پردازش سیگنال دیجیتال

درس ۱۶

فاز خطی تعمیم‌یافته

Generalized Linear Phase

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Generalized Linear Phase

Linear Phase System

- **Ideal Delay System**

$$H_{id}\left(e^{j\omega}\right) = e^{-j\omega\alpha} \qquad |\omega| < \pi$$

- Magnitude, phase, and group delay

$$\left|H_{id}\left(e^{j\omega}\right)\right| = 1$$

$$\angle H_{id}\left(e^{j\omega}\right) = -\omega\alpha$$

$$grd\left[H_{id}\left(e^{j\omega}\right)\right] = \alpha$$

- Impulse response

$$h_{id}[n] = \frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)}$$

- If $\alpha = n_d$ is integer

$$h_{id}[n] = \delta[n - n_d]$$

- For integer α linear phase system delays the input

$$y[n] = x[n] * h_{id}[n] = x[n] * \delta[n - n_d] = x[n - n_d]$$

Linear Phase Systems

- For non-integer α the output is an interpolation of samples
- Easiest way of representing is to think of it in **continuous**

$$h_c(t) = \delta(t - \alpha T) \qquad \text{and} \qquad H_c(j\Omega) = e^{-j\Omega\alpha T}$$

- This representation can be used even if $x[n]$ was not originally derived from a continuous-time signal
- The output of the system is

$$y[n] = x(nT - \alpha T)$$

- Samples of a time-shifted, band-limited interpolation of the input sequence $x[n]$
- A **linear phase system** can be thought as

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{-j\omega\alpha}$$

- A zero-phase system output is delayed by α

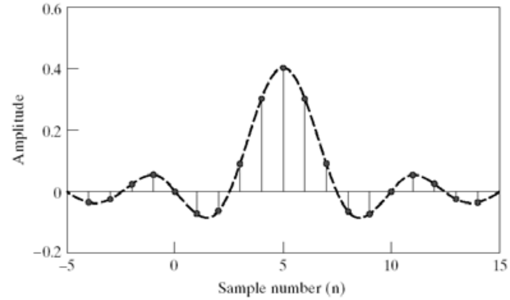
Symmetry of Linear Phase Impulse Responses

- Linear-phase systems

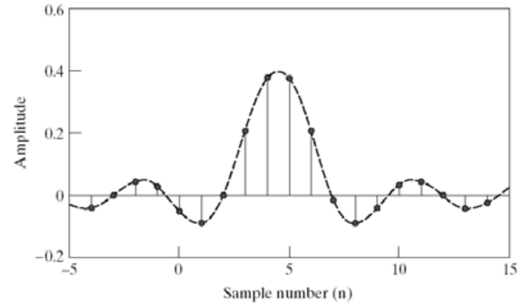
$$H\left(e^{j\omega}\right)=\left|H\left(e^{j\omega}\right)\right|e^{-j\omega\alpha}$$

- If 2α is integer
  - Impulse response symmetric

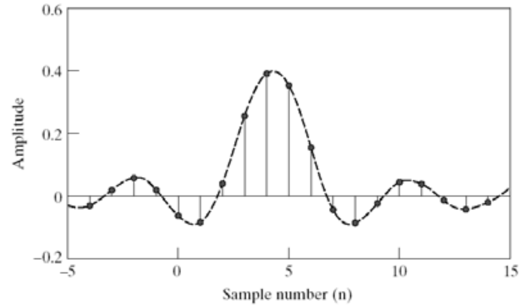
$$h[2\alpha-n]=h[n]$$



$\alpha=5$



$\alpha=4.5$



$\alpha=4.3$

Generalized Linear Phase System

- Generalized Linear Phase

$$H\left(e^{j\omega}\right) = A\left(e^{j\omega}\right)e^{-j\omega\alpha + j\beta}$$

$A\left(e^{j\omega}\right)$: Real function of ω

α and β constants

- Additive constant in addition to linear term
- Has constant group delay

$$\tau(\omega) = grd\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left(\arg\left[H\left(e^{j\omega}\right)\right]\right) = \alpha$$

- And linear phase of general form

$$\arg\left[H\left(e^{j\omega}\right)\right] = \beta - \omega\alpha \qquad 0 \le \omega < \pi$$

Condition for Generalized Linear Phase

- We can write a generalized linear phase system response as

$$H\left(e^{j\omega}\right)=A\left(e^{j\omega}\right)e^{-j\omega\alpha+j\beta}=A\left(e^{j\omega}\right)\cos(\beta-\omega\alpha)+jA\left(e^{j\omega}\right)\sin(\beta-\omega\alpha)$$

$$\boxed{H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}=\sum_{n=-\infty}^{\infty}h[n]\cos(\omega n)-j\sum_{n=-\infty}^{\infty}h[n]\sin(\omega n)}$$

- The phase angle of this system is

$$\tan\left(\arg\left[H\left(e^{j\omega}\right)\right]\right)=\tan(\beta-\omega\alpha)=\frac{\sin(\beta-\omega\alpha)}{\cos(\beta-\omega\alpha)}=\frac{-\sum_{n=-\infty}^{\infty}h[n]\sin(\omega n)}{\sum_{n=-\infty}^{\infty}h[n]\cos(\omega n)}$$

- Cross multiply to get necessary condition for generalized linear phase

$$\sum_{n=-\infty}^{\infty}h[n]\cos(\omega n)\sin(\beta-\omega\alpha)+\sum_{n=-\infty}^{\infty}h[n]\sin(\omega n)\cos(\beta-\omega\alpha)=0$$

$$\sum_{n=-\infty}^{\infty}h[n][\cos(\omega n)\sin(\beta-\omega\alpha)+\sin(\omega n)\cos(\beta-\omega\alpha)]=0$$

$$\sum_{n=-\infty}^{\infty}h[n]\sin(\beta-\omega\alpha+\omega n)=\sum_{n=-\infty}^{\infty}h[n]\sin[\beta+\omega(n-\alpha)]=0$$

Symmetry of Generalized Linear Phase

- **Necessary condition** for generalized linear phase

$$\forall\omega \quad \sum_{n=-\infty}^{\infty} h[n]\sin[\beta + \omega(n-\alpha)] = 0$$

- For $\beta = 0$ or π

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha)] = 0 \longrightarrow h[2\alpha - n] = h[n]$$

- For $\beta = \pi/2$ or $3\pi/2$

$$\sum_{n=-\infty}^{\infty} h[n]\cos[\omega(n-\alpha)] = 0 \longrightarrow h[2\alpha - n] = -h[n]$$

Causal Generalized Linear-Phase System

- If the system is **causal** and generalized linear-phase

$$h[M-n] = \mp h[n]$$

- Since $h[n] = 0$ for $n < 0$ we get

$$h[n] = 0 \qquad n < 0 \quad \text{and} \quad n > M$$

An FIR impulse response of length $M + 1$ is generalized linear phase
if
it is **symmetric**

- Here M is an even integer

Type I FIR Linear-Phase System

- **Type I** system is defined with symmetric impulse response

$$h[n] = h[M-n] \quad \text{for } 0 \le n \le M$$

  - M is an **even** integer

- The frequency response can be written as

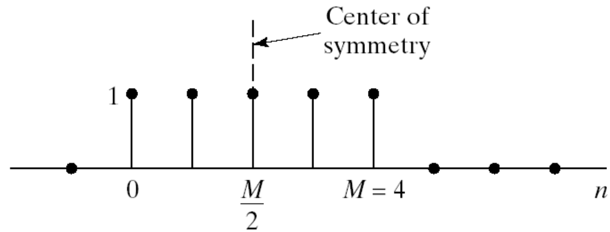
$$H\left(e^{j\omega}\right) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

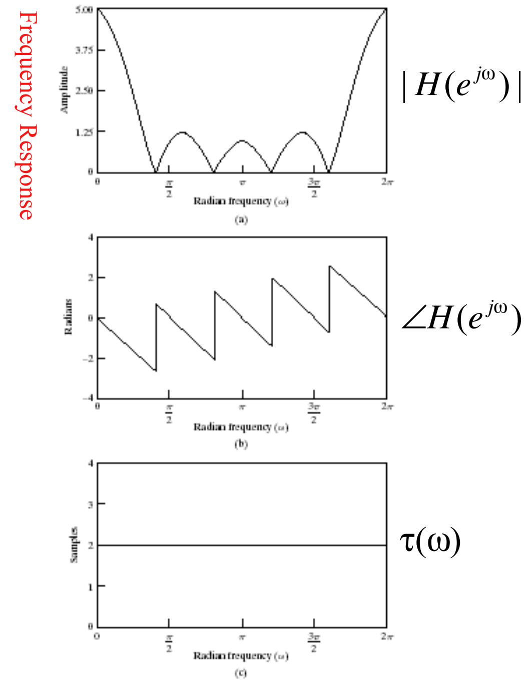
$$= e^{-j\omega M/2}\left[\sum_{n=0}^{M/2} a[n]\cos(\omega n)\right]$$

- Where

$$a[0] = h[M/2]$$

$$a[k] = 2h[M/2-k] \quad \text{for } k = 1,2,\ldots,M/2$$





Type II FIR Linear-Phase System

- **Type II** system is defined with symmetric impulse response

$$h[n] = h[M-n] \quad \text{for } 0 \le n \le M$$

  - M is an **odd** integer

- The frequency response can be written as

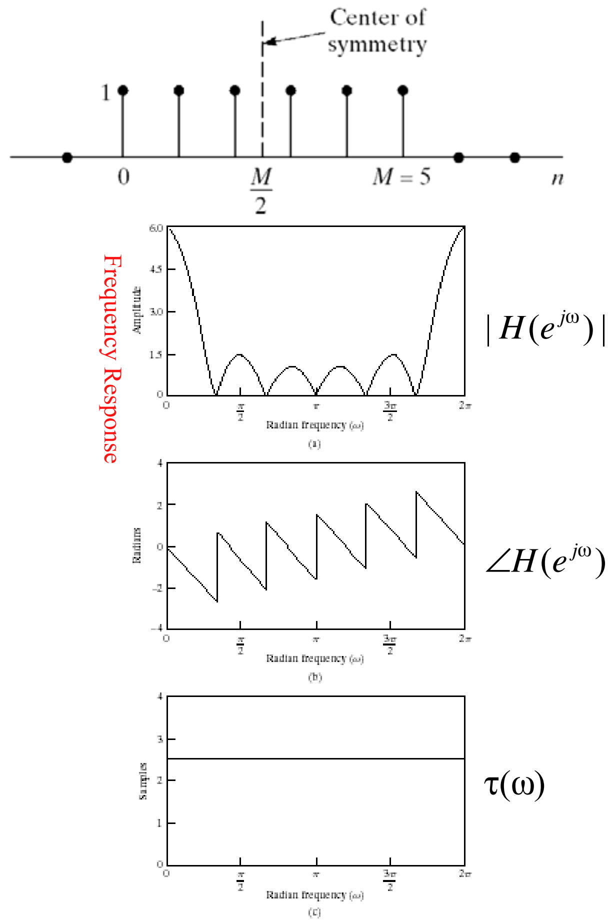
$$H\left(e^{j\omega}\right) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

$$= e^{-j\omega M/2}\left[\sum_{n=1}^{(M+1)/2} b[n]\cos\left(\omega\left(n - \frac{1}{2}\right)\right)\right]$$

- Where

$$b[k] = 2h[(M+1)/2 - k]$$

$$\text{for } k = 1,2,\ldots,(M+1)/2$$



Type III FIR Linear-Phase System

- **Type III** system is defined with symmetric impulse response

$$h[n] = -h[M-n] \quad \text{for } 0 \le n \le M$$

  – M is an **even** integer

- The frequency response can be written as

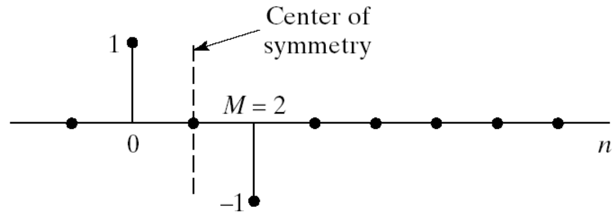
$$H\left(e^{j\omega}\right) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

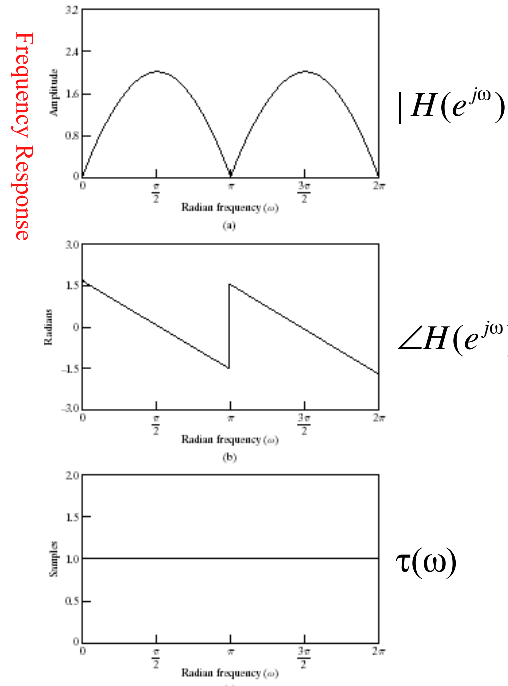
$$= je^{-j\omega M/2}\left[\sum_{n=1}^{M/2} c[n]\sin(\omega n)\right]$$

- Where

$$c[k] = 2h[M/2-k]$$

for $k = 1,2,\ldots, M/2$





Type IV FIR Linear-Phase System

- **Type IV** system is defined with symmetric impulse response

$$h[n] = -h[M-n] \quad \text{for } 0 \le n \le M$$

  – M is an **odd** integer

- The frequency response can be written as

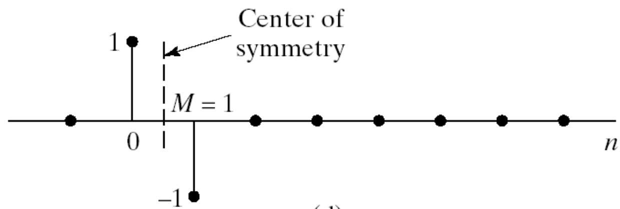
$$H\left(e^{j\omega}\right) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

$$= j e^{-j\omega M/2} \left[\sum_{n=1}^{(M+1)/2} d[n] \sin\left(\omega \left(n - \frac{1}{2} \right) \right) \right]$$

- Where

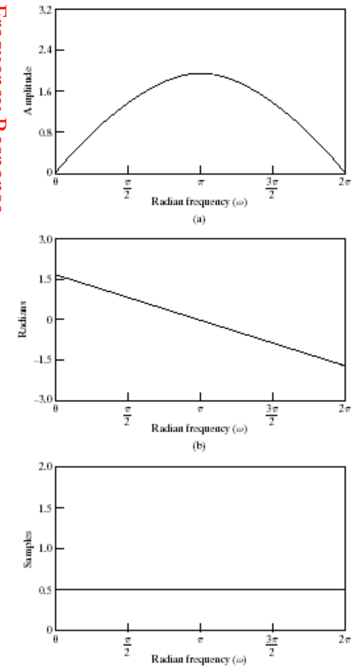
$$d[k] = 2h\left[(M+1)/2 - k\right]$$

$$\text{for } k = 1,2,\ldots,(M+1)/2$$



(d)

Frequency Response



$|H(e^{j\omega})|$

$\angle H(e^{j\omega})$

$\tau(\omega)$

Location of Zeros for Symmetric Cases

- For type **I** and **II** we have

$$h[n] = h[M-n] \xrightarrow{z} H(z) = z^{-M} H(z^{-1})$$

- So if z_0 is a zero $1/z_0$ is also a zero of the system
- If $h[n]$ is **real** and z_0 is a zero z_0^* is also a zero
- So for **real** and **symmetric** $h[n]$ zeros come in sets of four
- Special cases where zeros come in pairs
  - If a zero is on the unit circle reciprocal is equal to conjugate
  - If a zero is real conjugate is equal to itself
- Special cases where a zero come by itself
  - If $z = \pm 1$ both the reciprocal and conjugate is itself
- Particular importance of $z = -1$

$$H(-1) = (-1)^M H(-1)$$

  - If M is odd implies that

$$H(-1) = 0$$

  - Cannot design high-pass filter with symmetric FIR filter and M **odd**

Location of Zeros for Antisymmetric Cases

- For type **III** and **IV** we have

$$h[n] = -h[M-n] \xrightarrow{z} H(z) = -z^{-M} H\left(z^{-1}\right)$$

- All properties of symmetric systems holds
- Particular importance of both $z = +1$ and $z = -1$
  - If $z = 1$

$$H(1) = -H(1) \Rightarrow H(1) = 0$$

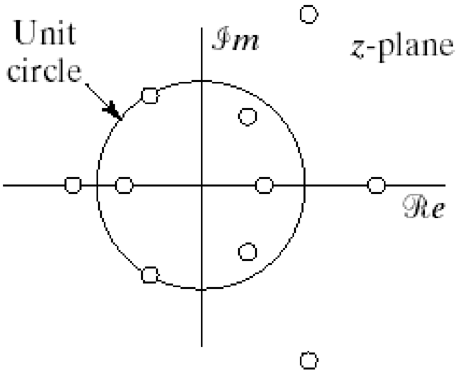
    - Independent from M: odd or even
  - If $z = -1$

$$H(-1) = (-1)^{M+1} H(-1)$$

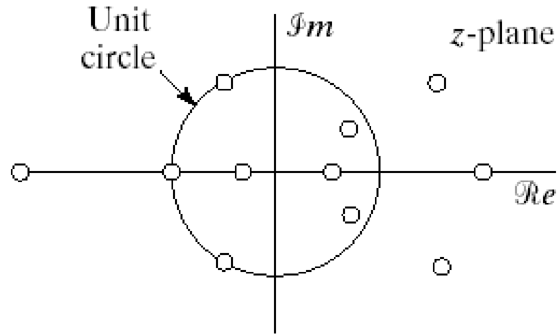
    - If $M + 1$ is odd implies that

$$H(-1) = 0$$

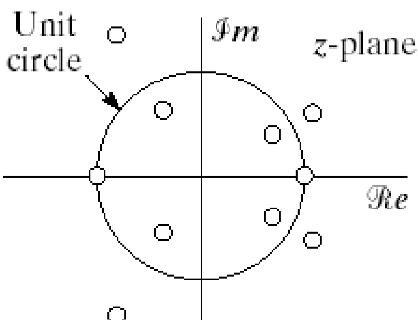
Typical Zero Locations



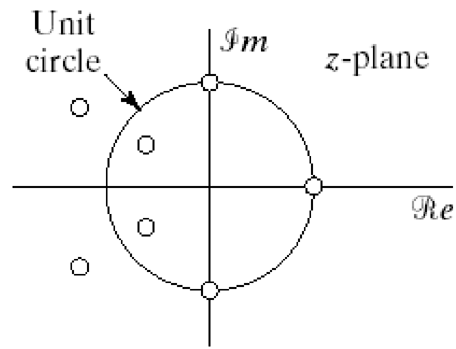
type **I**



type **II**



type **III**



type **IV**

Relation of FIR Linear Phase to Minimum-Phase

- In general a linear-phase FIR system is not minimum-phase
- We can always write a linear-phase FIR system as

$$H(z) = H_{\min}(z) H_{uc}(z) H_{\max}(z)$$

- Where

$$H_{\max}(z) = H_{\min}\left(z^{-1}\right) z^{-M_i}$$

- And M_i is the number of zeros
- $H_{\min}(z)$ covers all zeros **inside the unit circle**
- $H_{\mathrm{uc}}(z)$ covers all zeros on the **unit circle**
- $H_{\max}(z)$ covers all zeros **outside the unit circle**