پردازش سیگنال دیجیتال

درس ۹

پردازش گسسته–زمان سیگنال‌های پیوسته–زمان

Discrete-Time Processing of Continuous-Time Signals

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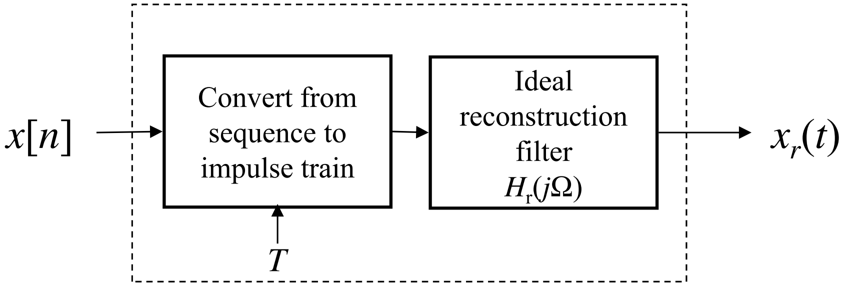
Reconstruction of Bandlimited Signal From Samples

- **Sampling** can be viewed as modulating with impulse train
- If Sampling Theorem is satisfied
  - The original continuous-time signal can be recovered
  - By **filtering** sampled signal with an ideal low-pass filter (LPF)
- Impulse-train modulated signal

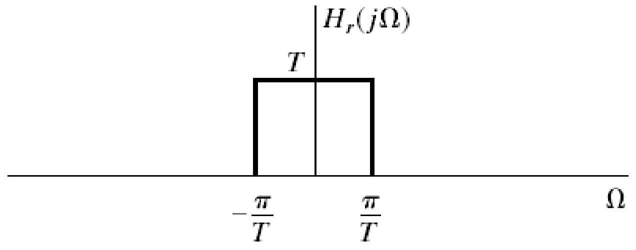
$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

- Pass through LPF with impulse response $h_r(t)$ to reconstruct

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

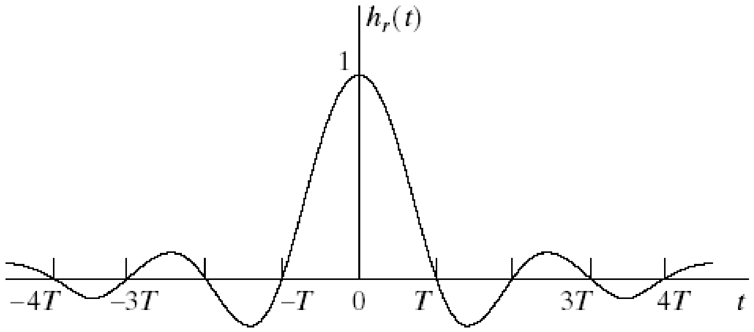


Ideal Reconstruction Filter



(b)

$$H_r(j\Omega) = \text{rect}[-\tfrac{\pi}{T}, \tfrac{\pi}{T}]$$



(c)

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

sinc function

Reconstructed Signal

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

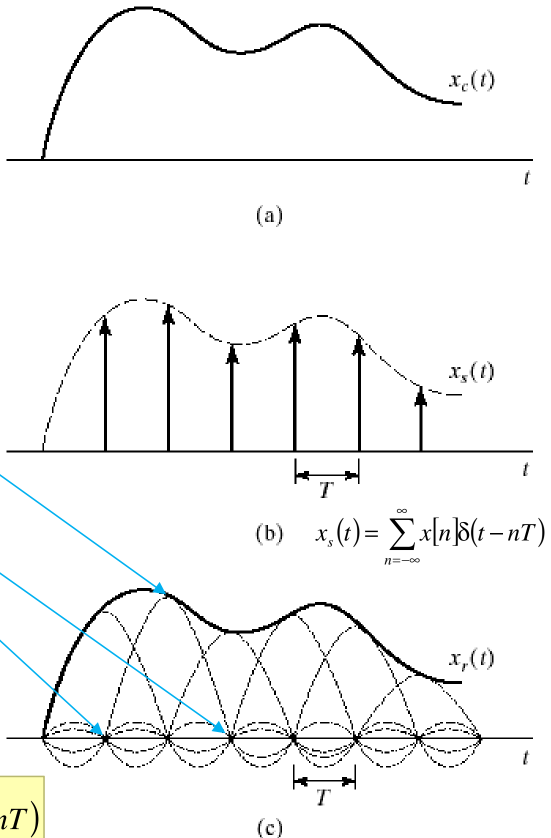
sinc function is 1 at $t = 0$

sinc function is 0 at $t = nT$

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_r(j\Omega) e^{-j\Omega T n}$$

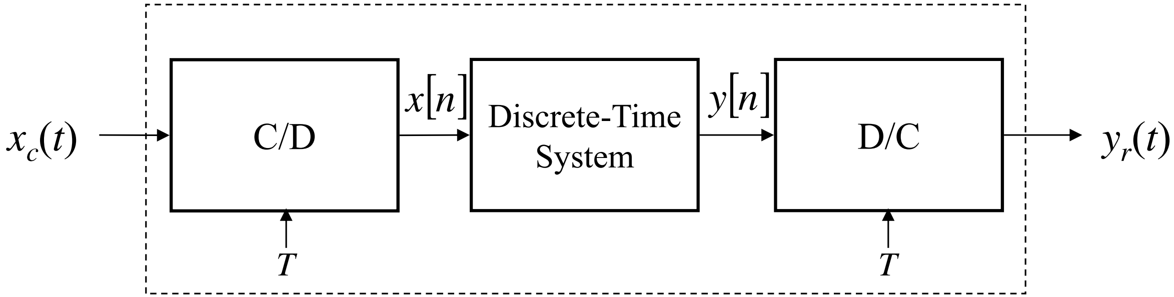
$$= H_r(j\Omega) \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n}$$

$$X_r(j\Omega) = X\left(e^{j\Omega T}\right) H_r(j\Omega)$$



$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t-nT)$$

Discrete-Time Processing of Continuous-Time Signals



- **Overall system** is equivalent to a continuous-time system
  - Input and output is continuous-time
- The continuous-time system depends on
  - Discrete-time system
  - Sampling rate
- We're interested in the equivalent frequency response
  - First step is the relation between $x_c(t)$ and $x[n]$
  - Next between $x[n]$ and $y[n]$
  - Finally between $y[n]$ and $y_r(t)$

Effective Frequency Response

- **Input** **continuous-time** **to** **discrete-time**

$$x[n] = x_c(nT)$$

ω $\Omega = \omega / T$

$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

- Assume a **discrete-time LTI system**

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{T}H(e^{j\omega})\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

- **Output** **discrete-time** to **continuous-time**

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}) = \begin{cases} TY(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & \text{otherwise} \end{cases}$$

- Output frequency response

$$Y_r(j\Omega) = \begin{cases} T \cdot H(e^{j\Omega T})X(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & \text{otherwise} \end{cases} = \begin{cases} H(e^{j\Omega T})X_c(j\Omega) & |\Omega| < \pi / T \\ 0 & \text{otherwise} \end{cases}$$

- Effective Frequency Response

$$Y_r(j\Omega) = H_{eff}(j\Omega)X_c(j\Omega)$$

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & \text{otherwise} \end{cases}$$

Effective Frequency Response

$$Y_r(j\Omega) = H_r(j\Omega)Y\left(e^{j\Omega T}\right) = \begin{cases} TY\left(e^{j\Omega T}\right) & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases}$$

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right) = \frac{1}{T}H\left(e^{j\omega}\right)\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

⇩

$$\begin{aligned} Y_r(j\Omega) &= H_r(j\Omega)H\left(e^{j\Omega T}\right)X\left(e^{j\Omega T}\right) \\ &= H_r(j\Omega)H\left(e^{j\Omega T}\right)\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right) \\ &= \begin{cases} H\left(e^{j\Omega T}\right)X_c(j\Omega) & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$X_c(j\Omega) = 0, |\Omega| \geq \pi/T$

only $k = 0$ remains in Σ

$H_r(j\Omega) = T, |\Omega| \leq \pi/T$

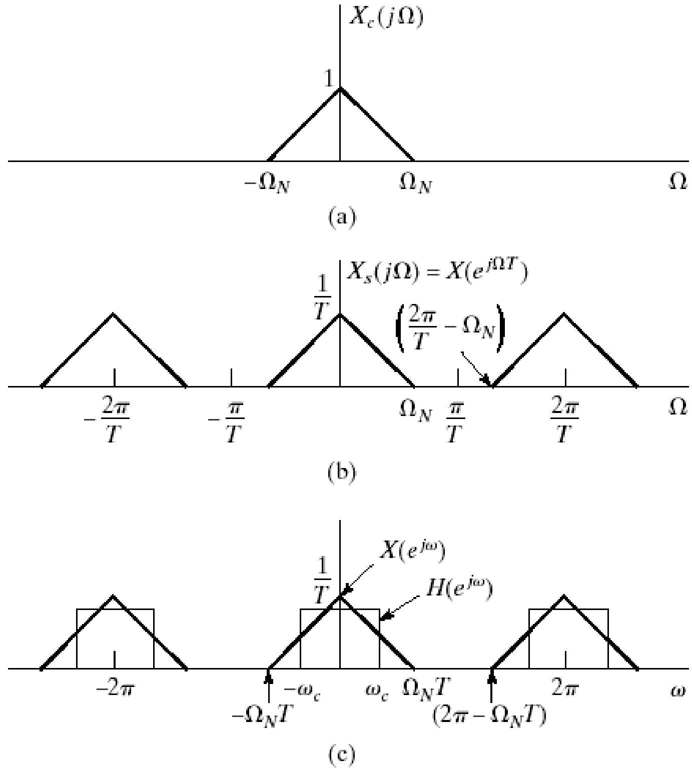
Example

- Ideal low-pass filter implemented as a discrete-time system

Continuous-time input signal

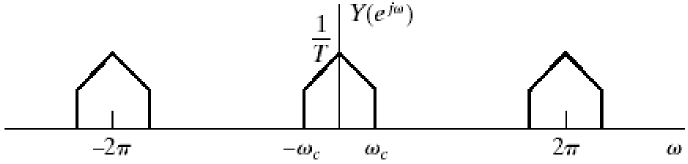
Sampled continuous-time input signal

Apply discrete-time LPF



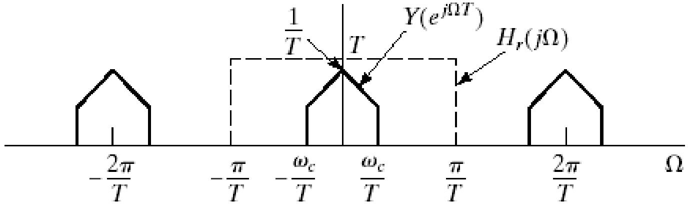
Example Continued

Signal after discrete-time LPF is applied



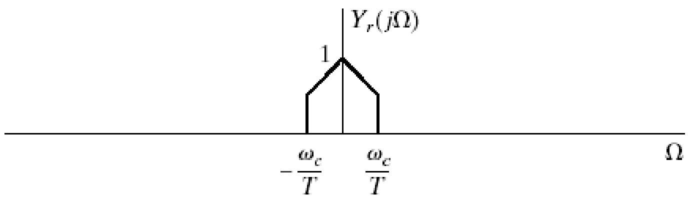
(d)

Application of reconstruction filter



(e)

Output continuous-time signal after reconstruction



(f)

Impulse Invariance

- Given a continuous-time system $H_c(j\Omega)$
  - how to choose discrete-time system response $H(e^{j\omega})$
  - so that effective response of discrete-time system $H_{eff}(j\Omega) = H_c(j\Omega)$
- Answer:

$$H(e^{j\omega}) = H_c(j\Omega) = H_c(j\omega / T) \qquad |\omega| < \pi$$

- Condition:

$$H_c(j\Omega) = 0 \qquad |\Omega| \geq \pi / T$$

- Given these conditions the discrete-time impulse response can be written in terms of continuous-time impulse response as

$$h[n] = Th_c(nT)$$

- Resulting system is
  the **impulse-invariant** version of the continuous-time system

Example: Impulse Invariance

- Ideal low-pass discrete-time filter by impulse invariance

$$H_c(j\Omega) = \begin{cases} 1 & |\Omega| < \Omega_c \\ 0 & \text{otherwise} \end{cases}$$

- The impulse response of continuous-time system is

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

- Obtain discrete-time impulse response via impulse invariance

$$h[n] = Th_c(nT) = T\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$

- The frequency response of the discrete-time system is

$$H_c(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$