پردازش سیگنال دیجیتال

درس ۲۵

تحلیل فوریه با استفاده از DFT

Fourier Analysis Using the DFT

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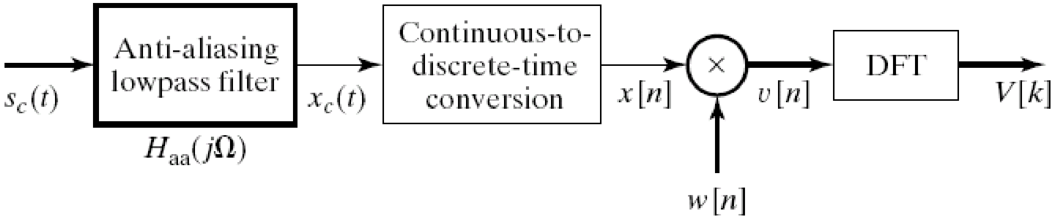
http://courses.fouladi.ir/dsp

Fourier Analysis Using the DFT

Fourier Analysis of Signals Using DFT

- One major application of the DFT: analyze signals
- Let's analyze frequency content of a continuous-time signal



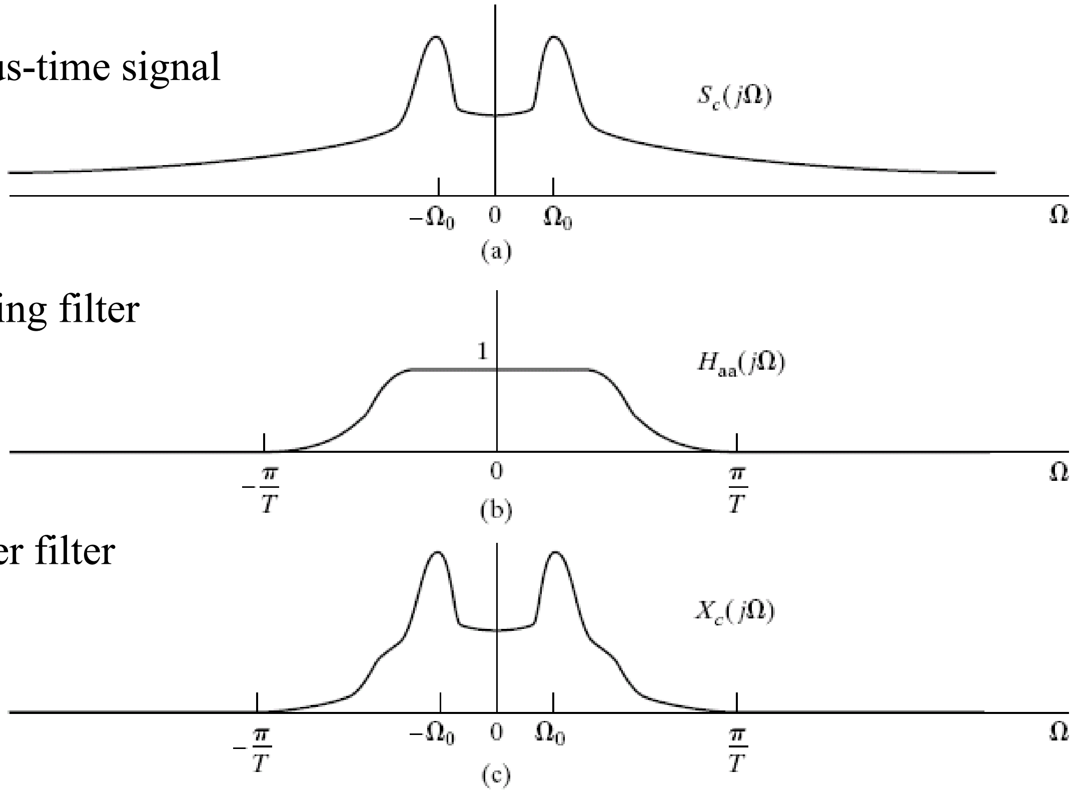
- **Steps to analyze signal with DFT**
  - Remove high-frequencies to prevent aliasing after sampling
  - Sample signal to convert to discrete-time
  - Window to limit the duration of the signal
  - Take DFT of the resulting signal

Example

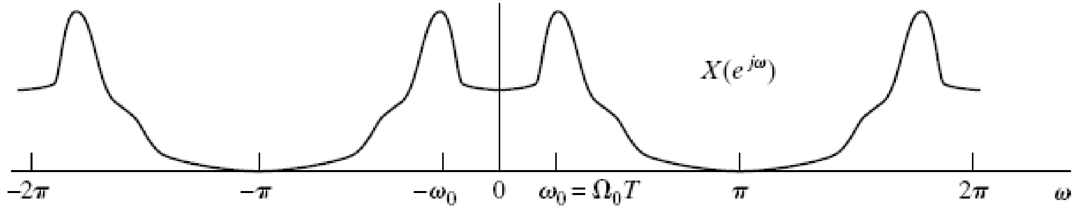
Continuous-time signal

Anti-aliasing filter

Signal after filter

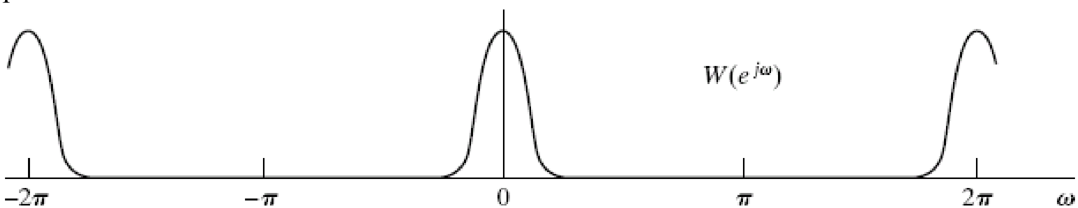


Sampled discrete-time signal



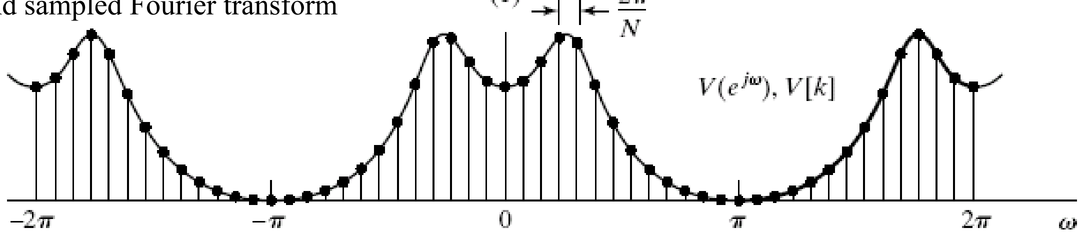
(d)

Frequency response of window



(e)

Windowed and sampled Fourier transform



Effect of Windowing on Sinusoidal Signals

- The effects of anti-aliasing filtering and sampling is known
- We will analyze the effect of **windowing**
- Choose a simple signal to analyze this effect: sinusoids

$$s_c(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 \cos(\Omega_1 t + \theta_1)$$

- Assume ideal sampling and no aliasing we get

$$x[n] = A_0 \cos(\omega_0 n + \theta_0) + A_1 \cos(\omega_1 n + \theta_1)$$

- And after windowing we have

$$v[n] = A_0 w[n] \cos(\omega_0 n + \theta_0) + A_1 w[n] \cos(\omega_1 n + \theta_1)$$

- Calculate the DTFT of $v[n]$ by writing out the cosines as

$$v[n] = \frac{A_0}{2} w[n] e^{j\theta_0} e^{j\omega_0 n} + \frac{A_0}{2} w[n] e^{-j\theta_0} e^{-j\omega_0 n} + \frac{A_1}{2} w[n] e^{j\theta_1} e^{j\omega_1 n} + \frac{A_1}{2} w[n] e^{-j\theta_1} e^{-j\omega_1 n}$$

$$V(e^{j\omega}) = \frac{A_0}{2} e^{j\theta_0} W\left(e^{j(\omega-\omega_0)n}\right) + \frac{A_0}{2} e^{-j\theta_0} W\left(e^{j(\omega+\omega_0)n}\right)$$
$$+ \frac{A_1}{2} e^{j\theta_1} W\left(e^{j(\omega-\omega_1)n}\right) + \frac{A_1}{2} e^{-j\theta_1} W\left(e^{j(\omega+\omega_1)n}\right)$$

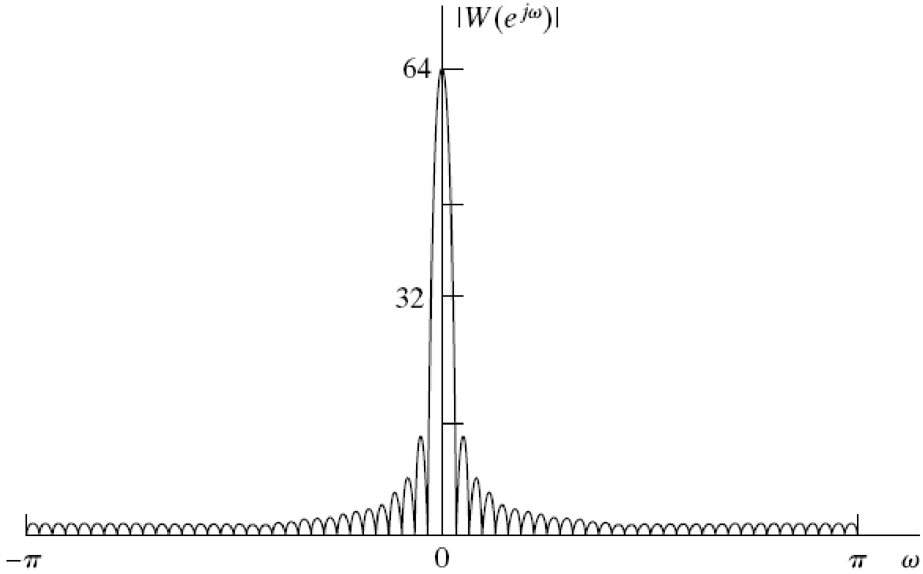
DTFT of windowed signal: consists of the **Fourier transform of the window**, shifted to the frequencies $\pm\omega_0$ and $\pm\omega_1$ and scaled by the complex amplitudes of the individual complex exponentials that make up the signal.

Example

- Consider a rectangular window $w[n]$ of length 64
- Assume $1/T = 10$ kHz, $A_0 = 1$ and $A_1 = 0.75$ and phases to be zero

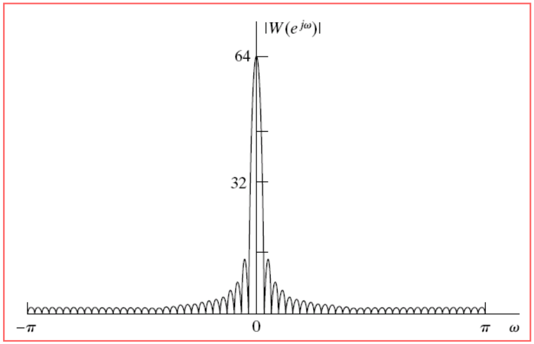
$$s_c(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 \cos(\Omega_1 t + \theta_1) \quad A_0 = 1, A_1 = 0.75, \theta_1 = \theta_2 = 0$$

- Magnitude of the DTFT of the window

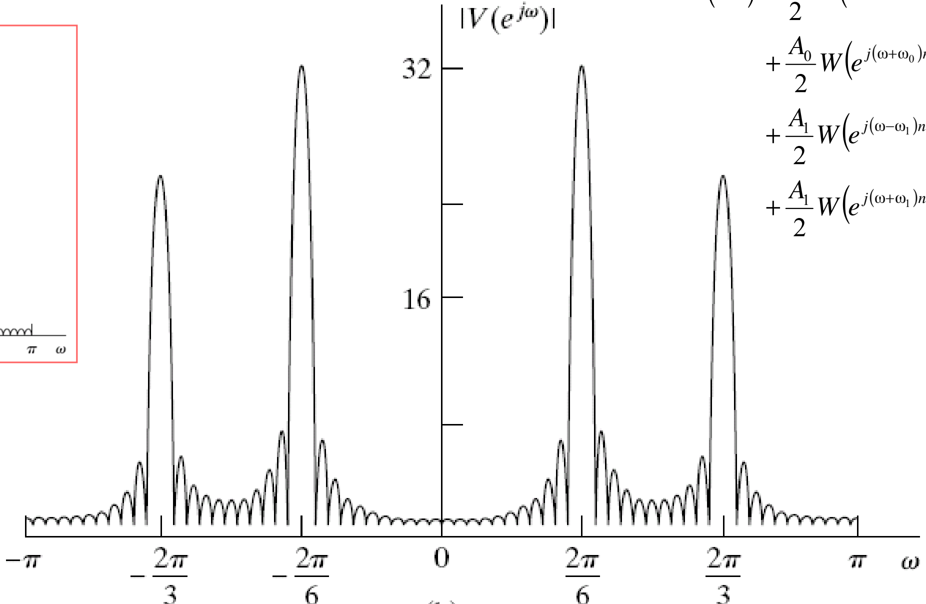


- Magnitude of the DTFT of the sampled signal

$$V\left(e^{j\omega}\right) = \frac{A_0}{2}W\left(e^{j(\omega-\omega_0)n}\right) + \frac{A_0}{2}W\left(e^{j(\omega+\omega_0)n}\right) + \frac{A_1}{2}W\left(e^{j(\omega-\omega_1)n}\right) + \frac{A_1}{2}W\left(e^{j(\omega+\omega_1)n}\right)$$



$$v[n] = \frac{A_0}{2}w[n]e^{j\omega_0 n} + \frac{A_0}{2}w[n]e^{-j\omega_0 n} + \frac{A_1}{2}w[n]e^{j\omega_1 n} + \frac{A_1}{2}w[n]e^{-j\omega_1 n}$$



(b)

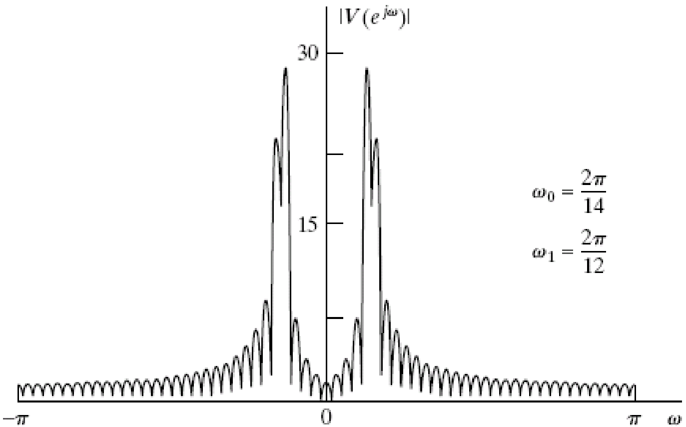
- We expect to see dirac function at input frequencies
- Due to windowing we see, instead, the response of the window
- Note that both **tones** will affect each other due to the smearing
  - This is called leakage: pretty small in this example

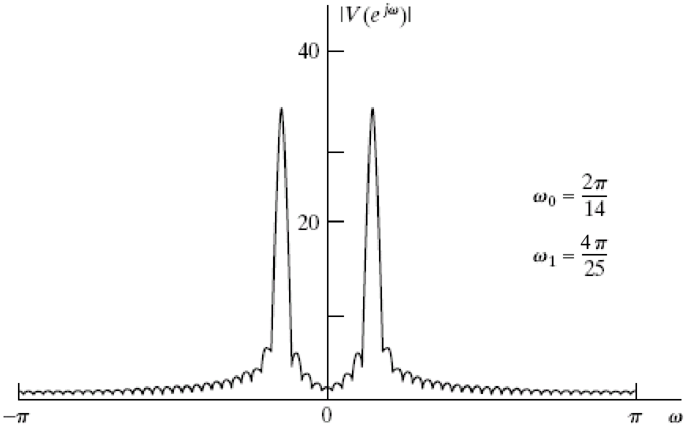
$$\omega_0 = \frac{2\pi}{6}$$

$$\omega_1 = \frac{2\pi}{3}$$

Example Cont'd

- If we the input **tones** are close to each other



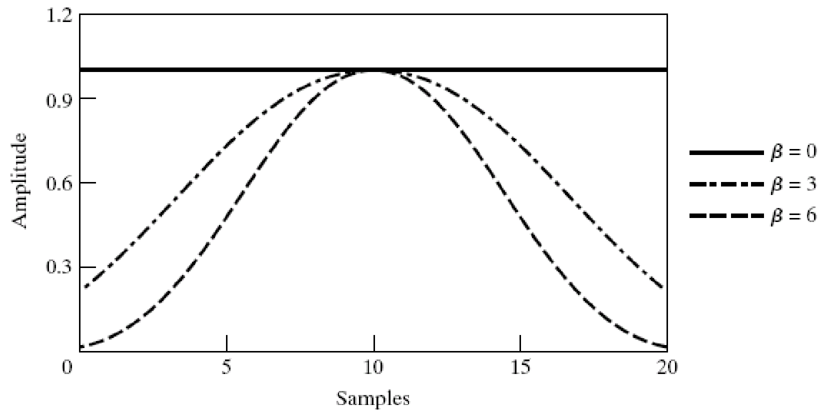


- On the left: the tones are so close that they have considerable affect on each others magnitude

- On the right: the tones are too close to even separate in this case
  - They cannot be resolved using this particular window

Window Functions

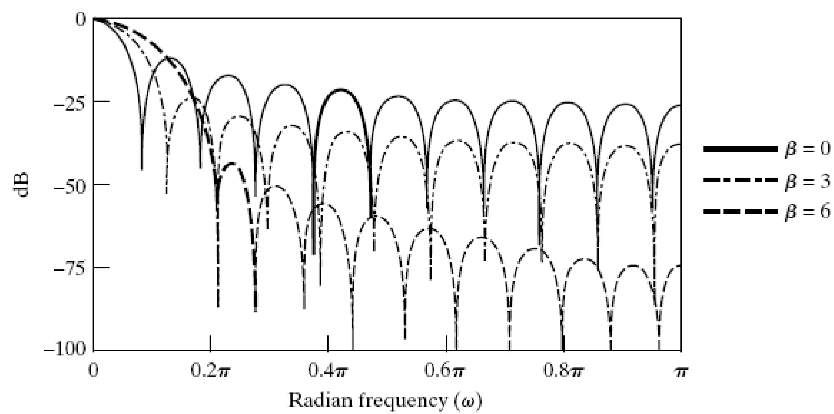
- Two factors are determined by the window function
  - **Resolution:** influenced mainly by the main lobe width
  - **Leakage:** relative amplitude of side lobes versus main lobe



(a)

- We know from filter design chapter that we can choose various windows to trade-off these two factors

- Example:
  - Kaiser window



The Effect of Spectral Sampling

- DFT samples the DTFT with N equally spaced samples at

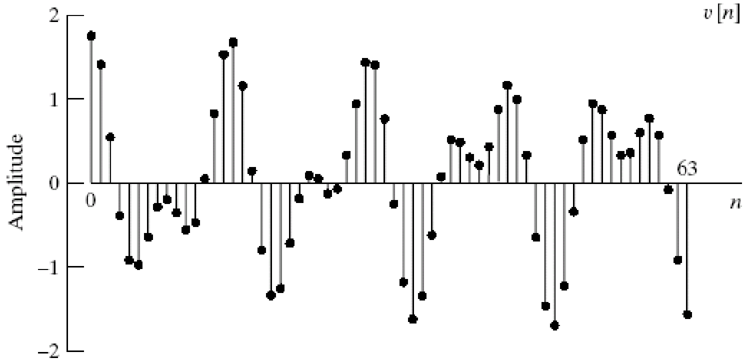
$$\omega_k = \frac{2\pi k}{N} \qquad k = 0,1,\ldots,N-1$$

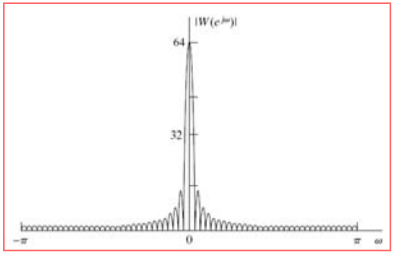
- Or in terms of continuous-frequency

$$\Omega_k = \frac{2\pi k}{NT} \qquad k = 0,1,\ldots,N/2$$

- Example: Signal after windowing

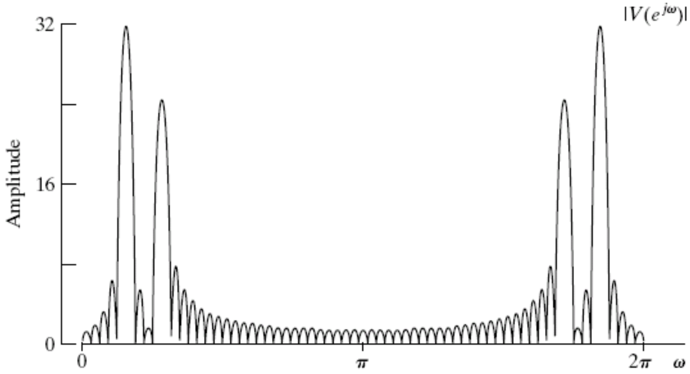
$$v[n] = \begin{cases} \cos\left(\frac{2\pi}{14}n\right) + 0.75\cos\left(\frac{4\pi}{15}n\right) & 0 \le n \le 63 \\ 0 & \text{otherwise} \end{cases}$$

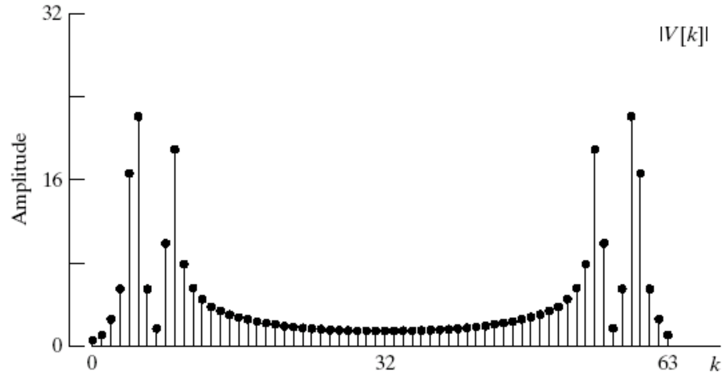




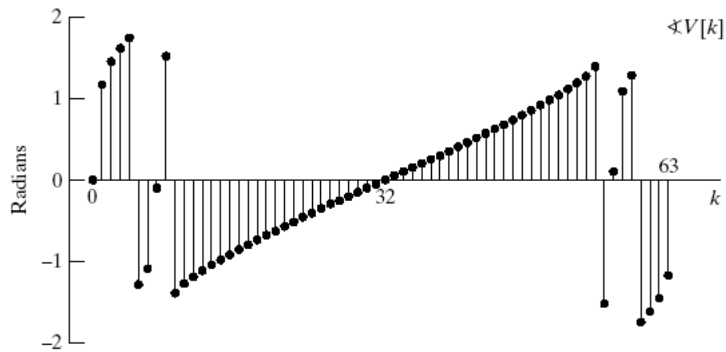
w[n]:
a rectangular window with
64 samples width.

Example Cont'd





(d)



- Note the peak of the DTFTs are in between samples of the DFT

$$\omega_1 = \frac{2\pi}{14} = \frac{2\pi}{64}k \Rightarrow k = 4.5714$$

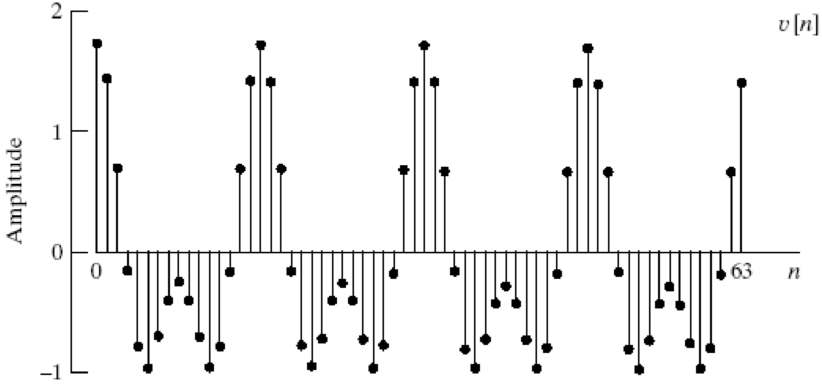
$$\omega_2 = \frac{4\pi}{15} = \frac{2\pi}{64}k \Rightarrow k = 8.5333$$

- DFT doesn't necessary reflect real magnitude of spectral peaks

Example Cont'd

- Let's consider another sequence after windowing we have

$$v[n] = \begin{cases} \cos\left(\frac{2\pi}{16}n\right) + 0.75\cos\left(\frac{2\pi}{8}n\right) & 0 \le n \le 63 \\ 0 & \text{otherwise} \end{cases}$$

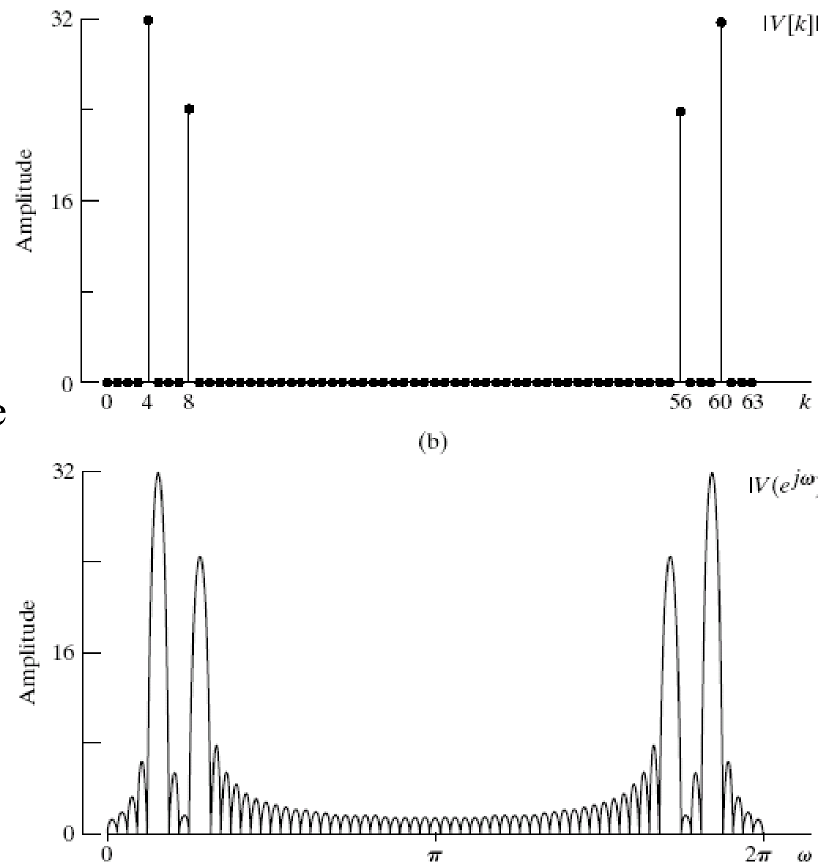


$$\omega_1 = \frac{2\pi}{16} = \frac{2\pi}{64}k \Rightarrow k = 4$$

$$\omega_2 = \frac{2\pi}{8} = \frac{2\pi}{64}k \Rightarrow k = 8$$

Example Cont'd

- In this case N samples cover exactly 4 and 8 periods of the tones
- The samples correspond to the peak of the lobes
- The magnitude of the peaks are accurate
- Note that we don't see the side lobes in this case

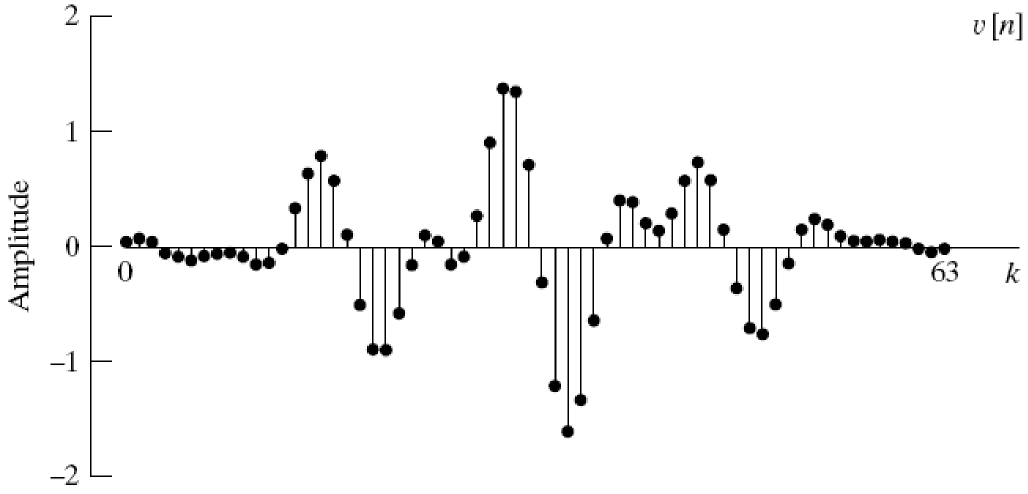


Example: DFT Analysis with Kaiser Window

- The windowed signal is given as

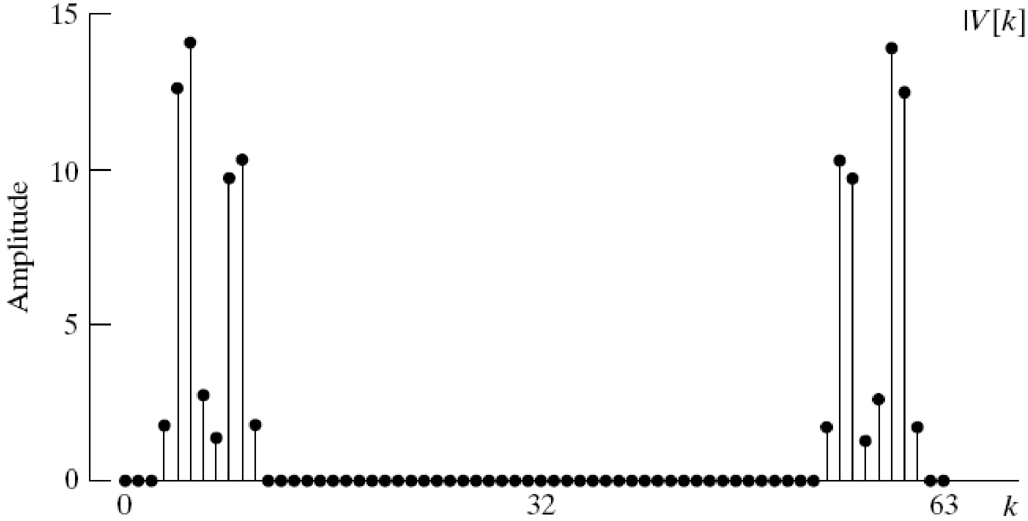
$$v[n] = w_K[n]\cos\left(\frac{2\pi}{14}n\right) + 0.75 w_K[n]\cos\left(\frac{4\pi}{15}n\right)$$

- Where $w_K[n]$ is a Kaiser window with $\beta = 5.48$ for a relative side lobe amplitude of -40 dB
- The windowed signal



Example Cont'd

- DFT with this Kaiser window



- The two tones are clearly resolved with the Kaiser window