)\| \leq \frac{L}{k}\|u - v\|_{X_k} e^{kt},$$

which proves that Φ is a Lipschitz function in X_k, with Lipschitz constant L/k. In particular, if we choose $k > L$, we obtain that Φ is a strictly contracting function on the Banach space X_k and hence admits a fixed point u which is unique and which is from the space $C^0([0, \infty); X)$. In addition, u verifies the following equation:

$$u(t) = u_0 - \int_0^t F(u(s))\mathrm{d}s, \ \forall t \geq 0. \tag{15.1.8}$$

Since the functions $u : [0, \infty) \to X$ and $F : X \to X$ are continuous, from (15.1.8) we deduce that u is differentiable on $(0, \infty)$ and its derivative is

$$u'(t) = -F(u(t)) \ \forall t \geq 0.$$

Then, we deduce immediately that u is a differentiable function on the right-hand side of point $t = 0$ and then we conclude that $u \in C^1([0, \infty); X)$. The proof of the existence is concluded. Let us demonstrate now the uniqueness of the solution of the problem given in Eqs. (15.1.2) and (15.1.3). We suppose by reduction to the absurd that the problem (15.1.2), (15.1.3) admits two solutions which will be denoted by u_1 and u_2. Define the function u_3 by $u_3 = u_1 - u_2$. It is easy to verify that

$$u_3(t) = -\int_0^t [F(u_1(s)) - F(u_2(s))]\mathrm{d}s, \ \forall t > 0 \ u_3(0) = 0,$$

from where we deduce that

$$\|u_3(t)\| \leq L\int_0^t \|u_3(s)\|\mathrm{d}s \ \forall t \geq 0.$$

Now, define the function Ψ by

$$\Psi(t) = \|u_3(t)\|, \ \forall t \geq 0. \tag{15.1.9}$$

It is easy to see that Ψ is a continuous function on the internal $[0, \infty)$ and its values are not negative.

Also, Ψ becomes null for $t = 0$ and it verifies the inequality

$$\Psi(t) \leq L\int_0^t \Psi(s)\mathrm{d}s \ \forall t \geq 0.$$

If we now define the function φ by

$$\varphi(t) = e^{-Lt} \int_0^t \Psi(s) ds, \ \forall t \geq 0, \tag{15.1.10}$$

then we find that $\varphi \in C^1([0, \infty); R)$ and it verifies the inequality

$$\varphi'(t) \leq 0, \ \forall t \geq 0.$$

In this way, we deduce that the function φ is decreasing on the interval $[0, \infty)$ and $\varphi(0) = 0$ which means that φ is identical null on $[0, \infty)$. Then from (15.1.10) it can be concluded that the function Ψ is identical null on the interval $[0, \infty)$ and finally, from (15.1.9), u_3 is identical null on $[0, \infty)$, that is, u_1 and u_2 coincide on this interval. This concludes the proof of the uniqueness of the solution and hence the proof of Theorem 15.1.1 is concluded. ■

We now approach the problem of the heat conduction, in a particular case. Namely, we take into account the rectangle $D = [0, T] \times [0, l]$ and consider the following mixed initial boundary value problem:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \ \forall (t, x) \in D, \\ &u(t, 0) = u(t, l) = 0, \ \forall t \in [0, t), \\ &u(0, x) = \varphi(x), \ \forall x \in [0, l], \end{aligned} \tag{15.1.11}$$

where the function φ is assumed to be continuously differentiable on the segment $[0, l]$. It is known that the function φ can be expanded in a Fourier series in the form

$$\varphi(x) = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi}{l} x, \tag{15.1.12}$$

in which the Fourier coefficients a_k have the expressions

$$a_k = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{k\pi}{l} x dx, \ k = 1, 2, \ldots$$

By using the well-known Bernoulli–Fourier method, applied in the case of Eq. $(15.1.11)_1$, we obtain the solutions

$$u_k(t, x) = e^{-\frac{\pi^2 k^2}{l^2} t} \sin \frac{k\pi}{l} x,$$

which satisfy the conditions

$$u_k(t, 0) = u_k(t, l) = 0, \ u_k(0, x) = \sin \frac{k\pi}{l} x.$$

It is not difficult to prove that the function $u(t, x)$, defined by

$$u(t, x) = \sum_{k=1}^{\infty} a_k e^{-\frac{\pi^2 k^2}{l^2} t} \sin \frac{k\pi}{l} x, \tag{15.1.13}$$

is a solution of the problem (15.1.11).

For $t > 0$, the uniform convergence and the absolute convergence of the series from (15.1.13), in the neighborhood of a point (t, x) from the rectangle D, can be obtained from the fact that

$$\lim_{k\to\infty} \left(\frac{k\pi}{l}\right)^m e^{-\frac{\pi^2 k^2}{l^2} t} = 0,\ m = 0, 1, \ldots$$

Also, for the same reason, we obtain the uniform convergence and the absolute convergence of the series obtained from the series (15.1.13), by twice differentiating with respect to the spatial variable x and with respect to t, respectively.

If the initial condition $(15.1.11)_3$ is given on a line segment $t = t_0$, and the boundary condition $(15.1.11)_2$ is given for $t_0 \leq t \leq T$, then the solution of the mixed initial boundary value problem, considered for the rectangle $0 < x < l,\ t_0 < t < T$, can be expressed, also, by the formula (15.1.13), in which t is replaced with $t - t_0$.

However, we must mention that the series from (15.1.13) may not make sense for $\forall t < t_0$.

The considerations above remain valid also in the more general case when we have more spatial variables. The only difference consists in the fact that the series from (15.1.12) and (15.1.13) is replaced by multiple series.

15.2 The Problem of Heat Conduction in $\mathbb{R}^n$

We take into account the problem

$$\frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) = 0,\ \forall (t, x) \in (0, \infty) \times \mathbb{R}^n,$$
$$u(0, x) = u_0(x),\ \forall x \in \mathbb{R}^n.$$

A first approach of this problem is to use the Fourier transform which, as is well known, has the form

$$\hat{u}(t, \xi) = \int_{\mathbb{R}^n} e^{-2i\pi x.\xi} u(t, x) \mathrm{d}x.$$

Then, formally, the problem of heat conduction becomes

$$\frac{\partial \hat{u}}{\partial t} + 4\pi^2\|\xi\|^2\hat{u}(t,\xi) = 0,\ \forall (t,\xi) \in (0,\infty)\times \mathbb{R}^n,$$
$$\hat{u}(0,\xi) = \hat{u_0}(\xi),\ \forall \xi \in \mathbb{R}^n,$$

whose solution is

$$\hat{u}(t,\xi) = \hat{u_0}(\xi)e^{-4\pi^2|\xi|t},\ \forall t > 0. \tag{15.2.1}$$

Starting from this formal solution of the problem of heat conduction, we introduce the kernel of the heat.

Definition 15.2.1 For $\forall t > 0$, we define the kernel of Gauss or the kernel of the heat by

$$K(t,x) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}.$$

We can prove, without difficulty, the following properties of the kernel of the heat, contained in the next proposition.

Proposition 15.2.1 *The following relations hold true:*

$$\hat{K}(t,\xi) = e^{-4\pi^2|\xi|^2 t},\ \forall t > 0,$$
$$\frac{\partial K}{\partial t}(t,x) = \frac{\partial \hat{K}(t,x)}{\partial t}, \forall t > 0,\ \forall x \in \mathbb{R}^n.$$

By taking into account the expression of the formal solution established in the form (15.2.1) and by taking into account the properties of the product of convolution and of the Fourier transform, we are led to the idea of writing the solution of the problem of the heat in the form

$$u(t,x) = (f * K(t,.))(x). \tag{15.2.2}$$

The following theorem offers an additional argument of necessity of representation of the solution in the form (15.2.2).

Theorem 15.2.1 *(i) We suppose that $u_0 \in L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. Then the function u defined by*

$$u(t,x) = (u_0 * K(t,.))(x), \tag{15.2.3}$$

satisfies the equation

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0,\ \forall (t,x) \in (0,\infty)\times \mathbb{R}^n.$$

(ii) *We assume that* $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then the function* u *defined in (15.2.3) is from the space* $C^0([0, \infty) \times \mathbb{R}^n)$ *and verifies the condition*

$$u(0, x) = u_0(x).$$

(iii) *We suppose that* $u_0 \in L^p(\mathbb{R}^n)$ *with* $p < \infty$. *Then the function* u *defined in (15.2.3) has the property*

$$\lim_{t \to 0} u(t, .) = u_0, \text{ in } L^p(\mathbb{R}^n).$$

Proof (Sketch) If $u_0 \in L^p(\mathbb{R}^n)$, where $1 \le p \le \infty$, then we can deduce the existence of the function $u(t, x)$ defined by

$$u(t, x) = (u_0 + K(t, .))(x), \ \forall t > 0, \ \forall x \in \mathbb{R}^n,$$

by using the rapid decrease of the kernel of Gauss K, with regard to x. It is easy to justify that we can differentiate under the integral sign and then from Theorem 15.2.1, point (ii) we deduce that u is a solution of the heat conduction equation.

After that, we have to verify the initial conditions in each of the cases (i), (ii), and (iii) to complete the proof of the theorem. ■

In the following theorem, we have a result of uniqueness for solution of the heat conduction problem.

Theorem 15.2.2 *We suppose that the solution* u *of the heat conduction problem previously considered is continuous on* $[0, \infty) \times \mathbb{R}^n$ *and* $u \in C^2((0, \infty) \times \mathbb{R}^n)$.

In addition, we suppose that the function u *verifies the conditions*
$\forall \varepsilon > 0, \exists C > 0$ *so that*

$$|u(t, x)| \le Ce^{\varepsilon |x|^2};$$
$$| \nabla u(t, x)| \le Ce^{\varepsilon |x|^2}.$$

Then u *is identical null on* $[0, \infty) \times \mathbb{R}^n$.

Proof Let x_0 be a fixed point in $\mathbb{R}^n$ and a real number $t_0 > 0$. Define the function g by

$$g(t, x) = K(t_0 - t, x - x_0).$$

By direct calculations, we can find that the function g verifies the equation

$$\frac{\partial g}{\partial t}(t, x) + \Delta g(t, x) = 0, \ \forall t < t_0, \ \forall x \in \mathbb{R}^n.$$

Therefore, we deduce that

$$0 = g(\frac{\partial u}{\partial t} - \partial u) + u(\frac{\partial g}{\partial t} + \partial g)$$
$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[u\frac{\partial g}{\partial x_i} - g\frac{\partial u}{\partial x_i}] + \frac{\partial}{\partial t}(ug) = div(F),$$

in which we denoted by F the following vector field:

$$F = (u\frac{\partial g}{\partial x_1} - g\frac{\partial u}{\partial x_1}, u\frac{\partial g}{\partial x_2} - g\frac{\partial u}{\partial x_2}, ..., u\frac{\partial g}{\partial x_n} - g\frac{\partial u}{\partial x_n}).$$

Let us consider the domain Q defined by

$$Q = \{(t, x) : 0 < a < t < b < t_0; |x| < r\}$$

and we apply the Ostrogradsky formula so that we are led to the calculations

$$0 = \int_{\partial Q} F\nu d\sigma = \int_{|x|\le r} u(b, x)K(t_0 - b, x - x_0)\mathrm{d}x$$
$$- \int_{|x|\le r} u(a, x)K(t_0 - a, x - x_0)\mathrm{d}x$$
$$+ \sum_{i=1}^{n}\int_a^b \int_{|x|\le r} u(t, x)\frac{\partial K}{\partial x_i}(t_0 - t, x - x_0)\frac{x_i}{r}d\sigma(x)\mathrm{d}t$$
$$- \sum_{i=1}^{n}\int_a^b \int_{|x|\le r} \frac{\partial u}{\partial x_i}(t, x)K(t_0 - t, x - x_0)\frac{x_i}{r}d\sigma(x)\mathrm{d}t.$$

If in the last relation, we pass to the limit with $r \to \infty$ and use the hypothesis of growth which was imposed in the statement on the function u, we obtain that the last two terms tend to zero. So, we obtain the relation

$$0 = (u(b, .) * K(t_0 - b, .))(x_0) = (u(a, .) * K(t_0 - a, .))(x_0).$$

If we pass to the limit with $b \to t_0$, then the first term tends to $u(t_0, x_0)$. If we pass to the limit with $a \to 0$, then the second term tends to zero.

In conclusion, $u(t_0, x_0) = 0$ and the proof is completed. ■

15.3 The Problem of Heat for an Open Set

In this paragraph, we will denote by Ω an arbitrary set from $\mathbb{R}^n$ which is open, bounded and with regular boundary.

The result from the following theorem is a principle of maximum for the classical solution of the heat conduction problem.

Theorem 15.3.1 *We suppose that the function $u \in C^0([0,T] \times \overline{\Omega})$ is continuously differentiable with respect to t on $(0,T) \times \Omega$ and twice continuously differentiable with respect to x on $[0,T) \times \Omega$. In addition, we suppose that u satisfies the equation*

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0, \ \forall (t,x) \in Q_T = (0,T) \times \Omega.$$

Then

$$\max_{(t,x)\in Q_T} u(t,x) = \max_{(t,x)\in P} u(t,x),$$

where $Q_T = (0,T) \times \Omega$, and $P = \{0\} \times \overline{\Omega} \cup [0,T] \times \partial\Omega$.

Proof First, we want to mention that P is called *the parabolic boundary* of the cylinder $Q_T = (0,T) \times \Omega$. Define the function v by

$$v(t,x) = u(t,x) + \varepsilon |x|^2, \ \varepsilon > 0.$$

It is clear that v is continuous on $\overline{Q_T}$ and verifies the equation

$$\forall (t,x) \in (0,T) \times \Omega : \frac{\partial v}{\partial t}(t,x) - \Delta v(t,x) = -2\varepsilon n < 0. \qquad (15.3.1)$$

We suppose that there exists a point $(x_0, t_0) \in \overline{Q_T} \setminus P$ in which v reaches its maximum. Then, it is easy to verify that

$$\Delta v(t_0, x_0) \leq 0,$$
$$\frac{\partial v}{\partial t}(t_0, x_0) = 0,$$
$$\frac{\partial v}{\partial t}(T, x_0) \geq 0,$$

for $t_0 < T$, from where we deduce that

$$\frac{\partial v}{\partial t}(t_0, x_0) - \Delta v(t_0, x_0) \geq 0,$$

which is in contradiction with (15.3.1). In conclusion, we have

$$\max_{\overline{Q_T}} v = \max_P v \leq \max_P u + \varepsilon \sup_{\overline{\Omega}} |x|^2,$$

and because $u \leq v$ we conclude that

$$\max_P u \le \max_{\overline{Q_T}} u \le \max_{\overline{Q_T}} v \le \max_P u + \varepsilon \sup_{\overline{\Omega}} |x|^2, \ \forall \varepsilon > 0.$$

The desired result is obtained by passing to the limit with $\varepsilon \to 0$. ■

In the following, we will approach the problem of heat conduction in the form

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) = 0, \ \forall (t, x) \in (0, T) \times \Omega, \\
&u(t, x) = 0, \ \forall (t, x) \in (0, T) \times \partial\Omega, \\
&u(0, x) = u_0(x), \ \forall x \in \Omega.
\end{aligned}$$

In compliance with the Lax–Milgram theorem (Theorem 15.2.1, Chap. 6), we have that if the function $f \in L^2(\Omega)$, then there exists only one function u_f, from the space $H_0^1(\Omega) \cap H^2(\Omega)$ so that

$$\begin{aligned}
&-\Delta u_f(x) + u_f(x) = 0, \ a.e. \ x \in \Omega, \\
&u_f = 0, \ \text{on } \partial\Omega.
\end{aligned}$$

By using the reasoning from the Lax–Milgram theorem, it is easy to prove the following proposition.

Proposition 15.3.1 *We suppose that the function $f \in L^2(\Omega)$ and we take into account a real number $\lambda > 0$. Then, there is only one weak solution $u_{\lambda,f} \in H_0^1(\Omega)$ of the problem*

$$\begin{aligned}
&-\lambda\Delta u_{\lambda,f}(x) + u_{\lambda,f}(x) = 0, \ a.p.t. \ x \in \Omega \\
&u_{\lambda,f} = 0, \ \text{on } \partial\Omega.
\end{aligned}$$

In addition, $u_{\lambda,f} \in L^2(\Omega)$.

Definition 15.3.1 The weak solution $u_{\lambda,f}$ defined above will be denoted by $J_\lambda(f)$ and it is called the resolvent of the Laplace's operator, computed for the function f.

The quantity

$$\frac{1}{\lambda}(f - J_\lambda(f)), \lambda > 0,$$

will be denoted by $\Delta_{\lambda,f}$ and it is called the Yosida approximation of the Laplacian computed for the function f.

Observation 15.3.1 *If we take into account the operator*

$$-\lambda\Delta + I = \lambda\left(\frac{1}{\lambda}I - \Delta\right),$$

as a linear operator defined (not necessarily everywhere) on the space $L^2(\Omega)$, then the resolvent J_λ of the Laplacian can be interpreted as the reverse of this operator.

Proposition 15.3.2 *The following statements hold true:*

(i) The resolvent of the Laplacian is a linear operator

$$J_\lambda : L^2(\Omega) \to D(\Delta),$$

in which $D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$ and $D(\Delta)$ is the domain of the Laplacian.

(ii) The Yosida approximation Δ_λ is a linear operator

$$\Delta_\lambda : L^2(\Omega) \to L^2(\Omega).$$

Proof The proof for both these results is immediate, by taking into account the definition of the two operators. ■

The main properties of the resolvent J_λ and of the Yosida approximation Δ_λ are given in following theorem.

Theorem 15.3.2 *The resolvent and approximate Yosida have the following main properties:*

1^o. *J_λ is defined everywhere on $L^2(\Omega)$, $\forall\lambda > a$;*
2^o. $\forall f \in L^2(\Omega),\ \forall\lambda > 0 : \Delta_\lambda(f) = -\Delta(J_\lambda(f))$;
3^o. $\forall f \in D(\Delta),\ \forall\lambda > 0 : \Delta_\lambda(f) = -(J_\lambda(\Delta f))$;
4^o. $\forall f \in D(\Delta),\ \forall\lambda > 0 : \|\Delta_\lambda(f)\|_{L^2(\Omega)} \le \|\Delta f\|_{L^2(\Omega)}$;
5^o. $\forall f \in L^2(\Omega) : \lim\limits_{\lambda\to 0} J_\lambda(\Delta f)$ *in the strong topology of $L^2(\Omega)$;*
6^o. $\forall f \in D(\Delta) : \lim\limits_{\lambda\to 0} \Delta_\lambda(f) = -\Delta f$;
7^o. $\forall f \in L^2(\Omega) : \langle\Delta_\lambda(f), f\rangle_{L^2(\Omega)} \ge 0$;
8^o. $\forall f \in L^2(\Omega),\ \forall\lambda > 0 : \|\Delta_\lambda(f)\|_{Ł^2(\Omega)} \le \frac{1}{\lambda}\|f\|_{Ł^2(\Omega)}$;
9^o.

$$\begin{aligned}\forall f,\ g \in L^2(\Omega) : \int_\Omega \Delta_\lambda(f)g\mathrm{d}x &= \int_\Omega \Delta_\lambda(g)f\mathrm{d}x\\ &= \lambda\int_\Omega \Delta_\lambda(f)\Delta_\lambda(g)\mathrm{d}x + \lambda\int_\Omega \nabla_\lambda(f)\nabla_\lambda(g)\mathrm{d}x.\end{aligned}$$

Proof 1^o. $J_\lambda(f)$ is from the space $H_0^1(\Omega) \cap H^2(\Omega)$ and verifies the following variational formulation:

$$\begin{aligned}\forall v \in H_0^1(\Omega) : \lambda\int_\Omega &\nabla(J_\lambda)(f)) \cdot \nabla v\mathrm{d}x\\ &+ \int_\Omega J_\lambda(f)v\mathrm{d}x = \int_\Omega f v\mathrm{d}x.\end{aligned}$$

In particular, if we take the test function v even $J_\lambda(f)$, this variational formulation becomes

$$\lambda \int_\Omega |\nabla (J_\lambda(f))|^2 \mathrm{d}x + \int_\Omega (J_\lambda(f))^2 \mathrm{d}x$$
$$= \int_\Omega f \cdot J_\lambda(f))\mathrm{d}x \leq \|f\|_{Ł^2(\Omega)} \|J_\lambda(f)\|_{Ł^2(\Omega)},$$

from where we deduce that

$$\int_\Omega (J_\lambda(f))^2 \mathrm{d}x \leq \|f\|_{Ł^2(\Omega)} \|J_\lambda(f)\|_{Ł^2(\Omega)}.$$

2^o. It is easy to verify that

$$-\Delta(J_\lambda(f)) = -\frac{1}{\lambda}(f(x)) - J_\lambda(f)(x)), \text{ almost everywhere on } \Omega,$$

so that

$$-\Delta(J_\lambda(f))(x) = -\Delta_\lambda(f)(x).$$

3^o. If f is an arbitrary element in the domain of the Laplacian, $D(\Delta)$, we have

$$(I - \lambda\Delta)J_\lambda(f) = f,$$

and then

$$-\Delta f = \frac{1}{\lambda}\{(I - \lambda\Delta)(f) - f\} = \frac{1}{\lambda}(I - \lambda\Delta)(f - J_\lambda(f)),$$

from where we deduce that

$$J_\lambda(-\Delta f) = \frac{1}{\lambda}(f - J_\lambda(f)) = \Delta_\lambda(f).$$

4^o. This point can be easily proven by using the points 3^o and 1^o.
5^o. First, we assume that f is from the domain of the Laplacian. Then, we have

$$\|f - J_\lambda(f)\|_L^2(\Omega) \leq \lambda \|\Delta_\lambda(f)\|_L^2(\Omega) \leq \lambda \|\Delta(f)\|_{L^2(\Omega)},$$

from where we deduce that

$$\lim_{\lambda \to 0} \|f - J_\lambda(f)\| = 0.$$

Now, we assume that $f \in L^2(\Omega)$. Then, there exists a sequence $\{f_k\}_k$ of the functions from $C_0^\infty(\Omega)$ which is convergent to f_0 in $L^2(\Omega)$.

But $C_0^\infty(\Omega) \subset D(\Delta)$. Moreover, $C_0^\infty(\Omega)$ is even dense in $D(\Delta)$. Then, for each k, the sequence $\{J_\lambda(f_k)\}_k$ is convergent to f_k in the topology of $L^2(\Omega)$. On the other hand, because J_λ is a contraction, we have

$$\begin{aligned}\|f - J_\lambda(f)\|_L^2(\Omega) &\leq \|f - f_k\|_L^2(\Omega)\\ &+ \|f_k - J_\lambda(f_k)\|_L^2(\Omega) + \|J_\lambda(f_k) - J_\lambda(f)\|_L^2(\Omega)\\ &\leq 2\|f - f_k\|_L^2(\Omega) + \|f_k - J_\lambda(f_k)\|_L^2(\Omega).\end{aligned}$$

The final result is obtained by the procedure of passing to the limit.

6^o. The result is obtained without difficulty if we observe that

$$\Delta_\lambda(f) = -J_\lambda(\Delta f),$$

and by using the point 5^o.

7^o. By direct calculations, we have

$$\begin{aligned}\langle \Delta_\lambda(f), f\rangle_{L^2(\Omega)} =&\langle \Delta_\lambda(f), f - J_\lambda(f)\rangle_{L^2(\Omega)} + \langle \Delta_\lambda(f), J_\lambda(f)\rangle_{L^2(\Omega)}\\ =&\lambda\|\Delta_\lambda(f)\|^2_{L^2(\Omega)} - \langle \Delta(J_\lambda(f)), J_\lambda(f)\rangle_{L^2(\Omega)}\\ =&\ \lambda\|\Delta_\lambda(f)\|^2_{L^2(\Omega)} + \int_\Omega |\bigtriangledown J_\lambda(f)|^2 \mathrm{d}x,\end{aligned}$$

and then it is clear that

$$\langle \Delta_\lambda(f), f\rangle_{L^2(\Omega)} \geq 0.$$

8^o. Based on the inequality from the point 7^o, we have

$$\lambda\|\Delta_\lambda(f)\|^2_{L^2(\Omega)} \leq \|\Delta_\lambda(f)\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)},$$

from where we deduce that

$$\|\Delta_\lambda(f)\|_{L^2(\Omega)} \leq \frac{1}{\lambda}\|f\|_{L^2(\Omega)}.$$

This inequality proves that Δ_λ is a Lipschitz application defined everywhere on $L^2(\Omega)$, with Lipschitz constant $1/\lambda$.

9^o. It is not difficult to observe that

$$\int_\Omega \Delta_\lambda(f)g\mathrm{d}x = \int_\Omega \Delta_\lambda(f)(g - J_\lambda(g))\mathrm{d}x + \int_\Omega \Delta_\lambda(f)J_\lambda(g)\mathrm{d}x,$$

so that by using the point 2^o we obtain

$$\int_\Omega \Delta_\lambda(f) g \mathrm{d}x = \lambda \int_\Omega \Delta_\lambda(f)\Delta_\lambda(g)\mathrm{d}x - \int_\Omega \Delta(J_\lambda(f)) J_\lambda(g)\mathrm{d}x$$
$$= \lambda \int_\Omega \Delta_\lambda(f)\Delta_\lambda(g)\mathrm{d}x + \int_\Omega \nabla(J_\lambda(f)) \nabla (J_\lambda(g))\mathrm{d}x.$$

The last integral from the equality above is computed by parts and we obtain the desired result of symmetry. ■

As far as the solution of the heat conduction problem is concerned, the main result is proven in the following theorem.

Theorem 15.3.3 *We suppose that $u_o \in L^2(\Omega)$. Then, there exists only one solution, u, of the problem*

$$\begin{aligned} &\frac{\Delta u}{\Delta t}(t,x) - \Delta u(t,x) = 0,\ \forall (t,x) \in (0,T)\times\Omega, \\ &u(t,x) = 0,\ \forall (t,x) \in (0,T)\times\partial\Omega, \\ &u(0,x) = u_0(x),\ \forall x \in \Omega. \end{aligned} \tag{15.3.2}$$

In addition, as far as the regularity of the solution u is concerned, we have
1^o.

$$u \in C^0([0,\infty); L^2(\Omega)) \Leftrightarrow \forall t_0 \geq 0,\ \forall \{t_k\}_k, \lim_{k\to\infty} t_k = t_0 \Rightarrow$$
$$\Rightarrow \lim_{k\to\infty} \|u(t_k,x) - u(t_0,x)\|_{L^2(\Omega)} = 0;$$

2^o.

$$u \in C^0((0,\infty); D(\Delta)) \Leftrightarrow \forall t_0 \geq 0,\ \forall \{t_k\}_k, \lim_{k\to\infty} t_k = t_0 \Rightarrow$$
$$\Rightarrow \lim_{k\to\infty} \|u(t_k,x) - u(t_0,x)\|_{D(\Delta)} = 0;$$

3^o.

$$u \in C^1((0,\infty); L^2(\Omega)) \Leftrightarrow$$
$$\Leftrightarrow \exists \frac{\Delta u}{\partial t} \in C^1((0,\infty); L^2(\Omega)), \forall t_0 > 0, \forall \{h_k\}_k, \lim_{k\to\infty} h_k = 0 \Rightarrow$$
$$\Rightarrow \lim_{k\to\infty} \left\| u(t_0 + h_k, x) - u(t_0,x) - h_k \frac{\partial u}{\partial t}(t_0,x) \right\|_{L^2(\Omega)} = 0.$$

Proof For beginners, we approach the auxiliary problem

$$\begin{aligned} &\frac{\partial u_\lambda}{\partial t}(t,x) + \Delta_\lambda u_\lambda(t,x) = 0,\ \forall (t,x) \in (0,T)\times\Omega, \\ &u_\lambda(0,x) = 0,\ \forall x \in \Omega, \end{aligned} \tag{15.3.3}$$

where, as we mentioned before, Δ_λ is the Yosida approximation of the Laplacian. The existence and the uniqueness of u_λ, as a solution of the problem above, is obtained with Theorem 15.1.1, in which we take $X = L^2(\Omega)$ and $F = \Delta_\lambda$. It is clear that F, $F : X \to X$ is a linear application and a Lipschitz function with Lipschitz constant $1/\lambda$.

Thus, we deduce that $u_\lambda \in C^1([0,\infty); L^2(\Omega))$. Furthermore, we pass to the limit with $\lambda \to 0$. We want to remark the fact that the solution u_λ verifies the relation

$$\langle \frac{\partial u_\lambda}{\partial t}, u_\lambda\rangle_{L^2(\Omega)} + \langle \Delta_\lambda u_\lambda, u_\lambda\rangle_{L^2(\Omega)} = 0,$$

from where, by integration on $[0, T]$, we obtain

$$\frac{1}{2}\|u_\lambda\|^2_{L^2(\Omega)} + \int_0^T \langle \Delta_\lambda u_\lambda, u_\lambda\rangle_{L^2(\Omega)}\mathrm{d}t$$
$$= \frac{1}{2}\|u_\lambda(0,\cdot)\|^2_{L^2(\Omega)} = \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

We deduce that the sequence $\{u_\lambda(T,\cdot)\}_\lambda$ is bounded in $L^2(\Omega)$. We multiply, in both members of Eq. (15.3.3), by $t \cdot \frac{\partial u_\lambda}{\partial t}$ and we integrate the obtained relation on the interval $[0, T]$. We are led to

$$\int_0^T t\left\|\frac{\partial u_\lambda}{\partial t}\right\|^2_{L^2(\Omega)}\mathrm{d}t + \int_0^T t\langle \Delta_\lambda u_\lambda, \frac{\partial u_\lambda}{\partial t}\rangle_{L^2(\Omega)}\mathrm{d}t = 0. \qquad (15.3.4)$$

On the other hand, by using point 9^o from Theorem 15.3.2, we have

$$\frac{\partial}{\partial t}\left(\langle \Delta_\lambda u_\lambda, u_\lambda\rangle\right)_{L^2(\Omega)} = \left\langle \Delta_\lambda\left(\frac{\partial u_\lambda}{\partial t}\right), u_\lambda\right\rangle_{L^2(\Omega)}$$
$$+\left\langle \Delta_\lambda u_\lambda, \frac{\partial u_\lambda}{\partial t}\right\rangle_{L^2(\Omega)} = 2\left\langle \Delta_\lambda u_\lambda, \frac{\partial u_\lambda}{\partial t}\right\rangle_{L^2(\Omega)},$$

so that, by using the relation (15.3.4) we obtain

$$\int_0^T t\langle \Delta_\lambda u_\lambda, \frac{\partial u_\lambda}{\partial t}\rangle_{L^2(\Omega)}\mathrm{d}t = \frac{1}{2}\int_0^T t\frac{\partial}{\partial t}\langle \Delta_\lambda u_\lambda, u_\lambda\rangle_{L^2(\Omega)}\mathrm{d}t$$
$$= \frac{T}{2} < \Delta_\lambda u_\lambda(T,.), u_\lambda(T,.) > -\frac{1}{2}\int_0^T < \Delta_\lambda u_\lambda, u_\lambda >_{L^2(\Omega)} \mathrm{d}t.$$

Since u_λ is differentiable with respect to t, we can differentiate in Eq. $(15.3.3)_1$ so that the following equation is deduced:

$$\frac{\partial}{\partial t}\left(\frac{\partial u_\lambda}{\partial t}\right)(t,x) + \Delta_\lambda\left(\frac{\partial u_\lambda}{\partial t}\right)(t,x) = 0.$$

We multiply in this equality by $\partial u_\lambda/\partial t$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left\|\frac{\partial u_\lambda}{\partial t}\right\|^2_{L^2(\Omega)}\right) = -\left\langle \Delta_\lambda\left(\frac{\partial u_\lambda}{\partial t}\right), \frac{\partial u_\lambda}{\partial t}\right\rangle_{L^2(\Omega)} \le 0.$$

Therefore, we obtain that the function $t \mapsto \|\partial u_\lambda/\partial t\|^2_{L^2(\Omega)}$ is, also, decreasing on the interval $[0, T]$. Thus, we have the estimate

$$\int_0^T t\left\|\frac{\partial u_\lambda}{\partial t}\right\|^2_{L^2(\Omega)} \mathrm{d}t \ge T^2\left\|\frac{\partial u_\lambda}{\partial t}(T, \cdot)\right\|^2_{L^2(\Omega)},$$

so that the previous inequalities lead to the inequality

$$\frac{1}{2}\|u_\lambda(T,\cdot)\|^2_{L^2(\Omega)} + T\langle \Delta_\lambda u_\lambda(T,\cdot), u_\lambda(T,\cdot)\rangle_{L^2(\Omega)} + \frac{T^2}{2}\left\|\frac{\partial u_\lambda}{\partial t}(T,\cdot)\right\|^2_{L^2(\Omega)} \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Finally, we have

$$\left\|\frac{\partial u_\lambda}{\partial t}(T,\cdot)\right\|^2_{L^2(\Omega)} \le \frac{1}{T^2}\|u_0\|^2_{L^2(\Omega)}.$$

To continue the proof of the theorem, we need the result from the following proposition.

Proposition 15.3.3 *The following two statements are true:*

(i) We suppose that $u_0 \in D(\Delta)$. Then for any two real numbers λ and μ, with $\lambda,\ \mu > 0$, we have

$$\|u_\lambda(t,\cdot) - u_\mu(t,\cdot\|_{L^2(\Omega)} \le 2\sqrt{2(\lambda+\mu)t}\,\|\Delta u_0\|,_{L^2(\Omega)},\ \forall t > 0.$$

(ii) We suppose that $u_0 \in D(\Delta^2)$, that is, $u_0 \in D(\Delta)$ and $\Delta u_0 \in D(\Delta)$, then for any two real numbers λ and μ, with $\lambda,\ \mu > 0$, we have

$$\|\frac{\partial u_\lambda}{\partial t}(t,.) - \frac{\partial u_\mu}{\partial t}(t,.)\|_{L^2(\Omega)} \le 2\sqrt{2(\lambda+\mu)t}\,\|\Delta^2 u_0\|,_{L^2(\Omega)}.$$

Proof (i) We will write the fact that u_λ and u_μ verify the equation of heat conduction $(15.3.3)_1$ and subtract member by member the obtained relations

$$\frac{\partial u_\lambda}{\partial t}(t,x) + \Delta_\lambda u_\lambda(t,x) - \frac{\partial u_\mu}{\partial t}(t,x) - \Delta_\mu u_\mu(t,x) = 0.$$

In this equality, we multiply with $u_\lambda - u_\mu$, and the obtained relation is integrated on Ω. We obtain the equation

$$\frac{1}{2}\frac{\partial}{\partial t}(\|(u_\lambda(t,\cdot)-u_\mu(t,\cdot))\|^2_{L^2(\Omega)})+\langle \Delta_\lambda(u_\lambda)-\Delta_\mu(u_\mu), u_\lambda-u_\mu\rangle_{L^2(\Omega)}=0. \tag{15.3.5}$$

On the other hand,

$$\begin{aligned}
&\langle \Delta_\lambda(u_\lambda) - \Delta_\mu(u_\mu), u_\lambda - u_\mu\rangle_{L^2(\Omega)}\\
&\quad = \langle \Delta_\lambda(u_\lambda) - \Delta_\mu(u_\mu), u_\lambda - J_\lambda(u_\lambda) + J_\lambda(u_\lambda) - J_\mu(u_\mu) + J_\mu(u_\mu) - u_\mu\rangle_{L^2(\Omega)}\\
&\quad = \langle \Delta_\lambda(u_\lambda) - \Delta_\mu(u_\mu), \lambda\Delta_\lambda(u_\lambda) - \mu\Delta_\mu(u_\mu)\rangle_{L^2(\Omega)}\\
&\qquad - \langle \Delta(J_\lambda(u_\lambda) - J_\mu(u_\mu)), J_\lambda(u_\lambda) - J_\mu(u_\mu)\rangle_{L^2(\Omega)}\\
&\quad \geq \langle \Delta_\lambda(u_\lambda) - \Delta_\mu(u_\mu), \lambda\Delta_\lambda(u_\lambda) - \mu\Delta_\mu(u_\mu)\rangle_{L^2(\Omega)}.
\end{aligned}$$

Then from (15.3.5), we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\|(u_\lambda(t,\cdot) - u_\mu(t,\cdot))\|^2_{L^2(\Omega)})\\
&\quad \leq -\langle \Delta(J_\lambda(u_\lambda) - J_\mu(u_\mu)), \lambda\Delta_\lambda(u_\lambda) - \mu\Delta_\mu(u_\mu)\rangle_{L^2(\Omega)}\\
&\quad \leq (\lambda - \mu)\langle \Delta_\lambda(u_\lambda) - \Delta_\mu(u_\mu)\rangle_{L^2(\Omega)}\\
&\quad \leq (\lambda + \mu)\|\Delta_\lambda(u_\lambda)\|_{L^2(\Omega)}\|\Delta_\mu(u_\mu)\|_{L^2(\Omega)},
\end{aligned}$$

in which we used the fact that the application $t \mapsto \|\Delta_\lambda(u_\lambda)\|_{L^2(\Omega)}$ is decreasing. It is clear that from the last inequality we can write the inequality

$$\frac{1}{2}\frac{\partial}{\partial t}(\|u_\lambda(t,\cdot) - u_\mu(t,\cdot)\|^2_{L^2(\Omega)}) \leq (\lambda - \mu)\|\Delta u_0\|_{L^2(\Omega)},$$

and then the proof of the point (i) is concluded.

(ii) By performing some calculations which are similar to those from above, we can provide the inequality

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left\|\frac{\partial u_\lambda}{\partial t}(t,\cdot) - \frac{\partial u_\mu}{\partial t}(t,\cdot)\right\|^2_{L^2(\Omega)}\right) \leq 2(\lambda - \mu)\|\Delta^2 u_0\|_{L^2(\Omega)}$$

which concludes the proof.

■

Proposition 15.3.4 *We suppose that $u_0 \in D(\Delta^2)$. Then*

(i) The sequence $\{u_\lambda\}_\lambda$ is convergent in the strong topology of the space $L^2(\Omega)$ to u, uniformly with respect to $t \in [0, T]$.
(ii) The sequence $\{\partial u_\lambda/\partial t\}_\lambda$ is convergent in the strong topology of $L^2(\Omega)$ to u, uniformly with respect to $t \in [0, T]$.
Also, u is a solution of the equation

$$\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0,\ \forall (t,x) \in Q_T = [0,T] \times \Omega.$$

Proof (i) The proof of this point is immediately obtained from the point (i) of Proposition 15.3.3.

(ii) We will use the point (ii) from Proposition 15.3.3. First, we have

$$\begin{aligned}\|J_\lambda(u_\lambda)(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)} &\le \|J_\lambda(u_\lambda)(t,\cdot) - J_\lambda(u)(t,\cdot)\|_{L^2(\Omega)} \\ &+ \|J_\lambda(u)(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)},\end{aligned}$$

from where we deduce that

$$\begin{aligned}\|J_\lambda(u_\lambda)(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)} &\le \|(u_\lambda)(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)} \\ &+ \|J_\lambda(u)(t,\cdot) - u(t,\cdot)\|_{L^2(\Omega)}.\end{aligned}$$

This estimate together with the point 5^o of Theorem 15.3.2 leads to the conclusion that

$$\lim_{\lambda\to 0} J_\lambda(u_\lambda)(t,\cdot) = u(t,\cdot).$$

Also, because

$$\frac{\partial u_\lambda}{\partial t}(t,x) - \Delta J_\lambda(u_\lambda)(t,x) = 0,$$

we deduce that

$$\lim_{\lambda\to 0} \frac{\partial u_\lambda}{\partial t} = v = \frac{\partial u}{\partial t},\ \lim_{\lambda\to 0} J_\lambda(u_\lambda) = u,$$
$$\left(\frac{\partial u_\lambda}{\partial t}, J_\lambda(u_\lambda)\right) \in G(\Delta),$$

in which we denoted by $G(\Delta)$ the graph of the Laplacian.
Let us demonstrate that the graph of the Laplacian is included in the set $L^2(\Omega) \times L^2(\Omega)$. Let $\{x_k, y_k\}_k$ be a sequence of elements from $G(\Delta)$, convergent to (x, y), in the strong topology of the product space $L^2(\Omega) \times L^2(\Omega)$. Then

$$x_k \in H^2(\Omega) \cap H_0^1(\Omega)\ ;\ y_k = \Delta x_k$$

and thus

$$x_k - \Delta x_k = x_k - y_k.$$

In particular, the operator $R(1, \Delta)$ exists and if we apply this operator in the equality above, we obtain

$$R(1, \Delta)(x_k - \Delta x_k) = R(1, \Delta)(x_k - y_k) = x_k.$$

We pass to the limit with $k \to \infty$, in this equality (which is allowed because the operator $R(1, \Delta)$ is continuous). We obtain

$$R(1, \Delta)(x - y) = x,$$

from where we deduce that

$$x \in H^2(\Omega) \cap H_0^1(\Omega).$$

In the last equality, we apply the operator $I - \Delta$ and we obtain the equation

$$x - y = x - \Delta x,$$

so that $y = \Delta x$ and the proof of Proposition 15.3.4 is concluded. ■

The Proof of the Theorem 15.3.1 (continuation) First, let us remark the fact that $D(\Delta^2)$ is dense in $L^2(\Omega)$. By using the inequalities from Proposition 15.3.3 (the inequalities are applied uniformly with respect to t), and the fact that the graph of Δ is included in $L^2(\Omega) \times L^2(\Omega)$, we ensure the existence of a weak solution of the problem of heat in the case in which the the right-hand side is null.

The uniqueness will be proven by reduction to the absurd. We assume that the problem of heat conduction admits two weak solutions u_1 and u_2, If we use the notation $u_3 = u_2 - u_1$, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\|u_3(t, \cdot)\|^2_{L^2(\Omega)}\right) + \int_\Omega \| \nabla u_3(t, x)\|^2_{L^2(\Omega)} \mathrm{d}x = 0,$$

from where we deduce that the function $t \mapsto \| u_3(t, \cdot)\|^2_{L^2(\Omega)}$ is decreasing.

At the moment $t = 0$, we have $u_3 = 0$ and then we deduce that u_3 is identical null and the proof is concluded. ■

In the following proposition, we have a demonstration of an energetic relation attached to a solution of the heat conduction problem.

Proposition 15.3.5 *We suppose that the conditions of Theorem 15.3.3 are satisfied. Then, the following equality holds true*

$$\frac{1}{2}\|u(T, \cdot)\|^2_{L^2(\Omega)} + \int_0^T \| \nabla u(t, x)\|^2_{L^2(\Omega)} \mathrm{d}t = \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

This relation is known as the equality of energy.

Proof It is easy to find that the function φ defined by

$$\varphi : t \mapsto \frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Omega)},$$

is differentiable on $(0,\infty)$. By using the heat conduction equation, it is immediately deduced that

$$\varphi'(t) = -\int_\Omega |\nabla u(t,x)|^2 \mathrm{d}x,$$

from where, by integration on the interval $[\varepsilon, T]$, with respect to t, we find

$$\varphi(T) - \varphi(\varepsilon) = -\int_\varepsilon^T \int_\Omega |\nabla u(t,x)|^2 \mathrm{d}x\mathrm{d}t,$$

and the desired result is obtained by passing to the limit with $\varepsilon \to 0$, which is allowed based on the continuity of the two members of the equality. This ends the proof. ■

In the following theorem, we will show, without proof, a result of regularity of the solution of the heat conduction problem.

Theorem 15.3.4 *The following two statements are true:*

(i) If the initial data $u_0 \in H_0^1(\Omega)$, then the solution u of the problem of the heat belongs to the space

$$C^0([0,\infty); H_0^1(\Omega)) \cup L^2(0,\infty; H^2(\Omega)).$$

Furthermore, the derivative of u with respect to t is from the space $L^2(0,\infty; L^2(\Omega))$ and satisfies the relation

$$\frac{1}{2}\|\nabla u(T,\cdot)\|^2_{L^2(\Omega)} + \int_0^T \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)}(t,\cdot)\mathrm{d}t = \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)}.$$

(ii) If the initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then the solution u of the problem of heat belongs to the space

$$C^0([0,\infty); H^2(\Omega) \cup H_0^1(\Omega)) \cap L^2(0,\infty; H^3(\Omega)),$$

and its derivative with respect to time is from the space $L^2(0,\infty; H^1(\Omega))$.

For details of the prof, the reader can consult the book of Brezis [9].

Observation 15.3.2 *Based on the properties of regularity from Theorem 15.3.4, we can prove that the solution u of the heat conduction problem is a regular function also with respect to x, even if the initial data u_0 is not very regular.*

In the following proposition, we formulate the principle of maximum for the weak solutions of the heat conduction problem.

Proposition 15.3.6 *We suppose that the initial data u_0 from the heat conduction problem is from the space $L^\infty(\Omega)$. Then for $\forall(t,x) \in [0,\infty) \times \Omega$, we have the estimate*

$$\min\left(0, \inf_{x\in\Omega} u_0\right) \le u(t,x) \le max\left(0, \sup_{x\in\Omega} u_0\right).$$

In the proof, it is used the method of truncation proposed by Stampacchia. For details, the reader can consult the book of Brezis [9].

At the end of the chapter, we want to recall a method of solving (sketch) the heat conduction problem in the case in which the equation of the heat is nonhomogeneous (that is, the right-hand side f is not identically null).

Theorem 15.3.5 *We assume that $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$. Then, the nonhomogeneous heat conduction problem*

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = f(t,x),\ \forall(t,x) \in Q_T,\\
&u(t,x) = 0,\ \forall(t,x) \in (0,T) \times \Delta\Omega,\\
&u(0,x) = u_0(x), \forall x \in \Omega,
\end{aligned}$$

admits only one weak solution, given by the formula

$$u(t,x) = S(t)(u_0)(x) + \int_0^T S(t-s)(f(s,x))\mathrm{d}s,$$

in which S is the operator of truncation proposed by Stampacchia.

In the proof made in detail in the book of Brezis [9], it is used a method which is similar to the method of variation of constants used in the theory of ordinary differential equations.

Chapter 16
Weak Solutions for Hyperbolic Equations

16.1 The Problem of the Infinite Vibrant Chord

In principle, in this chapter we will study the wave equation, which constitutes the prototype of the hyperbolic equations.

Let Ω be an open set from $\mathbb{R}^n$ and T a real number $T > 0$. Then, the Cauchy problem, associated with the wave equation, consists of

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) = 0,\ \forall (t,x) \in Q_T,$$
$$u(0,x) = u_0(x),\ \forall x \in \Omega,$$
$$\frac{\partial u}{\partial t}(0,x) = u_1(x),\ \forall x \in \Omega,$$

where Q_T is the notation for the cylinder $Q_T = (0,T) \times \Omega$.

If the set Ω is bounded having the boundary $\partial\Omega$, the problem above is completed with the boundary conditions of Dirichlet type.

The operator

$$\frac{\partial^2}{\partial t^2} - \Delta$$

called the operator of D‘Alembert is a hyperbolic operator, and will be studied in the following.

We will approach, for beginners, the particular case $n = 1$ and we take $\Omega = \mathbb{R}$. Then, the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x),\ \forall (t,x) \in Q_T = (0,T) \times \mathbb{R}.$$

M. Marin and A. Öchsner, *Essentials of Partial Differential Equations*,
https://doi.org/10.1007/978-3-319-90647-8_16

Theorem 16.1.1 *The classical solutions of the waves equation are of the form*

$$u(t, x) = \Phi(x + t) + \Psi(x - t),$$

where Φ *and* Ψ *are arbitrary functions and are assumed be twice continuously differentiable on* $\mathbb{R}$.

Proof We will use a non-singular transform of coordinates
$(t, x) \to (\xi, \eta)$, given by

$$\xi = x + t,$$
$$\eta = x - t,$$

which satisfies the condition

$$\left| \frac{D(\xi, \eta)}{D(t, x)} \right| \neq 0.$$

Then, the operator of D'Alembert, which in this particular case is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2},$$

becomes

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4 \frac{\partial^2 u}{\partial \xi \partial \eta}.$$

Hence, if we use the notation $u(t, x) = U(\xi, \eta)$, the wave equation becomes

$$\frac{\partial^2 U}{\partial \xi \partial \eta}(\xi, \eta) = 0,$$

in which $\xi > \eta$ because $t > 0$. We integrate the equation with respect to ξ and obtain

$$\frac{\partial U}{\partial \eta}(\xi, \eta) = g(\eta).$$

Now, we integrate again, but this time with respect to η and we are led to the relation

$$u(\xi, \eta) = G(\eta) + F(\xi),$$

where we denoted by G a primitive of the function g.

If we return to the initial variables, from the system $\xi = x + t$, $\eta = x - t$, and use the substitution $G = \Psi$, $F = \Phi$, we obtain the formula of the solution mentioned above.

The hypotheses of regularity imposed on the functions Φ and Ψ allow us to perform the partial derivatives above. ■

Corollary 16.1.1 *Assume that* $u_0 \in C^2(R)$ *and* $u_1 \in C^1(R)$. *Then, the Cauchy problem associated with the wave equation in the case* $n = 1$ *and* $\Omega = \mathbb{R}$ *receives the form*

$$\frac{\partial^2 u}{\partial t^2}(t, x) - \frac{\partial^2 u}{\partial x^2}(t, x) = 0, \ \forall (t, x) \in Q_T,$$
$$u(0, x) = u_0(x), \ \forall x \in \mathbb{R},$$
$$\frac{\partial u}{\partial t}(0, x) = u_1(x), \ \forall x \in \mathbb{R}.$$

This problem admits the following classical solution:

$$u(t, x) = \frac{1}{2}[u_0(x + t) + u_0(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\mathrm{d}s. \tag{16.1.1}$$

Proof Based on Theorem 16.1.1, the solution of the wave equation has the representation

$$u(t, x) = \Phi(x + t) + \Psi(x - t).$$

We impose the initial conditions

$$u(x, 0) = u_0(x) = \Phi(x) + \Psi(x),$$
$$\frac{\partial u}{\partial t}(0, x) = u_1(x) = \Phi\prime(x) + \Psi\prime(x),$$

and obtain

$$\Phi(x) = \frac{1}{2}\left[u_0(x) + \int_0^x u_1(s)\mathrm{d}s\right],$$
$$\Psi(x) = \frac{1}{2}\left[u_0(x) - \int_0^x u_1(s)\mathrm{d}s\right],$$

from where the following form of the solution is obtained:

$$u(t, x) = \frac{1}{2}[u_0(x + t) + u_0(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\mathrm{d}s + C_1 + C_2. \tag{16.1.2}$$

With the help of initial conditions deduce that $C_1 + C_2 = 0$, so that the relation (16.1.2) receives the desired form (16.1.1) of the solution. ■

We approach now the general case $\mathbb{R}^n$, $n \geq 2$ and we take into account that the open set Ω is the whole space $\mathbb{R}^n$.

For any function F which is integrable on any compact set from $\mathbb{R}^+ \times \mathbb{R}^n$, we define the function M_F by

$$M_F(t, x, r) = \frac{1}{\omega_n} \int_{S(0,1)} F(t, x, ry) d\sigma(y), \tag{16.1.3}$$

where ω_n is the area of the unit sphere $S(0, 1) \subset \mathbb{R}^n$.

In the following proposition, the Cauchy problem associated with the wave equation in the n-dimensional case will be transformed into a Cauchy problem associated with the wave equation in the 1-dimensional case.

Proposition 16.1.1 *Let u be a classical solution of the Cauchy problem associated with the wave equation. Then, M_u is the solution of the following Cauchy problem*

$$\begin{aligned} &\frac{\partial^2 M_u}{\partial t^2}(t, u, r) = \frac{\partial^2 M_u}{\partial r^2}(t, x, r) \\ &\quad + \frac{n-1}{r}\frac{\partial M_u}{\partial r}(t, x, r), \ \forall (t, x, r) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}, \\ &M_u(0, x, r) = M_{u_0}(x, r), \ \forall (x, r) \in \mathbb{R}^n \times \mathbb{R}, \\ &\frac{\partial M_u}{\partial t}(0, x, r) = M_{u_1}(x, r) \ \forall (x, r) \in \mathbb{R}^n \times \mathbb{R}. \end{aligned} \tag{16.1.4}$$

Proof Taking into account definition (16.1.3), we have

$$M_u(t, x, r) = \frac{1}{\omega_n} \int_{S(0,1)} u(t, x + ry) \mathrm{d}\sigma(y).$$

Based on property of u, we can differentiate this integral so that we have

$$\begin{aligned} \frac{\partial M_u}{\partial r}(t, x, r) &= \frac{1}{\omega_n}\frac{\partial}{\partial r} \int_{S(0,1)} u(t, x + ry) \mathrm{d}\sigma(y) \\ &= \frac{1}{\omega_n} \int_{S(0,1)} \nabla u(t, x+ry) y \mathrm{d}\sigma(y) = \frac{1}{\omega_n} \int_{S(0,1)} \frac{\partial u}{\partial \gamma}(t, x+ry) \mathrm{d}\sigma(y) \\ &= \frac{r}{\omega_n} \int_{B(0,1)} \Delta u(t, x + ry) y \mathrm{d}\sigma(y), \end{aligned} \tag{16.1.5}$$

in which the last integral is obtained with the help of Green's formula.

Using the change of variable $x + ry = z$, we obtain

$$\frac{\partial M_u}{\partial r}(t, x, r) = \frac{1}{r^{n-1}\omega_n} \int_{B(x,r)} \Delta u(t, z) dz$$
$$= \frac{1}{r^{n-1}\omega_n} \int_{B(x,r)} \frac{\partial^2 u}{\partial t^2}(t, z) dz = \frac{1}{r^{n-1}\omega_n} \int_0^r \int_{S(x,\rho)} (t, z) d\sigma(z) d\rho,$$

in which we take into account that u satisfies the wave equation. Also, we used the Theorem of Fubini which allows us to reverse the order of integration.

We will use again (16.1.3) and if we take into account relation (16.1.5), we are led to the equation

$$\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial M_u}{\partial r}\right)(t, x, r) = \frac{1}{\omega_n} \int_{S(x,r)} \frac{\partial^2 u}{\partial t^2}(t, z) \mathrm{d}\sigma(z)$$
$$= \frac{r^{n-1}}{\omega_n} \int_{S(0,1)} \frac{\partial^2 u}{\partial t^2}(t, x + ry) d\sigma(y) = r^{n-1}\frac{\partial^2 M_u}{\partial t^2}(t, x, r),$$

from which we immediately obtain Eq. $(16.1.4)_1$.

The boundary conditions $(16.1.4)_2$ and $(16.1.4)_3$ are obtained without difficulty, by taking into account (16.1.3) and the boundary conditions which are satisfied by the function u. ■

In the following theorem, we expose the form of the classical solution of the Cauchy problem associated with the wave equation, in the case when n is an odd integer number.

Theorem 16.1.2 *We suppose that the following hypotheses are satisfied:*

- *n is an odd integer number;*
- *$u_0 \in C^{(n+3)/2}(\mathbb{R}^n)$;*
- *$u_1 \in C^{(n+1)/2}(\mathbb{R}^n)$.*

Then, the classical solution of the Cauchy problem associated with the wave equation is given by the formula

$$u(t, x) = \frac{1}{1.3.5\ldots(n-2)\omega_n}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\right)^{(n-3)/2} \times$$
$$\times \left(t^{n-2}\int_{S(0,1)} u_0(x + t, y)\mathrm{d}\sigma(y)\right)$$
$$+ \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}\left(t^{n-2}\int_{S(0,1)} u_1(x + t, y)\mathrm{d}\sigma(y)\right).$$

Proof Because n is an odd number, we will write $n = 2k + 1$.

In the following, we will use the functions $v(t, x, r)$, $\Phi(x, r)$ and $\Psi(x, r)$, defined as follows:

$$\begin{aligned} v(t, x, r) &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}M_u(t, x, r)\right), \\ \Phi(x, r) &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}M_{u_0}(x, r)\right), \\ \Psi(x, r) &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}M_{u_1}(x, r)\right), \end{aligned} \tag{16.1.6}$$

in which the definition of M_u is analogous to that of (16.1.3).

We can verify, by recurrence after values of k, the relations

(i)

$$\frac{\partial^2}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}F(r)\right) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left(r^{2k}\frac{\partial F}{\partial r}(r)\right);$$

(ii)

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}F(r)\right) = \sum_{i=0}^{k-1} c_i r^{i+1}\frac{\partial^i F}{\partial r^i}(r),$$

in which we denoted by F, an arbitrary function, assumed to be integrable on any compact set from $\mathbb{R}^n$. Also, c_i are real coefficients and c_0 has the value $c_0 = 1.3.5\ldots(2k-1)$.

By taking into account $(16.1.6)_1$, we have

$$\begin{aligned} \frac{\partial^2 v}{\partial r^2}(t_0, x, r) &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left(r^{2k}\frac{\partial M_u}{\partial r}(t, x, r)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^{2k}\frac{\partial M_u}{\partial r}(t, x, r)\right)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(2kr^{2k-1}\frac{\partial M_u}{\partial r}(t, x, r) + r^{2k}\frac{\partial^2 M_u}{\partial r^2}(t, x, r)\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}\frac{\partial^2 M_u}{\partial t^2}(t, x, r)\right) = \frac{\partial^2 v}{\partial t^2}(t, x, r). \end{aligned}$$

Now, we want to find the initial conditions satisfied by the function v. From $(16.1.6)_1$, for $t = 0$, we perform the following calculations:

$$v(0,x,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} M_u(0,x,r)\right)$$
$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} M_{u_0}(x,r)\right) = \Phi(x,r),$$
$$\frac{\partial v}{\partial t}(0,x,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} \frac{\partial M_u}{\partial t}(0,x,r)\right)$$
$$= \frac{1}{r}\left(\frac{\partial}{\partial r}\right)^{k-1} \left(r^{2k-1} M_{u_1}(x,r)\right) = \Psi(x,r).$$

From these relations, we deduce that the function $v(t,x,r)$ verifies the wave equation from Theorem 16.1.1 and the same initial conditions as in Corollary 16.1.1. Thus, v receives the expression

$$v(t,x,r) = \frac{1}{2}[\Phi(x,r+t) + \Phi(x,r-t)] + \frac{1}{2}\int_{r-t}^{r+t} \Psi(x,s)\mathrm{d}s.$$

We will obtain the expression of the solution u, starting from this formula of v. First, let us observe that

$$u(t,x) = \lim_{r\to 0} M_n(t,x,r) = \lim_{r\to 0} \frac{1}{c_0 r} v(t,x,r).$$

If we take into account the expressions for M_{u_0} and M_{u_1}, as well as the fact that $n-1$ is an even number, we deduce that the functions M_{u_0} and M_{u_1} are even functions, as functions of variable r. Consequently, Φ and Ψ are odd functions of variable r, and we have

$$u(t,x) = \frac{1}{c_0 r} \lim_{r\to 0} \frac{1}{r}[\Phi(x,r+t) + \Phi(x,t-r)] + \Psi(t,x).$$

Therefore, if we substitute the functions Φ and Ψ by their expressions from the definition (16.1.6), we obtain the expression of the solution from the statement of the theorem. ∎

At the end of the paragraph, we will find the form of the solution of the Cauchy problem, associated with the wave equation, in the case in which n is an even number.

Theorem 16.1.3 *We suppose that n is an even number, $n = 2k$. Also, we assume that $u_0 \in C^{k+2}(\mathbb{R}^{2k})$ and $u_1 \in C^{k+1}(\mathbb{R}^{2k})$. Then, the classical solution of the Cauchy problem associated with the wave equation is given by the following formula:*

$$u(t,x) = \frac{2}{1\cdot 3\cdot 5...(2k-1)\omega_{2k+1}}\times$$
$$\times\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\right)^{k-1}\left(t^{2k-1}\int_{B(0,1)}\frac{u_0(x+t,y)}{\sqrt{1-|y|^2}}\mathrm{d}y\right)$$
$$+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1}\int_{B(0,1)}\frac{u_1(x+t,y)}{\sqrt{1-|y|^2}}\mathrm{d}y\right).$$

Proof We grow the dimension of the space with 1 and we consider that the functions u_0 and u_1 are defined on the space $\mathbb{R}^{n+1}$, but they depend only on the first n variables. Therefore, we can use Theorem 16.1.2, and then the solution has the form

$$u(t,x,x_{n+1}) = \frac{1}{1\cdot 3\cdot 5\ldots(2n-1)\omega_{n+1}}\times$$
$$\times\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\right)^{k-1}\left(t^{2k-1}\int_{S(0,1)}u_0(x+t,y)\mathrm{d}\sigma(y\prime,y_{n+1})\right)$$
$$+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1}\left(t^{2k-1}\int_{S(0,1)}u_1(x+t,y)\mathrm{d}\sigma(y\prime,y_{n+1})\right),$$

where $S(0,1)$ is the unit sphere from the space $\mathbb{R}^{n+1}$.

The desired result is obtained if we consider that the functions u_0 and u_1 are independent of the variable y_{n+1} and we use the theorem of Fubini regarding the inversion of the integration order.

Also, for each half of a sphere, we have

$$\mathrm{d}y' = \left(1+|y'|^2\right)d\sigma\left(y',y_{n+1}\right),$$

by using the known elementary theorem of Pitagora. ■

16.2 Weak Solutions of the Wave Equation

Let Ω be an arbitrary open set from $\mathbb{R}^n$ whose boundary will be denoted by $\partial\Omega$. We are looking for weak solutions of the problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) = 0,\ \forall(t,u)\in Q_T,\\
&u(0,x) = u_0(x)\ \forall x\in\Omega,\\
&\frac{\partial u}{\partial t}(0,x) = u_1(x),\ \forall x\in\Omega,\\
&u(t,x) = 0,\ \forall x\in\partial\Omega,\ \forall t>0,
\end{aligned}\tag{16.2.1}$$

where Q_T is the notation for the cylinder $Q_T = (0, T) \times \Omega$.

The main result of this paragraph, regarding the existence and the uniqueness of the solution of the problem (16.2.1), is included in the following theorem.

Theorem 16.2.1 *We suppose that* $u_0 \in H^2(\Omega) \cup H_0^1(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$. *Then, there exists only one weak solution of the wave equation with initial data* u_0 *and* u_1 *and this weak solution is from the space*

$$C^0([0, \infty); H^2(\Omega)) \cap C^1([0, \infty); H_0^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)).$$

In addition, for any $t \geq 0$, u *satisfies the equation*

$$\int_\Omega |\frac{\partial u}{\partial t}|^2(t, x)\mathrm{d}x + \int_\Omega |\bigtriangledown u|^2(t, x)\mathrm{d}x$$
$$= \int_\Omega |u_1|^2(x)\mathrm{d}x + \int_\Omega |\bigtriangledown u_0|^2(x)\mathrm{d}x.$$

Proof We will use the notations

$$v = \frac{\partial u}{\partial t}, \ U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Thus, the problem (16.2.1) can be written in matrix form

$$\frac{\partial U}{\partial t}(t, x) + \mathcal{A}(U)(t, x) = 0,$$

where the matrix operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}.$$

By taking into account the boundary conditions and the initial conditions, we will look for a weak solution U which belongs to the space $H_0^1(\Omega) \times L^2(\Omega)$.

Furthermore, we can proceed analogously as in the proof of the theorem of existence of a weak solution of the heat conduction problem.

For this purpose, we introduce the operator $\mathcal{B}$ defined by

$$\mathcal{B} = \begin{pmatrix} U_1 - U_2 \\ -\Delta U_1 + U_2 \end{pmatrix}.$$

The operator $\mathcal{B}$ will replace the operator $-\Delta$ in the heat conduction problem.

To complete the proof of Theorem 16.2.1, we need the results from the following two lemmas. ■

Lemma 16.2.1 *For any $\lambda > 0$ and any two functions $f_1,\ f_2 \in L^2(\Omega)$, there exists only one solution $U_{\lambda,F}$ in the space $H_0^1(\Omega) \times L^2(\Omega)$ of the equation*

$$\lambda \mathcal{B}(U_{\lambda,F}) + U_{\lambda,F} = F, \tag{16.2.2}$$

where

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

In addition, if $f_1 \in H_0^1(\Omega)$, then

$$U_{\lambda,F} \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega).$$

Proof We will write the matrix equation (16.2.2), by taking into account the matrix expressions of $\mathcal{B}$ and F, in the form of the following system:

$$\begin{aligned} &\lambda u_{\lambda,F} - \lambda v_{\lambda,F} + u_{\lambda,F} = f_1, \\ &-\lambda \Delta u_{\lambda,F} + \lambda v_{\lambda,F} + v_{\lambda,F} = f_2. \end{aligned}$$

This system can be rewritten in the form

$$\begin{aligned} &v_{\lambda,F} = \frac{1}{\lambda}\left[(\lambda+1)u_{\lambda,F} - f_1\right], \\ &-\lambda \Delta u_{\lambda,F} + \frac{\lambda+1}{\lambda}\left[(\lambda+1)u_{\lambda,F} - f_1\right] = f_2. \end{aligned} \tag{16.2.3}$$

Equation $(16.2.3)_2$ is equivalent to the equation

$$-\frac{\lambda^2}{(\lambda+1)^2}\Delta u_{\lambda,F} = a\frac{\lambda}{(\lambda+1)^2} f_2 + \frac{1}{(\lambda+1)} f_1. \tag{16.2.4}$$

It is clear that for any $f_1,\ f_2 \in L^2(\Omega)$, there exists only one function $u_{\lambda,F} \in H_0^1(\Omega)$ which is a solution of the equation (16.2.4).

After determining the function $u_{\lambda,F}$, the function $v_{\lambda,F}$ will receive the expression

$$v_{\lambda,F} = \frac{1}{\lambda}\left[(\lambda+1)u_{\lambda,F} - f_1\right],$$

and we have that $v_{\lambda,F} \in L^2(\Omega)$. It is clear that $u_{\lambda,F} \in H_0^1(\Omega) \cap H^2(\Omega)$.

Finally, if the function $f_1 \in H_0^1(\Omega)$, then the function $v_{\lambda,F} \in H_0^1(\Omega)$ and the proof of the lemma is concluded. ■

In the following, we use the operator

$$J_\lambda^{\mathcal{B}} : H_0^1(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) = \mathcal{D}(\mathcal{B}),$$

defined by

$$J_\lambda^{\mathcal{B}}(F) = U_{\lambda,F}.$$

Also, we will denote by H the product space $H = H_0^1(\Omega) \times L^2(\Omega)$, which will be endowed with the scalar product

$$\langle U, V\rangle_H = \int_\Omega \nabla U_1 \cdot \nabla U_2 \mathrm{d}x + \int_\Omega U_1 V_1 \mathrm{d}x + \int_\Omega U_2 V_2 \mathrm{d}x. \tag{16.2.5}$$

Here, we take into account that the open set, Ω, is not necessarily bounded.

The second result, necessary to finish the proof of Theorem 16.2.1, is included in the following lemma.

Lemma 16.2.2 *The operator $J_\lambda^{\mathcal{B}}$ is a contraction defined everywhere on the space H.*

Proof We multiply Eq. (16.2.2) with $U_{\lambda,F}$

$$\lambda\langle \mathcal{B}(U_{\lambda,F}), U_{\lambda,F}\rangle_H + \langle U_{\lambda,F}, U_{\lambda,F}\rangle_H = \langle F, U_{\lambda,F}\rangle_H,$$

which is equivalent to

$$\begin{aligned}
\lambda \int_\Omega \nabla u_{\lambda,F} \cdot \nabla u_{\lambda,F} \mathrm{d}x - \lambda \int_\Omega \nabla v_{\lambda,F} \cdot \nabla v_{\lambda,F} \mathrm{d}x - \lambda \int_\Omega \Delta u_{\lambda,F} v_{\lambda,F} \mathrm{d}x \\
+ \lambda \int_\Omega v_{\lambda,F} v_{\lambda,F} \mathrm{d}x + \lambda \int_\Omega u_{\lambda,F} u_{\lambda,F} \mathrm{d}x - \lambda \int_\Omega v_{\lambda,F} u_{\lambda,F} \mathrm{d}x \\
+ \int_\Omega |\nabla u_{\lambda,F}|^2 \mathrm{d}x + \int_\Omega (u_{\lambda,F})^2 \mathrm{d}x + \int_\Omega (v_{\lambda,F})^2 \mathrm{d}x \\
= \int_\Omega \nabla f_1 . \nabla u_{\lambda,F} \mathrm{d}x + \int_\Omega f_1 u_{\lambda,F} \mathrm{d}x + \int_\Omega f_2 v_{\lambda,F} \mathrm{d}x.
\end{aligned}$$

This equivalence is natural if we take into account (16.2.5).

We should remark the fact that

$$\lambda \int_\Omega v_{\lambda,F} u_{\lambda,F} \mathrm{d}x \le \frac{\lambda}{2} \int_\Omega (u_{\lambda,F})^2 \mathrm{d}x - \int_\Omega (v_{\lambda,F})^2 \mathrm{d}x.$$

On the other hand, if we integrate by parts the integral $\lambda \int_\Omega u_{\lambda,F} v_{\lambda,F} \mathrm{d}x$, we are led to the equality

$$\int_\Omega |\nabla u_{\lambda,F}|^2 \mathrm{d}x + \int_\Omega (u_{\lambda,F})^2 \mathrm{d}x + \int_\Omega (v_{\lambda,F})^2 \mathrm{d}x \le \|F\|_H \|U_{\lambda,F}\|_H$$

and from here we deduce the result of the lemma. ■

The Proof of the Theorem 16.2.1 (Continuation) We define the Yosida approximation $\mathcal{B}_\lambda$ of the operator $\mathcal{B}$

$$\mathcal{B}_\lambda(F) = \frac{1}{\lambda}(F - J_\lambda^{\mathcal{B}}(F)).$$

We can indicate some estimates on the Yosida approximation $\mathcal{B}_{\geq}$, similar to those proved in Theorem 8.3.2 from Chap. 8.

Also, we can reassume the proof of Theorem 8.3.3 from Chap. 8, to prove the existence of a weak solution for the following problem:

$$\begin{aligned}
&\frac{\partial U}{\partial t}(t,x) + \mathcal{B}(U)(t,x) = 0, \ \forall (t,x) \in (0,T) \times \Omega, \\
&U(0,x) = F(x), \ \forall x \in \Omega.
\end{aligned}$$

This solution can belong to the spaces $C^0([0,\infty); L^2(\Omega))$ or $C^0([0,\infty); D(\mathcal{B}))$ or $C^1([0,\infty); L^2(\Omega))$, if the necessary and sufficient conditions from Theorem 8.3.3, Chap. 8 are satisfied.

But the heat conduction equation, in the matrix form, which is verified by U can be written, equivalently, in the form of the system

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) + u(t,x) - v(t,x) = 0, \ \forall (t,x) \in (0,T) \times \Omega, \\
&\frac{\partial v}{\partial t}(t,x) - \Delta u(t,x) + v(t,x) = 0, \ \forall (t,x) \in (0,T) \times \Omega.
\end{aligned}$$

Also, the initial condition, satisfied by U, can be written in the form

$$U(0,x) = U_0(x) = \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix}, \ \forall x \in \Omega.$$

If we introduce the functions φ and ψ by

$$\begin{aligned}
\varphi(t,x) &= e^t u(t,x), \\
\psi(t,x) &= e^t v(t,x),
\end{aligned}$$

which then will be re-denoted by $u(x)$ and by $v(x)$, respectively, then we will obtain the following problem:

$$\frac{\partial u}{\partial t}(t,x) - v(t,x) = 0, \ \forall (t,x) \in (0,T) \times \Omega,$$
$$\frac{\partial v}{\partial t}(t,x) - \Delta u(t,x) = 0, \ \forall (t,x) \in (0,T) \times \Omega,$$
$$u(0,x) = u_0(x), \ \forall x \in \Omega,$$
$$\frac{\partial u}{\partial t}(0,x) = u_1(x), \ \forall x \in \Omega.$$

Therefore, we obtained that u is a (unique) weak solution of the Cauchy problem associated with the wave equation.

It is not so difficult to verify that the solution u belongs to the space indicated in the statement of the theorem. ■

Finally, we want to remark that there are more results regarding the regularity of the weak solution for the Cauchy problem, associated with the wave equation. For example, many of them are demonstrated in the book [9].

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