پردازش سیگنال دیجیتال

درس ۱۸

اثرات عددی دقت متناهی

Finite Precision Numerical Effects

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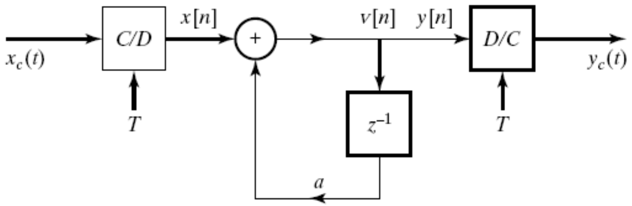
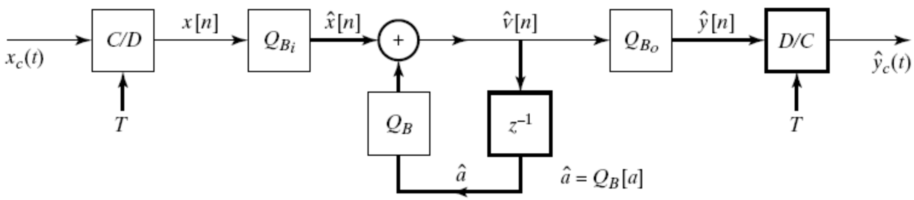
Finite Precision Numerical Effects

Quantization in Implementing Systems

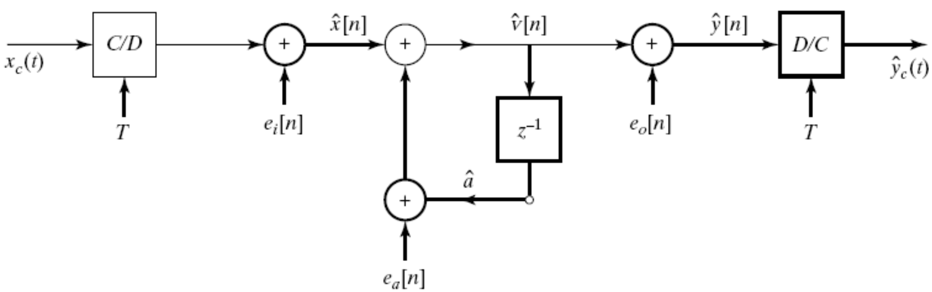
- Consider the following system:



- A more realistic model would be: (non-linear model)



- In order to analyze it we would prefer: (linearized model)



Effects of Coefficient Quantization in IIR Systems

- When the parameters of a rational system are **quantized**
  - The poles and zeros of the system function move
- If the system structure of the system is sensitive to perturbation of coefficients
  - The resulting system may no longer be **stable**
  - The resulting system may no longer **meet the original specs**
- We need to do a detailed **sensitivity analysis**
  - **Quantize** the coefficients and analyze frequency response
  - **Compare** frequency response to original response
- We would like to have a general sense of **the effect of quantization**

Effects on Roots

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}} \xrightarrow{\textbf{Quantization}} \hat{H}(z)=\frac{\sum_{k=0}^{M} \hat{b}_k z^{-k}}{1-\sum_{k=1}^{N} \hat{a}_k z^{-k}}$$

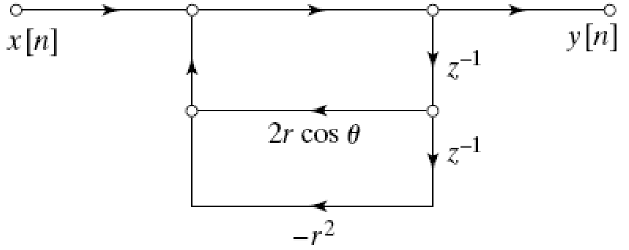
- Each root is affected by quantization errors in ALL coefficient
- Tightly clustered roots can be significantly effected
  - ⇒ Narrow-bandwidth lowpass or bandpass filters can be very sensitive to quantization noise

- The larger the number of roots in a cluster the **more sensitive** it becomes

- *This is the reason why* second order cascade structures *are less sensitive to quantization error than higher order system*
  - Each second order system is independent from each other

Poles of Quantized Second-Order Sections

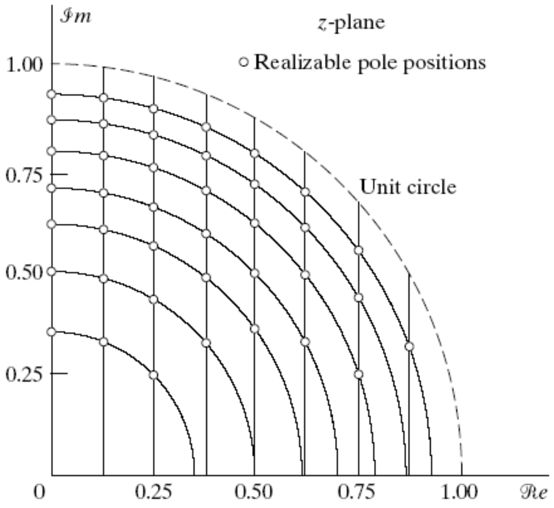
- Consider a 2nd order system with complex-conjugate pole pair



$z_1 = re^{j\theta}, z_2 = re^{-j\theta}$

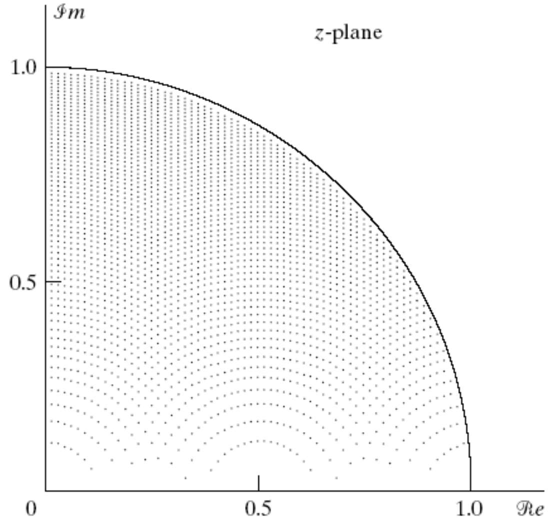
$-r^2, 2r\cos\theta$

- The pole locations after quantization will be **on the grid point**

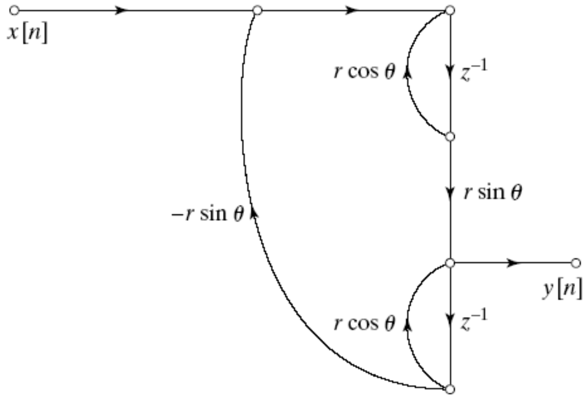


← 4-bits

7-bits →



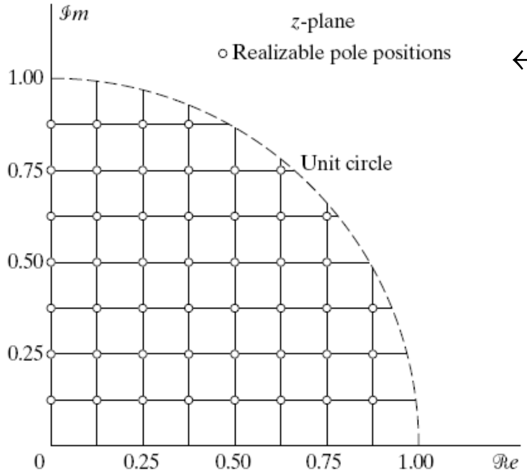
- Equivalent implementation of the second order system



$$z_1 = re^{j\theta}, z_2 = re^{-j\theta}$$

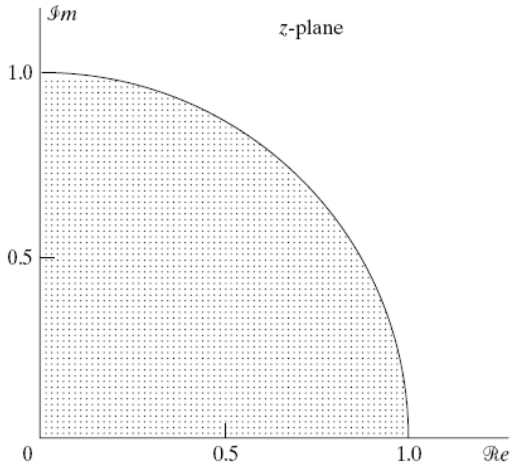
$$z_1 = re^{j\theta} = r\cos\theta + r\sin\theta$$

- But the quantization grid this time is



← 4-bits

7-bits →



Effects of Coefficient Quantization in FIR Systems

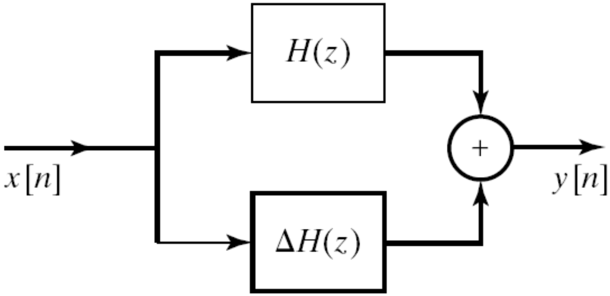
- No poles to worry about only zeros
- Direct form is commonly used for FIR systems

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} \qquad \hat{h}[n] = h[n] + \Delta h[n]$$

- Suppose the coefficients are quantized

$$\hat{H}(z) = \sum_{n=0}^{M} \hat{h}[n] z^{-n} = H(z) + \Delta H(z) \qquad \Delta H(z) = \sum_{n=0}^{M} \Delta h[n] z^{-n}$$

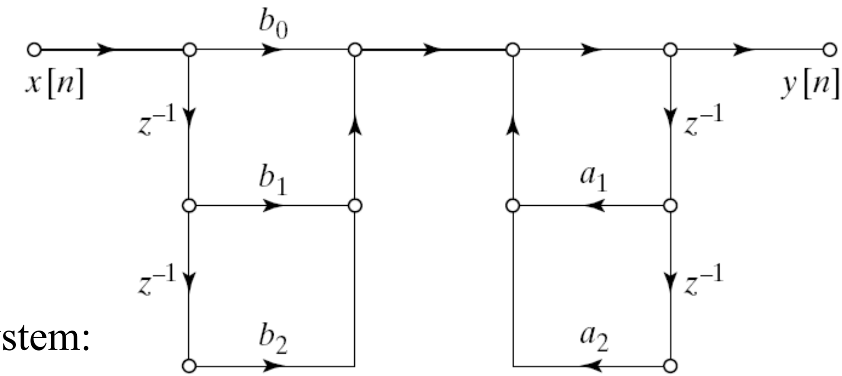
- Quantized system is linearly related to the quantization error



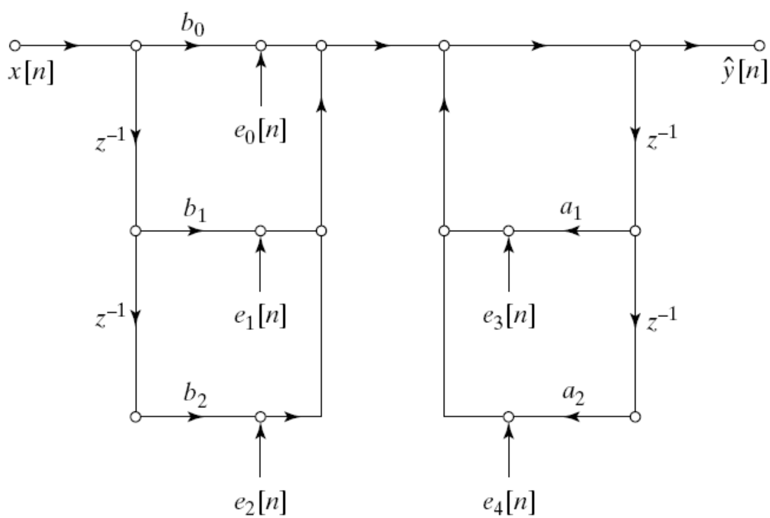
- Again quantization noise is higher for clustered zeros
- However, **most** FIR filters have <u>spread zeros</u>

- Difference equations implemented with finite-precision arithmetic are **non-linear** systems

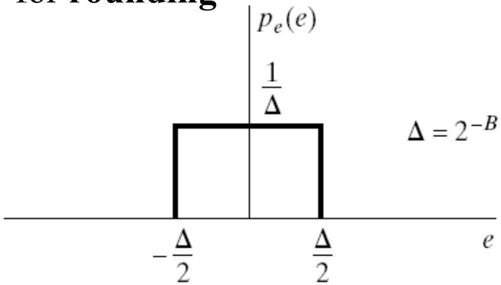
- Second order direct form I system:



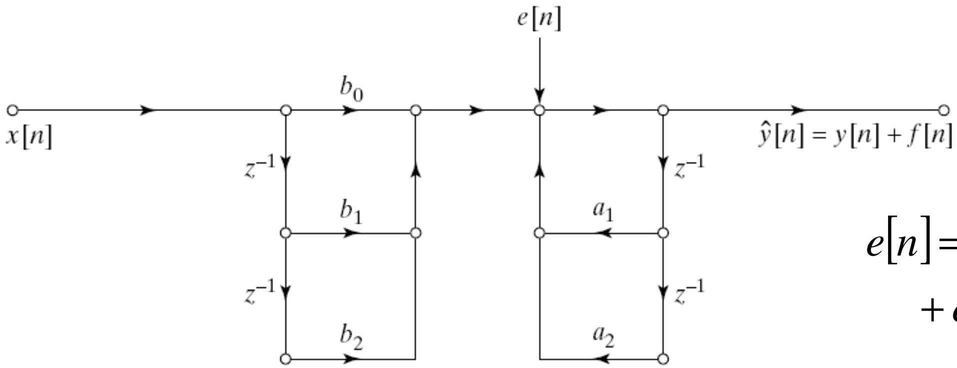
- Model with quantization effect:



- Density function error terms for **rounding**



- Combine all error terms to single location to get



$$e[n] = e_0[n] + e_1[n] + e_2[n] + e_3[n] + e_4[n]$$

- The variance of $e[n]$ in the general case is $\sigma_e^2 = (M+1+N)\dfrac{2^{-2B}}{12}$

- The contribution of $e[n]$ to the output is $f[n] = \sum_{k=1}^{N} a_k f[n-k] + e[n]$

- The variance of the output error term $f[n]$ is

$$\sigma_f^2 = (M+1+N)\frac{2^{-2B}}{12}\sum_{n=-\infty}^{\infty}\left|h_{ef}[n]\right|^2 \qquad H_{ef}(z) = 1/A(z)$$

Round-Off Noise in a First-Order System

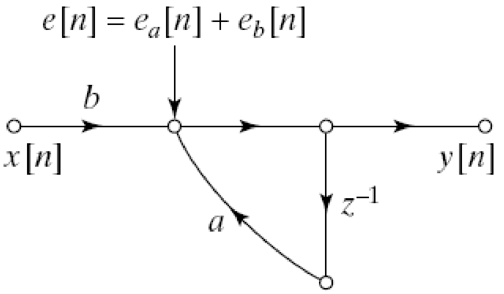
- Suppose we want to implement the following stable system

$$H(z) = \frac{b}{1 - az^{-1}} \qquad |a| < 1$$

- The quantization error noise variance is

$$\sigma_f^2 = (M + 1 + N)\frac{2^{-2B}}{12}\sum_{n=-\infty}^{\infty}|h_{ef}[n]|^2 = 2\frac{2^{-2B}}{12}\sum_{n=0}^{\infty}|a|^{2n} = 2\frac{2^{-2B}}{12}\left(\frac{1}{1-|a|^2}\right)$$

- Noise variance increases as $|a|$ gets closer to the unit circle
- As $|a|$ gets closer to 1 we have to use more bits to compensate for the increasing error

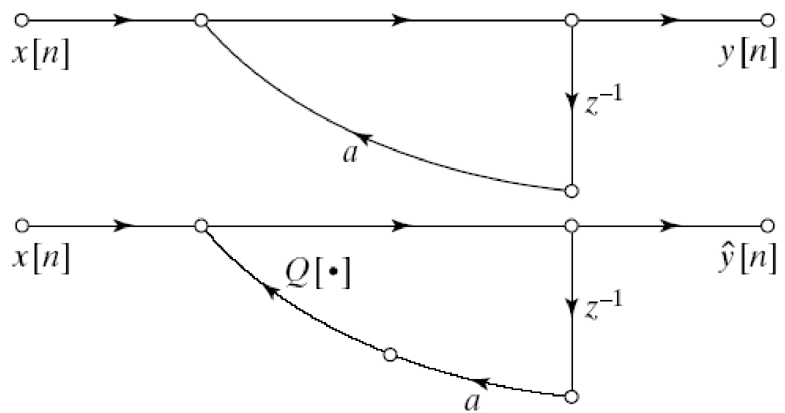


Zero-Input Limit Cycles in Fixed-Point Realization of IIR Filters

- For stable IIR systems the output will decay to zero when the input becomes zero
- A finite-precision implementation, however, may continue to oscillate indefinitely
- Nonlinear behaviour very difficult to analyze so we sill study by example
- Example: **Limite Cycle Behavior in First-Order Systems**

$$y[n] = ay[n-1] + x[n] \qquad |a| < 1$$

- Assume $x[n]$ and $y[n-1]$ implemented by 4 bit



Example Cont'd

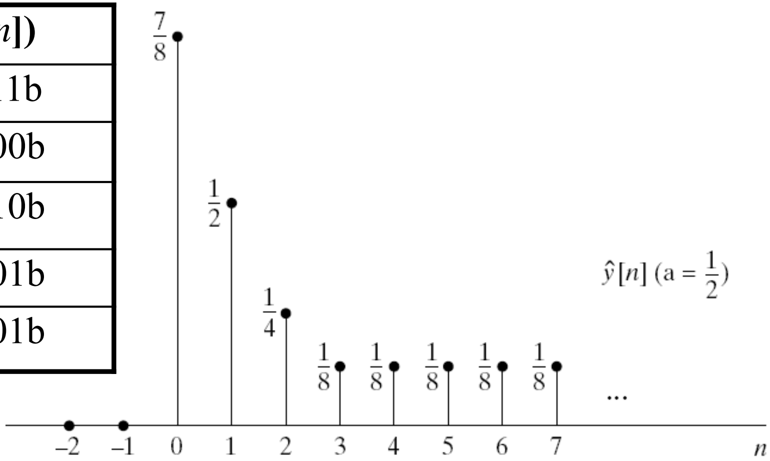
$$y[n] = ay[n-1] + x[n] \qquad |a| < 1$$

- Assume that a = ½ = 0.100b and the input is

$$x[n] = \frac{7}{8}\delta[n] = (0.111b)\delta[n]$$

- If we calculate the output for values of n

n	$y[n]$	$Q(y[n])$
0	7/8 = 0.111b	7/8 = 0.111b
1	7/16 = 0.011100b	1/2 = 0.100b
2	1/4 = 0.010000b	1/4 = 0.010b
3	1/8 = 0.001000b	1/8 = 0.001b
4	1/16 = 0.00010b	1/8 = 0.001b



- A finite input caused an oscillation with period 1

Example: Limite Cycles due to Overflow

- Consider a second-order system realized by

$$\hat{y}[n] = x[n] + Q(a_1 \hat{y}[n-1]) + Q(a_2 \hat{y}[n-2])$$

  - Where $Q()$ represents two's complement rounding
  - Word length is chosen to be 4 bits
- Assume $a_1 = 3/4 = 0.110\text{b}$ and $a_2 = -3/4 = 1.010\text{b}$
- Also assume

$$\hat{y}[-1] = 3/4 = 0.110b \text{ and } \hat{y}[-2] = -3/4 = 1.010b$$

- The output at sample $n = 0$ is

$$\hat{y}[0] = 0.110b \times 0.110\text{b} + 1.010b \times 1.010\text{b}$$
$$= 0.100100\text{b} + 0.100100\text{b}$$

- After rounding up we get

$$\hat{y}[0] = 0.101\text{b} + 0.101\text{b} = 1.010\text{b} = \text{-}3/4$$

- Binary carry overflows into the sign bit changing the sign
- When repeated for $n = 1$

$$\hat{y}[1] = 1.010\text{b} + 1.010\text{b} = 0.110 = 3/4$$

Avoiding Limite Cycles

- **Desirable to get zero output for zero input**: Avoid limit-cycles

- Generally adding more bits would avoid overflow

- Using double-length accumulators at **addition points** would decrease likelihood of limit cycles

- Trade-off between **limit-cycle avoidance** and **complexity**

- **FIR systems** cannot support zero-input limit cycles (no feedback!)
  - because they have no feedback paths. The output of an FIR system will be zero no later than $(M + 1)$ samples after the input goes to zero and remains there.
  - This is a major advantage of FIR systems in applications wherein limit cycle oscillations cannot be tolerated.