پردازش سیگنال دیجیتال

درس ۱۲

تحلیل تبدیل سیستم‌های LTI

Transform Analysis of LTI Systems

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Transform Analysis of LTI Systems

Quick Review of LTI Systems

- LTI Systems are uniquely determined by their impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[k] * h[k]$$

- We can write the input-output relation also in the z-domain

$$Y(z) = H(z)X(z)$$

- Or we can define an LTI system with its frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- $H(e^{j\omega})$ defines magnitude and phase change at each frequency
- We can define a magnitude response

$$\left|Y(e^{j\omega})\right| = \left|H(e^{j\omega})\right|\left|X(e^{j\omega})\right|$$

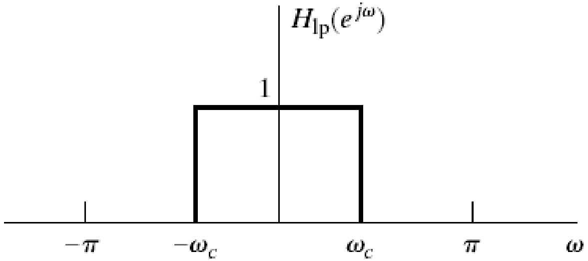
- And a phase response

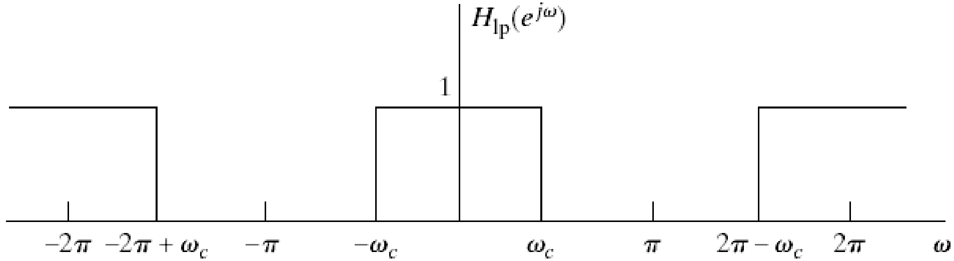
$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Ideal Low Pass Filter

- Ideal low-pass filter

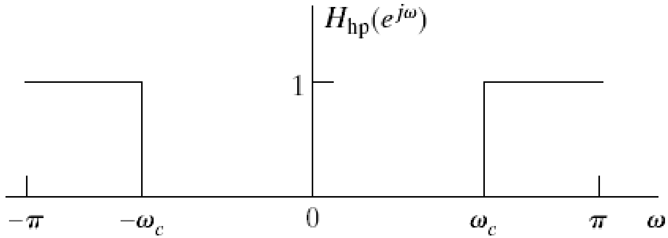
$$H_{lp}\left(e^{j\omega}\right)=\begin{cases}1 & |\omega|<\omega_c \\ 0 & \omega_c<|\omega|\le\pi\end{cases} \qquad h_{lp}[n]=\frac{\sin\omega_c n}{\pi n}$$





Ideal High-Pass Filter

$$H_{hp}\left(e^{j\omega}\right)=\begin{cases}0 & |\omega|<\omega_c \\ 1 & \omega_c<|\omega|\leq\pi\end{cases}$$



- Can be written in terms of a low-pass filter as

$$H_{hp}\left(e^{j\omega}\right)=1-H_{lp}\left(e^{j\omega}\right)$$

$$h_{hp}[n]=\delta[n]-h_{lp}[n]=\delta[n]-\frac{\sin\omega_c n}{\pi n}$$

Phase Distortion and Delay

- Remember the ideal delay system

$$h_{id}[n] = \delta[n - n_d] \xrightarrow{DTFT} H_{id}\left(e^{j\omega}\right) = e^{-j\omega n_d}$$

- In terms of magnitude and phase response

$$\left|H_{id}\left(e^{j\omega}\right)\right| = 1$$

$$\angle H_{id}\left(e^{j\omega}\right) = -\omega n_d \qquad |\omega| < \pi$$

- Delay distortion is generally acceptable form of distortion
  - Translates into a simple delay in time
- Also called a linear phase response
  - Generally used as target phase response in system design
- Ideal lowpass or highpass filters have zero phase response
  - Not implementable in practice

Ideal Low-Pass with Linear Phase

$$H_{lp}\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega n_d} & |\omega|<\omega_c \\ 0 & \omega_c<|\omega|\le\pi \end{cases}$$

- Delayed version of ideal impulse response

$$h_{lp}[n]=\frac{\sin\omega_c\left(n-n_d\right)}{\pi\left(n-n_d\right)}$$

- Filters high-frequency components and delays signal by n_d
- Linear-phase ideal lowpass filters is still not implementable
- **Group Delay**
  - Effect of phase on a narrowband signal: Delay
  - Derivative of the phase
  - Linear phase corresponds to constant delay
  - Deviation from constant indicated degree of **nonlinearity**

$$\tau(\omega)=\mathrm{grd}\left[H\left(e^{j\omega}\right)\right]=-\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\}$$

  - arg[] defines unwrapped or continuous phase

System Functions for Difference Equations

- Ideal systems are conceptually useful but not implementable
- **Constant-coefficient difference equations** are
  - general to represent most useful systems
  - Implementable
  - LTI and causal with zero initial conditions

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- The z-transform is useful in analyzing difference equations
- Let's take the z-transform of both sides

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

System Function

- Systems described as difference equations have system functions of the form

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}$$

- Example

$$H(z)=\frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}=\frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}}=\frac{Y(z)}{X(z)}$$

$$\left(1+\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)Y(z)=\left(1+2z^{-1}+z^{-2}\right)X(z)$$

$$y[n]+\frac{1}{4}y[n-1]+\frac{3}{8}y[n-2]=x[n]+2x[n-1]+x[n-2]$$

Stability and Causality

- A system function does not uniquely specify a system
  - Need to know the ROC
- Properties of system gives clues about the ROC
- **Causal** systems must be right sided
  - ROC is outside the outermost pole
- **Stable** system requires absolute summable impulse response

$$\sum_{k=-\infty}^{\infty} \left|h[n]\right| < \infty$$

  - Absolute summability implies existence of DTFT
  - DTFT exists if unit circle is in the ROC
  - Therefore, stability implies that the ROC includes the unit circle
- **Causal** AND **stable** systems have all poles inside unit circle
  - Causal hence the ROC is outside outermost pole
  - Stable hence unit circle included in ROC
  - This means outermost pole is inside unit circle
  - Hence all poles are inside unit circle

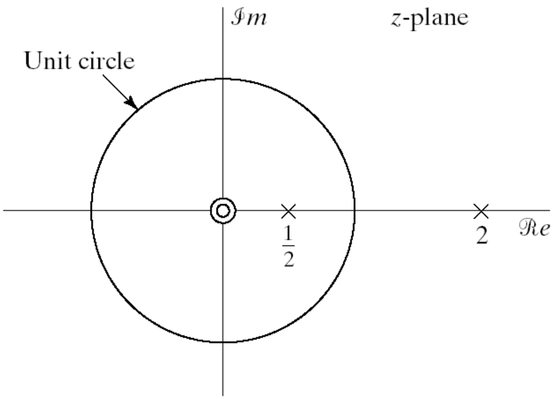
Example

- Let's consider the following LTI system

$$y[n]-\frac{5}{2}y[n-1]+y[n-2]=x[n]$$

- System function can be written as

$$H(z)=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$$



- Three possibilities for ROC
  - If ROC_1 causal but not stable
  - If ROC_2 stable but not causal
  - If ROC_3 not causal neither stable

$$\mathrm{ROC}_1 : |z| > 2$$

$$\mathrm{ROC}_2 : \frac{1}{2} < |z| < 2$$

$$\mathrm{ROC}_3 : |z| < \frac{1}{2}$$

Inverse System

- Given an LTI system H(z) the inverse system $H_i(z)$ is given as

$$H_i(z) = \frac{1}{H(z)}$$

- The cascade of a system and its inverse yields unity

$$G(z) = H(z)H_i(z) = 1 \qquad\qquad g[n] = h[n] * h_i[n] = \delta[n]$$

- If it exists, the frequency response of the inverse system is

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

- Not all systems have an inverse: zeros cannot be inverted
  - Example: Ideal lowpass filter
- The inverse of rational system functions

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)} \quad \rightarrow \quad H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}$$

- ROC of inverse has to overlap with ROC of original system

Examples: Inverse System

Example 1: Let's find the inverse system of

$$H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}} \qquad \text{ROC}: |z| > 0.9 \longrightarrow H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$

- The ROC of the inverse system is either $|z| > 0.5$ or $|z| < 0.5$
- Only $|z| > 0.5$ overlaps with original ROC

Examples: Inverse System

Example 2: Let's find the inverse system of

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}} \qquad \text{ROC}: |z| > 0.9 \longrightarrow H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5}$$

$$= \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}$$

- Again two possible ROCs $|z| > 2$ or $|z| < 2$
- This time both overlap with original ROC so both are valid
  - Two valid inverses for this system

$$h_{i,1}[n] = 2(2)^n u[-n-1] - 1.8(2)^{n-1} u[-n]$$ Stable and Non-Causal

$$h_{i,2}[n] = -2(2)^n u[n] + 1.8(2)^{n-1} u[n-1]$$ Non-Stable and Causal

Infinite Impulse Response (IIR) Systems

- Rational system function

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

- If at least one pole does not cancel with a zero
- There will at least one term of the form

$$a^n u[n] \quad \text{or} \quad -a^n u[-n-1]$$

- Therefore the impulse response will be **infinite length**

Infinite Impulse Response (IIR) Systems: Example

Example: Causal system of the form

$$y[n] - ay[n-1] = x[n]$$

- The impulse response from inverse transform

$$H(z) = \frac{1}{1 - az^{-1}} \qquad \text{ROC}: |z| > a \ \text{ from causality}$$

$$h[n] = a^n u[n]$$

Finite Impulse Response (FIR) Systems

- If transfer function does not have any poles except at $z = 0$
  - In this case $N = 0$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \sum_{k=0}^{M} b'_k z^{-k}$$

- No partial fraction expansion possible (or needed)
- The impulse response can be seen to be

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

- Impulse response is of **finite length**

Example: FIR System

- Consider the following impulse response

$$h[n] = \begin{cases} a^n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- The system function is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}$$

- Assuming a real and positive the zeros can be written as

$$z_k = ae^{j2\pi k/(M+1)} \quad \text{for} \quad k = 0,1,\ldots,M$$

- For $k = 0$ we have a zero at $z_0 = a$
- The zero cancels the pole at $z = a$
- We can write this system as

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]$$

- Or equivalently from $H(z)$ as

$$y[n] - ay[n-1] = x[n] - a^{M+1} x[n-M-1]$$