پردازش سیگنال دیجیتال

درس ۲۳

تبدیل فوریه‌ی گسسته

The Discrete Fourier Transform

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The Discrete Fourier Transform

Sampling the Fourier Transform

- Consider an aperiodic sequence with a Fourier transform

$$x[n] \xleftrightarrow{\text{DTFT}} X\left(e^{j\omega}\right)$$

- Assume that a sequence is obtained by **sampling** the **DTFT**

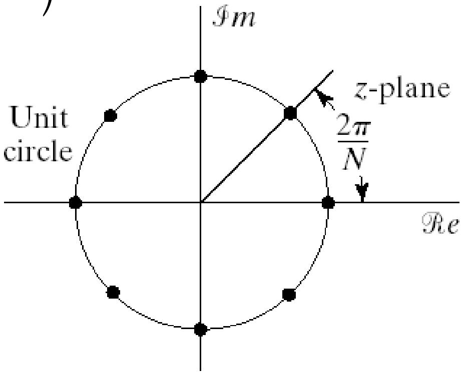
$$\tilde{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega=(2\pi/N)k} = X\left(e^{j(2\pi/N)k}\right)$$

- Since the DTFT is periodic resulting sequence is also periodic
- We can also write it in terms of the z-transform

$$\tilde{X}[k] = X(z)\Big|_{z=e^{(2\pi/N)k}} = X\left(e^{j(2\pi/N)k}\right)$$

- The sampling points are shown in figure
- $\tilde{X}[k]$ could be the DFS of a sequence
- Write the corresponding sequence

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}$$



Sampling the Fourier Transform Cont'd

- The only assumption made on the sequence is that DTFT exists

$$X\left(e^{j\omega}\right)=\sum_{m=-\infty}^{\infty}x[m]e^{-j\omega m} \qquad \tilde{X}[k]=X\left(e^{j(2\pi/N)k}\right) \qquad \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}$$

- Combine equation to get

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty}x[m]e^{-j(2\pi/N)km}\right]e^{j(2\pi/N)kn}$$

$$=\sum_{m=-\infty}^{\infty}x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1}e^{j(2\pi/N)k(n-m)}\right]=\sum_{m=-\infty}^{\infty}x[m]\tilde{p}[n-m]$$

- Term in the parenthesis is

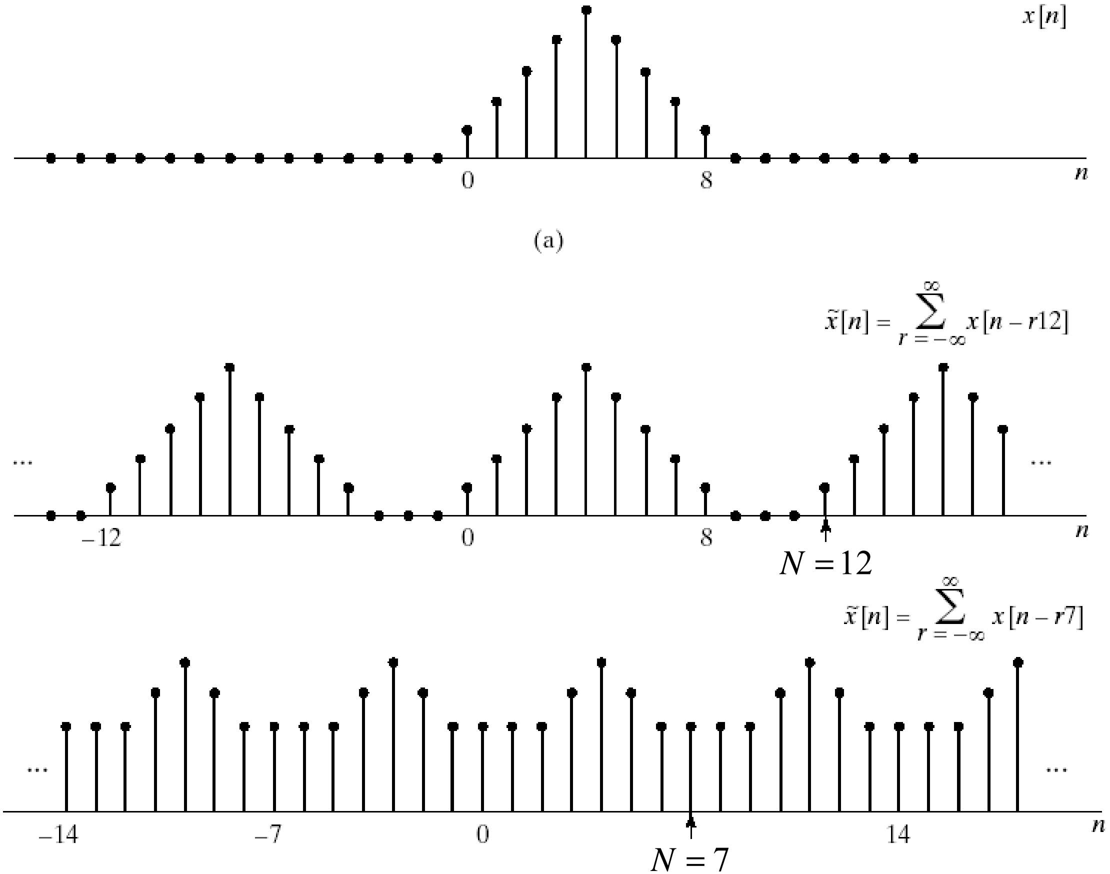
$$\frac{1}{N}\sum_{n=0}^{N-1}e^{j(2\pi/N)(k-r)n}=\begin{cases}1, & k-r=mN, \quad m \text{ an integer},\\ 0, & \text{otherwise}.\end{cases}$$

$$\tilde{p}[n-m]=\frac{1}{N}\sum_{k=0}^{N-1}e^{j(2\pi/N)k(n-m)}=\sum_{r=-\infty}^{\infty}\delta[n-m-rN]$$

- So we get

$$\tilde{x}[n]=x[n]*\sum_{r=-\infty}^{\infty}\delta[n-rN]=\sum_{r=-\infty}^{\infty}x[n-rN]$$

Sampling the Fourier Transform Cont'd



Sampling the Fourier Transform Cont'd

- Samples of the DTFT of an **aperiodic sequence**
  - can be thought of as DFS coefficients of a **periodic sequence**
  - obtained through summing periodic replicas of original sequence
- If the original sequence
  - is of finite length
  - and we take sufficient number of **samples of its DTFT**
  - the original sequence can be recovered by

$$x[n] = \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

- It is not necessary to know the DTFT at all frequencies
  - to recover the discrete-time sequence in time domain
- **Discrete Fourier Transform**
  - Representing a finite length sequence by samples of DTFT

The Discrete Fourier Transform

- Consider a finite length sequence $x[n]$ of length N

$$x[n] = 0 \quad \text{outside of} \quad 0 \le n \le N-1$$

- For given length-N sequence associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

- The DFS coefficients of the periodic sequence are samples of the DTFT of $x[n]$
- Since $x[n]$ is of length N there is no overlap between terms of $x[n-rN]$ and we can write the periodic sequence as

$$\tilde{x}[n] = x[(n \mod N)] = x[((n))_N]$$

- To maintain **duality** between time and frequency
  - We choose one period of $\tilde{X}[k]$ as the Fourier transform of $x[n]$

$$X[k] = \begin{cases} \tilde{X}[k] & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{X}[k] = X[(k \mod N)] = X[((k))_N]$$

The Discrete Fourier Transform Cont'd

- The DFS pair

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn} \qquad \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

- The summations involve only **one period** so we can write:

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn} & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn} & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

- The Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

- The DFT pair can also be written as

$$x[n] \stackrel{\text{DFT}}{\longleftrightarrow} X[k]$$

The Discrete Fourier Transform Cont'd

- The Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

$$k = 1,2,\ldots,N-1 \qquad n = 1,2,\ldots,N-1$$

- The DFT pair can also be written as

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

Example

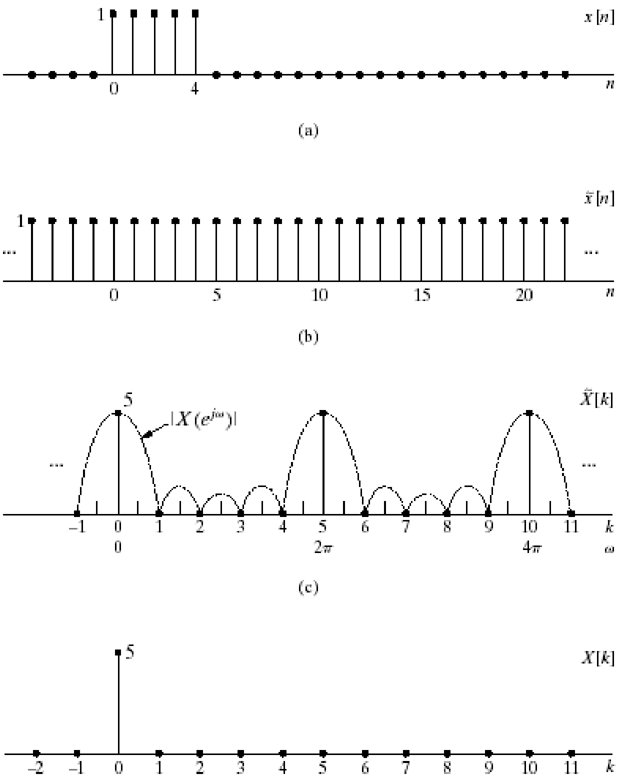
The DFT of a rectangular pulse

- $x[n]$ is of length 5
- We can consider $x[n]$ of any length greater than or equal to 5
- Let's pick $N = 5$
- Calculate the DFS of the periodic form of $x[n]$

$$\tilde{X}[k] = \sum_{n=0}^{4} e^{-j(2\pi k/5)n}$$

$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi k/5)}}$$

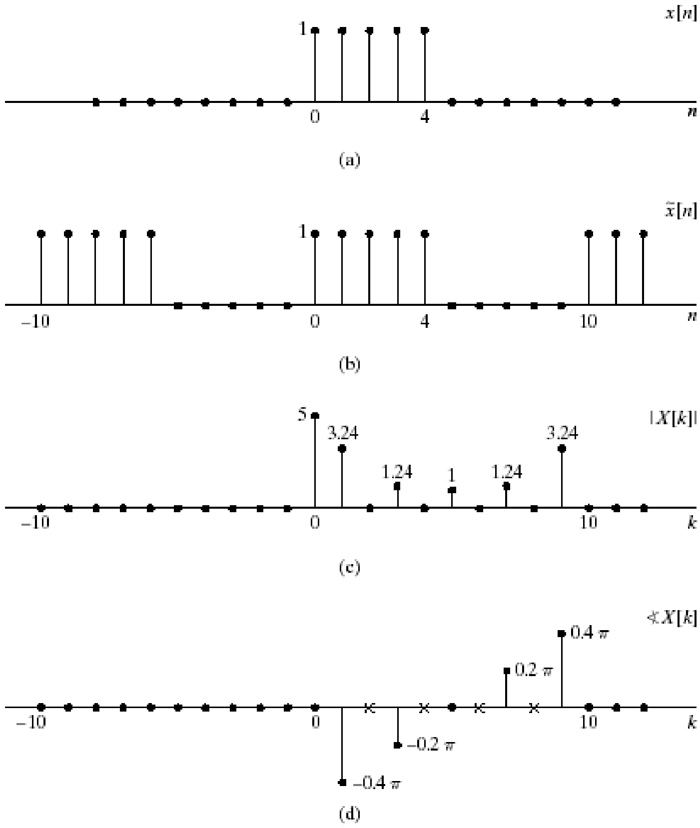
$$= \begin{cases} 5 & k = 0, \pm 5, \pm 10, \ldots \\ 0 & \text{otherwise} \end{cases}$$



Example Cont'd

The DFT of a rectangular pulse

- If we consider $x[n]$ of length 10
  - We get a different set of DFT coefficients
- Still samples of the DTFT but in different places



Properties of DFT

- **Linearity**

$$x_1[n] \overset{DFT}{\longleftrightarrow} X_1[k]$$

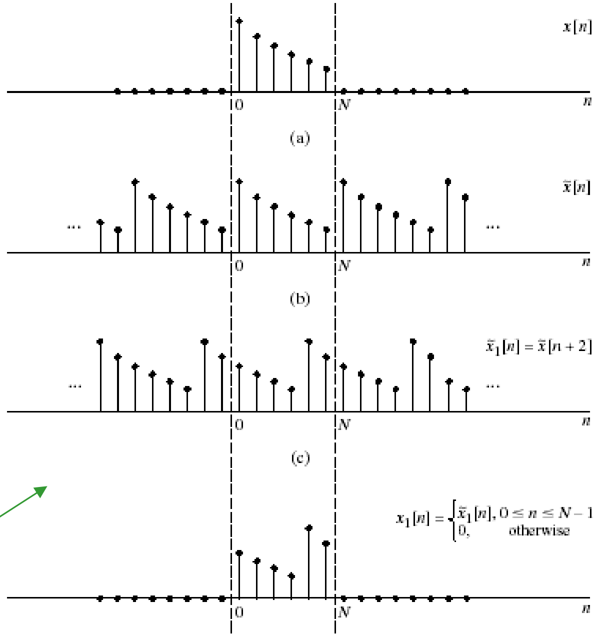
$$x_2[n] \overset{DFT}{\longleftrightarrow} X_2[k]$$

$$ax_1[n]+bx_2[n] \overset{DFT}{\longleftrightarrow} aX_1[k]+bX_2[k]$$

- **Duality**

$$x[n] \overset{DFT}{\longleftrightarrow} X[k]$$

$$X[n] \overset{DFT}{\longleftrightarrow} Nx[((-k))_N]$$

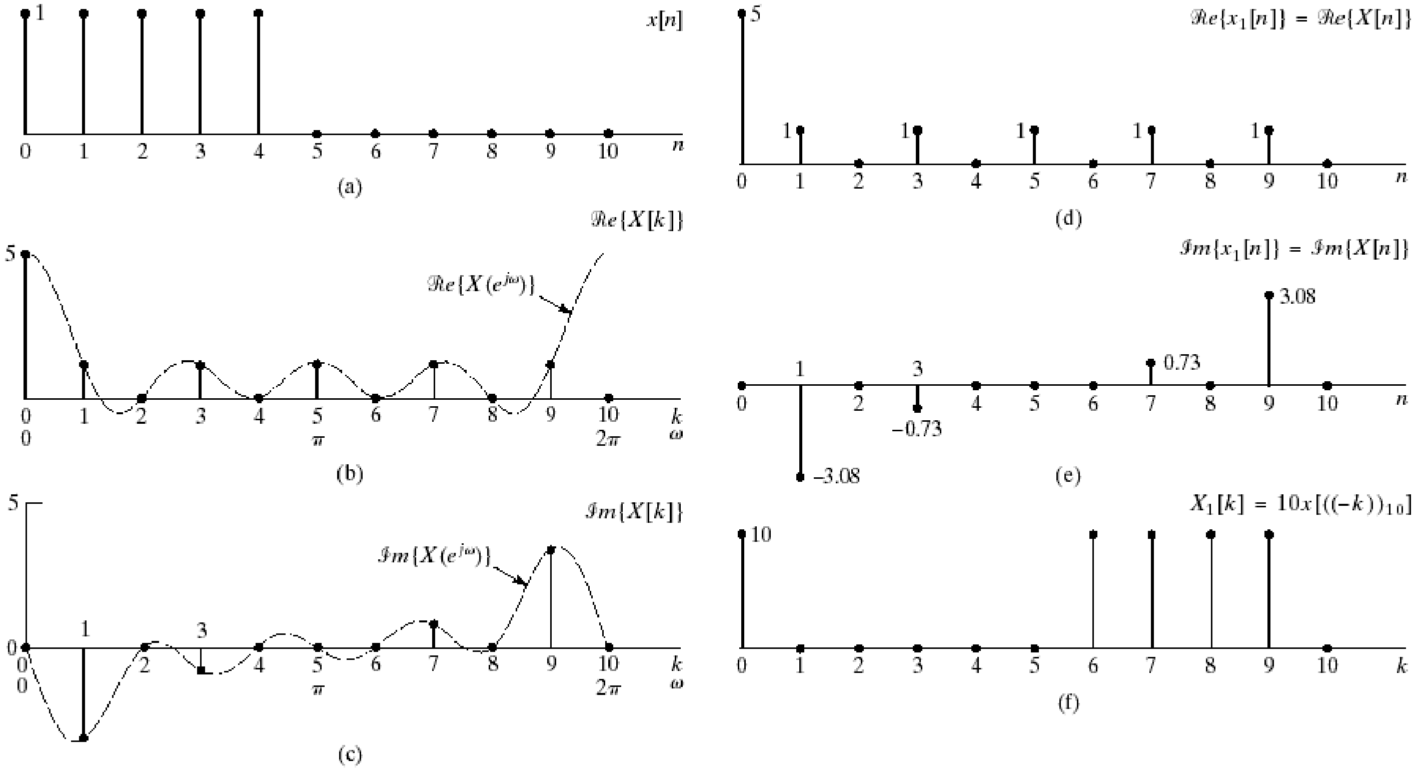


- **Circular Shift** of a Sequence

$$x[n] \overset{DFT}{\longleftrightarrow} X[k]$$

$$x[((n-m))_N] \quad 0 \le n \le N-1 \overset{DFT}{\longleftrightarrow} X[k]e^{-j(2\pi k/N)m}$$

Example: Duality



Properties of DFT

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$	$X_1[k] X_2[k]$
8.	$x_1[n] x_2[n]$	$\frac{1}{N} \sum_{\ell=0}^{N-1} X_1[\ell] X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$

Properties of DFT Cont'd

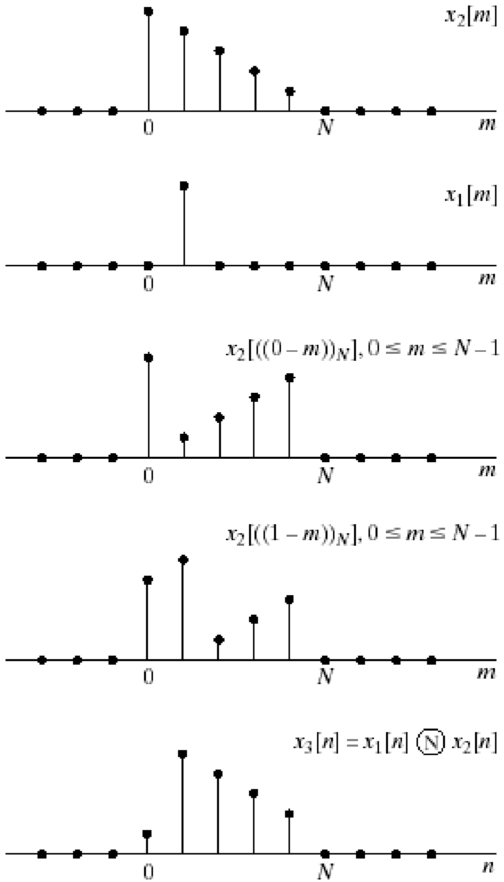
Finite-Length Sequence (Length N)	N-point DFT (Length N)
11. $\mathcal{Re}\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{Im}\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{Re}\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{Im}\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{Re}\{X[k]\} = \mathcal{Re}\{X[((-k))_N]\} \\ \mathcal{Im}\{X[k]\} = -\mathcal{Im}\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \angle\{X[k]\} = -\angle\{X[((-k))_N]\} \end{cases}$
16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{Re}\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{Im}\{X[k]\}$

- Circular convolution of two finite length sequences

$$x_3[n] = x_1[n] \text{ Ⓝ } x_2[n]$$

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

$$x_3[n] = \sum_{m=0}^{N-1} x_2[m] x_1[((n-m))_N]$$



Example

- Circular convolution of two rectangular pulses $L = N = 6$

$$x_1[n] = x_2[n] = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

- DFT of each sequence

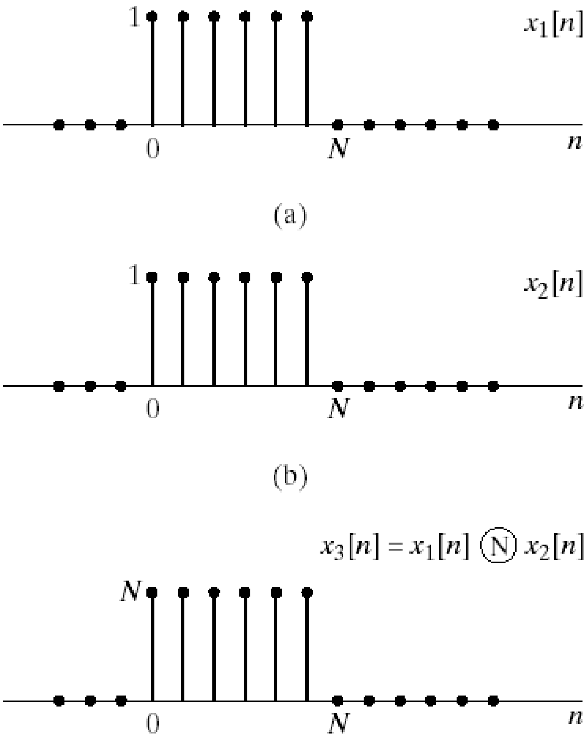
$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} = \begin{cases} N & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

- Multiplication of DFTs

$$X_3[k] = X_1[k]X_2[k] = \begin{cases} N^2 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

- And the inverse DFT

$$x_3[n] = \begin{cases} N & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$



Example

- We can augment zeros to each sequence $N = 2L = 12$

- The DFT of each sequence

$$X_1[k] = X_2[k] = \frac{1 - e^{-j\frac{2\pi Lk}{N}}}{1 - e^{-j\frac{2\pi k}{N}}}$$

- Multiplication of DFTs

$$X_3[k] = \left(\frac{1 - e^{-j\frac{2\pi Lk}{N}}}{1 - e^{-j\frac{2\pi k}{N}}} \right)^2$$