پردازش سیگنال دیجیتال

درس ۲۱

تقریب بهینه‌ی فیلترهای FIR

Optimum Approximation of FIR Filters

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Optimum Approximation of FIR Filters

Optimum Filter Design

- Filter design by windows is **simple** but **not optimal**
  - Like to design filters with minimal length

- **Optimality Criterion**
  - Window design with rectangular filter is optimal in one sense
    - Minimizes the mean-squared approximation error to desired response
    - But causes large error around discontinuities

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases} \qquad \varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_d\left(e^{j\omega}\right) - H\left(e^{j\omega}\right) \right|^2 d\omega$$

  - Alternative criteria can give better results
    - **Minimax**: Minimize maximum error
    - Frequency-weighted error

- Most popular method: *Parks-McClellan Algorithm*
  - Reformulates filter design problem as function approximation

Function Approximation

- Consider the design of type I FIR filter
  - Assume zero-phase for simplicity
  - Can delay by sufficient amount to make causal

$$h_e[n] = h_e[-n] \longrightarrow A_e\left(e^{j\omega}\right) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n}$$

  - Assume $L = M/2$ an integer

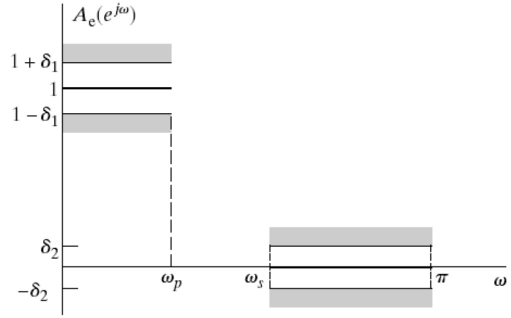
$$A_e\left(e^{j\omega}\right) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n)$$

- After delaying the resulting impulse response

$$h[n] = h_e[n - M/2] = h[M - n] \longrightarrow H\left(e^{j\omega}\right) = A_e\left(e^{j\omega}\right)e^{-j\omega M/2}$$

- **Goal** is to approximate a desired response with $A_e\left(e^{j\omega}\right)$

- Example approximation mask
  - Low-pass filter



Polynomial Approximation

- Using Chebyshev polynomials

$$\cos(\omega n) = T_n(\cos\omega) \quad \text{where} \quad T_n(x) = \cos\left(n\cos^{-1} x\right)$$

- Express the following as a sum of powers

$$A_e\left(e^{j\omega}\right) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

- Can also be represented as

$$A_e\left(e^{j\omega}\right) = P(x)\Big|_{x=\cos\omega} \quad \text{where} \quad P(x) = \sum_{k=0}^{L} a_k x^k$$

- *Parks and McClellan* fix ω_p, ω_s, and L
  - Convert filter design to an **approximation problem**
- The approximation error is defined as

$$E(\omega) = W(\omega)\left[H_d\left(e^{j\omega}\right) - A_e\left(e^{j\omega}\right)\right]$$

  - $W(\omega)$ is the weighting function
  - $H_d(e^{j\omega})$ is the desired frequency response; $A_e(e^{j\omega})$ is approximated frequency response
  - $W(\omega)$ and $H_d(e^{j\omega})$ defined only over the **passpand** and **stopband**
  - **Transition bands** are unconstrained

Lowpass Filter Approximation

- The weighting function for lowpass filter is

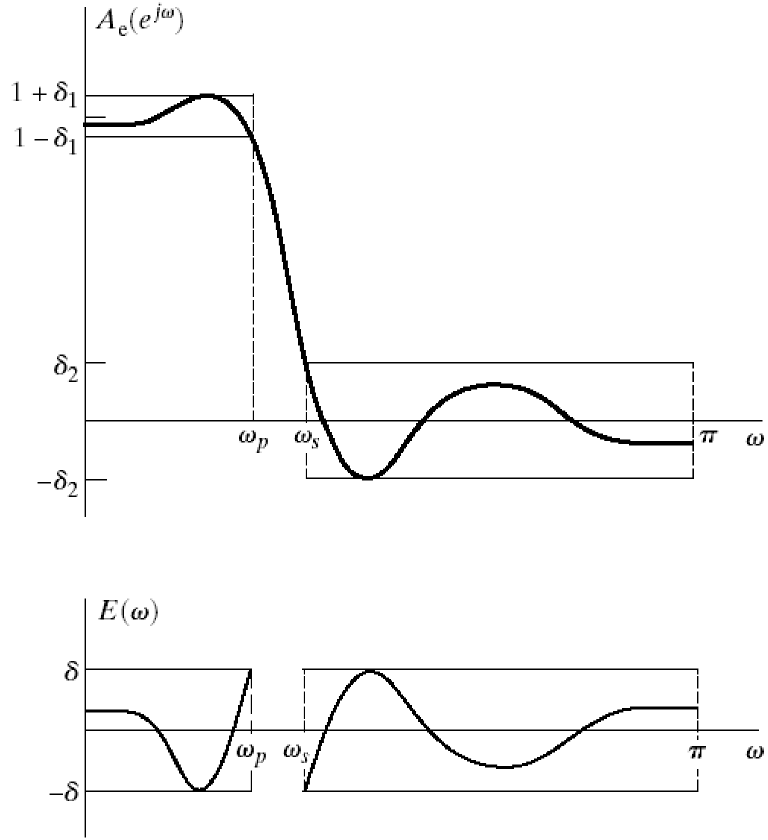
$$W(\omega)=\begin{cases}\dfrac{\delta_2}{\delta_1} & 0\le\omega\le\omega_p\\ 1 & \omega_s\le\omega\le\pi\end{cases}$$

- This choice will force the error to $\delta=\max\{|\delta_1|,|\delta_2|\}=\delta_2$ in both bands

- Criterion used is **minmax**

$$\min_{\{h_e[n]:0\le n\le L\}}\left(\max_{\omega\in F}|E(\omega)|\right)$$

- F is the set of frequencies the approximations is made over



Alternation Theorem (from Approximation Theory)

- F_p denote the closed subset
  - consisting of **the disjoint union of closed subsets of the real axis x**
- The following is an r[th] order polynomial

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

- $D_p(x)$ denotes given desired function that is continuous on F_p
- $W_p(x)$ is a positive function (weighting function) that is continuous on F_p
- The weighted error is given as

$$E_p(x) = W_p(x)\left[D_p(x) - P(x)\right]$$

- The maximum error is defined as

$$\|E\| = \max_{x \in F_p} \left|E_p(x)\right|$$

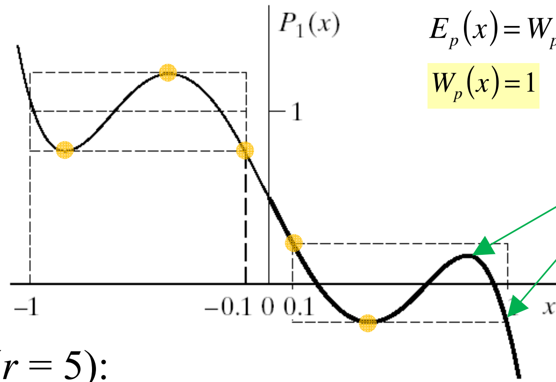
- A necessary and sufficient condition that $P(x)$ be the unique r^{th} order polynomial that minimizes $\|E\|$ is that $E_p(x)$ exhibit at least $(r + 2)$ alternations, i.e.:
- There must be at least $(r + 2)$ values x_i in F_p such that $x_1 < x_2 < \ldots < x_{r+2}$

$$E_p(x_i) = -E_p(x_{i+1}) = \mp\|E\| \;\text{ for }\; i = 1,2,\ldots,(r+2)$$

Alternation Theorem (from Approximation Theory): Example

- Examine polynomials $P(x)$ that approximate

$$D_p(x) = \begin{cases} 1 & \text{for} \quad -1 \le x \le -0.1 \\ 0 & \text{for} \quad 0.1 \le x \le 1 \end{cases}$$



$$E_p(x) = W_p(x)[D_p(x) - P(x)]$$

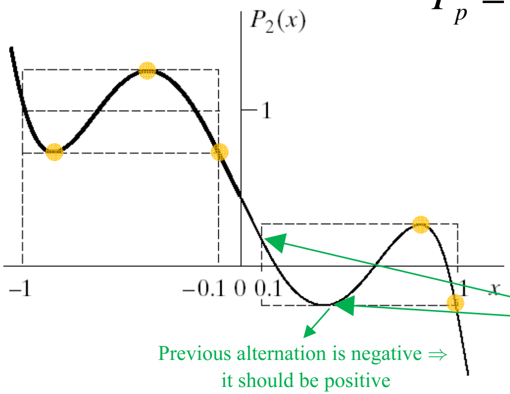
$$W_p(x) = 1$$

Not alternations

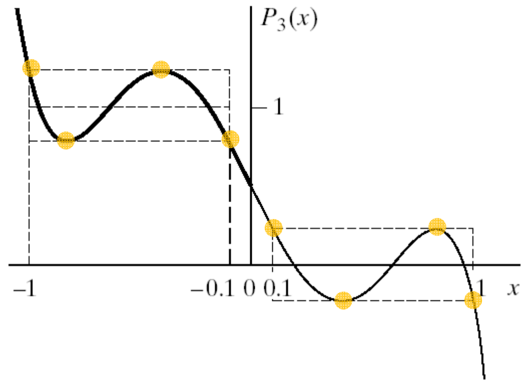
- Fifth order polynomials shown ($r = 5$):
- Which satisfy the theorem? (At least $r + 2 = 7$ alternations)

$$\|E\| = \max_{x \in F_p} |E_p(x)|$$

$$F_p = [-1,-0.1] \cup [0.1,1]$$



Not alternations



Optimal Type I Lowpass Filters

- In this case the $P(x)$ polynomial is the cosine polynomial

$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

- The desired lowpass filter **frequency response** ($x = \cos\omega$)

$$D_p(\cos\omega) = \begin{cases} 1 & \cos\omega_p \le \omega \le 1 \\ 0 & -1 \le \omega \le \cos\omega_s \end{cases}$$

- The weighting function is given as

$$W_p(\cos\omega) = \begin{cases} 1/K & \cos\omega_p \le \omega \le 1 \\ 1 & -1 \le \omega \le \cos\omega_s \end{cases}$$

- The approximation error is given as

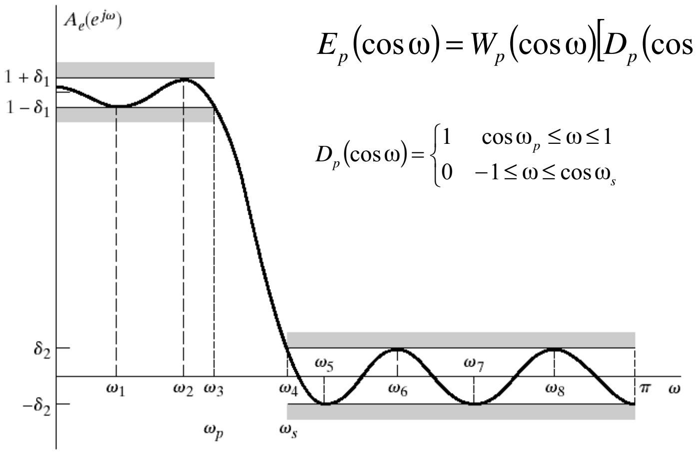
$$E_p(\cos\omega) = W_p(\cos\omega)\left[D_p(\cos\omega) - P(\cos\omega)\right]$$

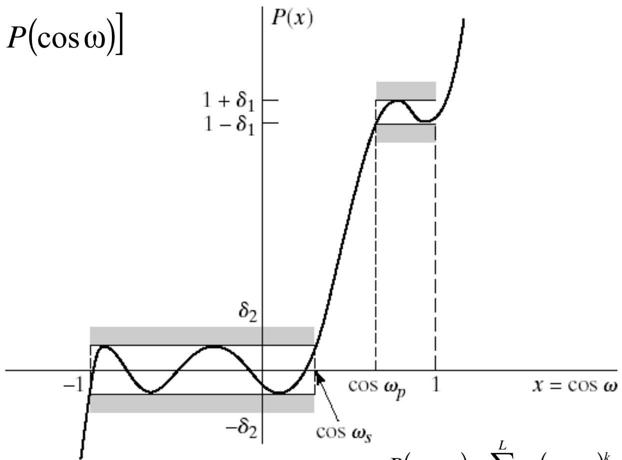
Min. number of alternations in F_p must be L + 2

Typical Example Lowpass Filter Approximation

$$E_p(\cos\omega) = W_p(\cos\omega)\left[D_p(\cos\omega) - P(\cos\omega)\right]$$

$$D_p(\cos\omega) = \begin{cases} 1 & \cos\omega_p \le \omega \le 1 \\ 0 & -1 \le \omega \le \cos\omega_s \end{cases}$$

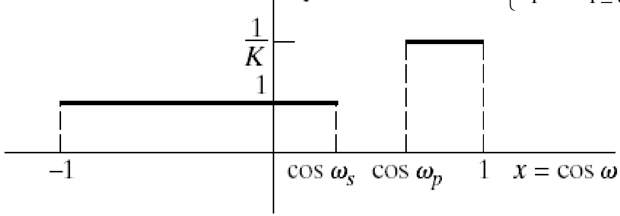


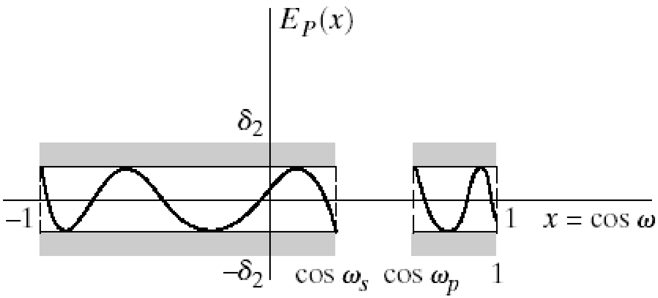


$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

- 7th order approximation

$$W_p(\cos\omega) = \begin{cases} 1/K & \cos\omega_p \le \omega \le 1 \\ 1 & -1 \le \omega \le \cos\omega_s \end{cases}$$





Properties of Type I Lowpass Filters

- Maximum possible number of alternations of the error is $L + 3$
- Alternations will always occur at ω_p and ω_s
- All points with zero slope inside the **passpand** and all points with zero slope inside the **stopband** will correspond to alternations
  - The filter will be equiripple except possibly at 0 and π

Flowchart of Parks-McClellan Algorithm