بسم الله الرحمن الرحیم

پردازش سیگنال دیجیتال

درس ۱۳

پاسخ فرکانسی سیستم‌های گویا

Frequency Response of Rational Systems

کاظم فولادی

دانشکده مهندسی برق و کامپیوتر

دانشگاه تهران

http://courses.fouladi.ir/dsp

Frequency Response of Rational Systems

Frequency Response of Rational System Functions

- DTFT of a stable and LTI rational system function

$$H\left(e^{j\omega}\right)=\frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1-d_k e^{-j\omega}\right)}$$

- Magnitude Response

$$\left|H\left(e^{j\omega}\right)\right|=\left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}\left|1-c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1-d_k e^{-j\omega}\right|}$$

- Magnitude Squared

$$\left|H\left(e^{j\omega}\right)\right|^2=H\left(e^{j\omega}\right)^* H\left(e^{j\omega}\right)=\left(\frac{b_0}{a_0}\right)^2\frac{\prod_{k=1}^{M}\left(1-c_k e^{-j\omega}\right)\left(1-c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1-d_k e^{-j\omega}\right)\left(1-d_k^* e^{j\omega}\right)}$$

Log Magnitude Response

- Log Magnitude in decibels (dB)

$$20\log_{10}\left|H\left(e^{j\omega}\right)\right| = 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1 - c_k e^{-j\omega}\right| - \sum_{k=1}^{N} 20\log_{10}\left|1 - d_k e^{-j\omega}\right|$$

$$\text{Gain in dB} = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

$$\text{Attenuation in dB} = -20\log_{10}\left|H\left(e^{j\omega}\right)\right| = -\text{Gain in dB}$$

- Example:
  - $|H(e^{j\omega})| = 0.001$ translates into –60dB gain or 60dB attenuation
  - $|H(e^{j\omega})| = 1$ translates into 0dB gain
  - $|H(e^{j\omega})| = 0.5$ translates into -6dB gain
- Output of system

$$20\log_{10}\left|Y\left(e^{j\omega}\right)\right| = 20\log_{10}\left|H\left(e^{j\omega}\right)\right| + 20\log_{10}\left|X\left(e^{j\omega}\right)\right|$$

Phase Response

- **Phase response** of a rational system function

$$\angle\left|H\left(e^{j\omega}\right)\right| = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} \angle\left(1 - c_k e^{-j\omega}\right) - \sum_{k=1}^{N} \angle\left(1 - d_k e^{-j\omega}\right)$$

- Corresponding **group delay**

$$\mathrm{grd}\left|H\left(e^{j\omega}\right)\right| = \sum_{k=1}^{N} \frac{d}{d\omega} \arg\left(1 - d_k e^{-j\omega}\right) - \sum_{k=1}^{M} \frac{d}{d\omega} \arg\left(1 - d_k e^{-j\omega}\right)$$

  - Here arg[.] represents the continuous (unwrapped) phase
  - Work it out to get

$$\mathrm{grd}\left|H\left(e^{j\omega}\right)\right| = \sum_{k=1}^{N} \frac{\left|d_k\right|^2 - \mathrm{Re}\left\{d_k e^{-j\omega}\right\}}{1 + \left|d_k\right|^2 - 2\,\mathrm{Re}\left\{d_k e^{-j\omega}\right\}} - \sum_{k=1}^{M} \frac{\left|c_k\right|^2 - \mathrm{Re}\left\{c_k e^{-j\omega}\right\}}{1 + \left|c_k\right|^2 - 2\,\mathrm{Re}\left\{c_k e^{-j\omega}\right\}}$$

Unwrapped (Continuous) Phase

- Phase is **ambiguous** When calculating the arctan(.) function on a computer
  - Values between -π and +π
  - Denoted in the book as ARG(.)

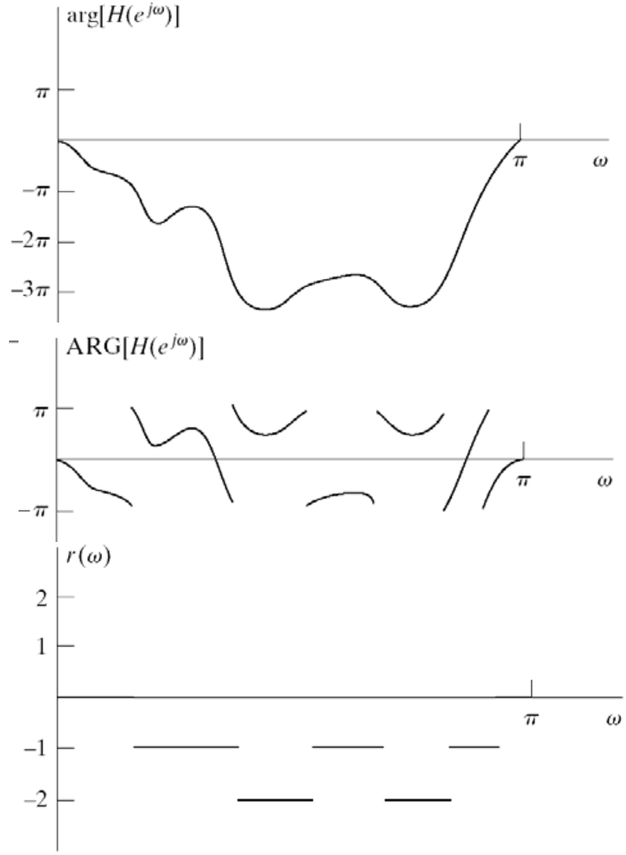
$$-\pi < \mathrm{ARG}\left[H\left(e^{j\omega}\right)\right] \leq \pi$$

  - Any multiple of 2π would give the same result

$$\angle H\left(e^{j\omega}\right) = \mathrm{ARG}\left[H\left(e^{j\omega}\right)\right] + 2\pi r(\omega)$$

  - Here $r(\omega)$ is an integer for any given value of ω
- **Group delay** is the derivative of the unwrapped phase

$$\mathrm{grd}\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left[\arg\left[H\left(e^{j\omega}\right)\right]\right]$$



Frequency Response of a Single Zero or Pole

- Let's analyze the effect of a single term

$$\left|1 - c_k e^{-j\omega}\right|^2 = \left|1 - re^{j\theta}e^{-j\omega}\right|^2 = 1 + r^2 - 2r\cos(\omega - \theta)$$

- If we represent it in dB

$$20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right| = 10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right]$$

- The phase term is written as

$$\mathrm{ARG}\left[1 - re^{j\theta}e^{-j\omega}\right] = \arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

- And the group delay obtained by differentiating the phase

$$\mathrm{grd}\left[1 - re^{j\theta}e^{-j\omega}\right] = \frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{r^2 - r\cos(\omega - \theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

- Maximum and minimum value of magnitude

$$10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right] = 10\log_{10}\left[1 + r^2 + 2r\right] = 20\log_{10}\left[1 + r\right]$$

$$10\log_{10}\left[1 + r^2 - 2r\cos(\omega - \theta)\right] = 10\log_{10}\left[1 + r^2 - 2r\right] = 20\log_{10}\left|1 - r\right|$$