پردازش سیگنال دیجیتال

درس ۶

تبدیل Z

The z-Transform

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The z-Transform

- Counterpart of the Laplace transform for discrete-time signals
- Generalization of the Fourier Transform
  - Fourier Transform does not exist for all signals
- The z-Transform is often time more convenient to use
- Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

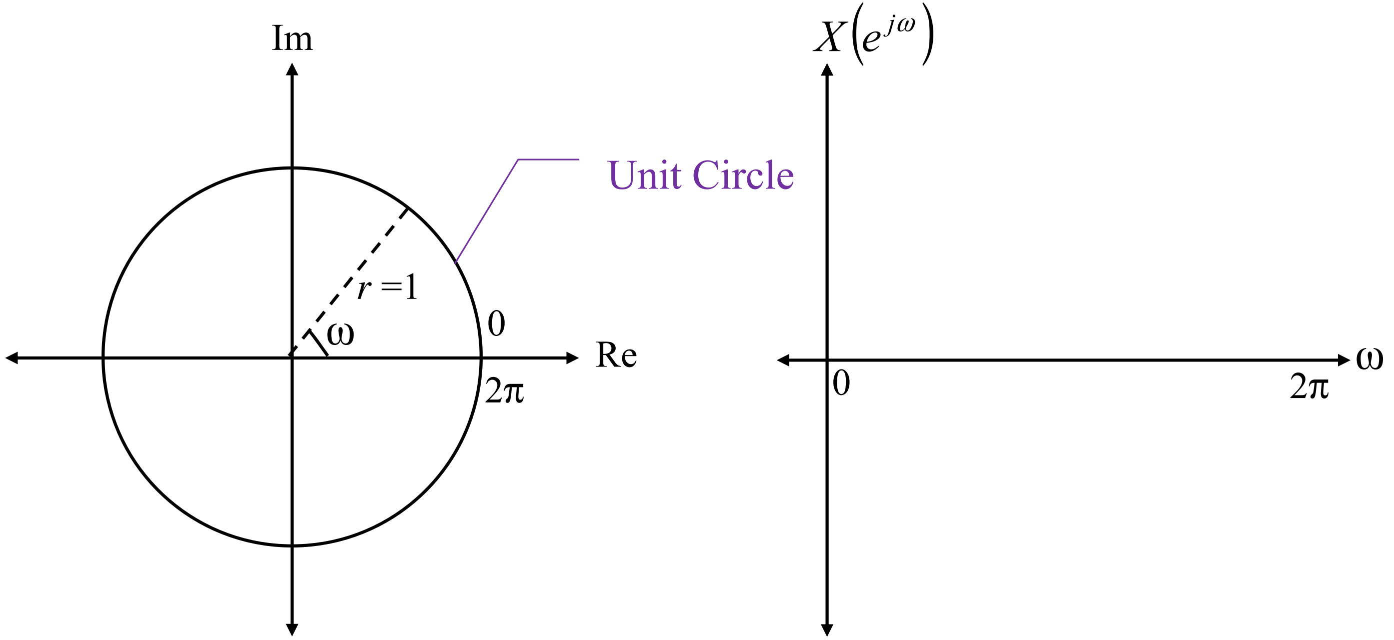
- Compare to DTFT definition:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z is a complex variable that can be represented as $z = r\, e^{j\omega}$
- Substituting $z = e^{j\omega}$ will reduce the z-transform to DTFT

The z-transform and the DTFT

- The z-transform is a function of the complex z variable
- Convenient to describe on the complex z-plane
- If we plot $z = e^{j\omega}$ for $\omega = 0$ to 2π we get the unit circle



Convergence of the z-Transform

- DTFT does not always converge

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - Infinite sum not always finite if $x[n]$ no absolute summable
  - Example: $x[n] = a^n u[n]$ for $|a| > 1$ does not have a DTFT

- Complex variable z can be written as $re^{j\omega}$ so the z-transform

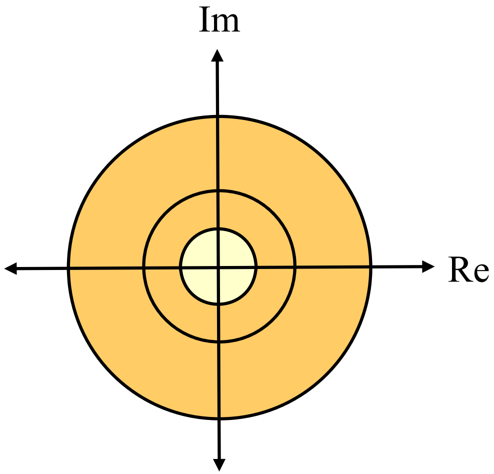
$$X\left(re^{j\omega}\right)=\sum_{n=-\infty}^{\infty} x[n]\left(\mathrm{r}e^{-j\omega}\right)^{-n}=\sum_{n=-\infty}^{\infty}\left(x[n]\mathrm{r}^{-n}\right)e^{-j\omega n}$$

- DTFT of $x[n]$ multiplied with exponential sequence r^{-n}
  - For certain choices of r the sum maybe made finite

$$\sum_{n=-\infty}^{\infty}\left|x[n]\,r^{-n}\right| < \infty$$

Region of Convergence

- The set of values of z for which the z-transform converges
- Each value of r represents a circle of radius r
- The region of convergence is made of circles



- Example: z-transform converges for values of $0.5 < r < 2$
  - ROC is shown on the left
  - In this example the ROC includes the unit circle, so DTFT exists
- **Not all sequence have a z-transform**
- Example: $x[n] = \cos(\omega_o n)$
  - Does not converge for any r
  - No ROC, No z-transform
  - But DTFT exists?!
  - Sequence has finite energy
  - DTFT converges in the mean-squared sense

Right-Sided Exponential Sequence Example

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- For Convergence we require

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

- Hence the ROC is defined as

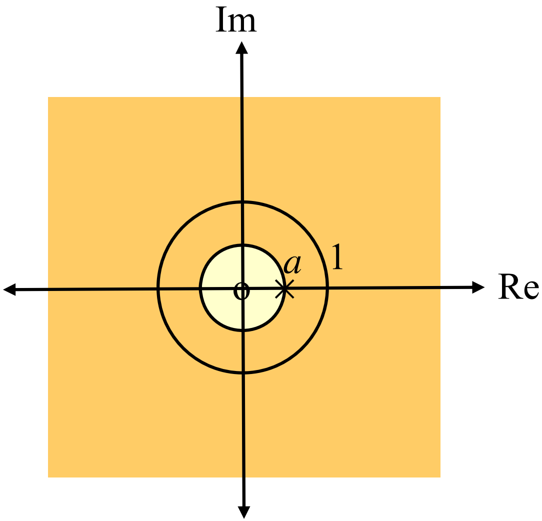
$$\left|az^{-1}\right|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1 - a}$$



- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

Same Example Alternative Way

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

$$\sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{\left(az^{-1}\right)^0 - \left(az^{-1}\right)^\infty}{1-az^{-1}} \qquad |z| > 2$$

- For the term with infinite exponential to vanish we need

$$\left|az^{-1}\right| < 1 \quad \Rightarrow \quad |a| < |z|$$

  - Determines the ROC (same as the previous approach)
- In the ROC the sum converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}}$$

Two-Sided Exponential Sequence Example

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^{\infty}}{1+\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^{0}}{1-\frac{1}{2}z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}}$$

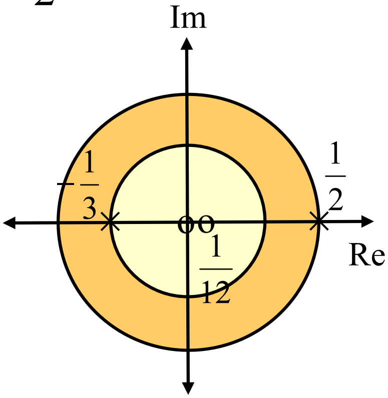
$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$\text{ROC}: \left|-\frac{1}{3}z^{-1}\right| < 1$$

$$\frac{1}{3} < |z|$$

$$\text{ROC}: \left|\frac{1}{2}z^{-1}\right| > 1$$

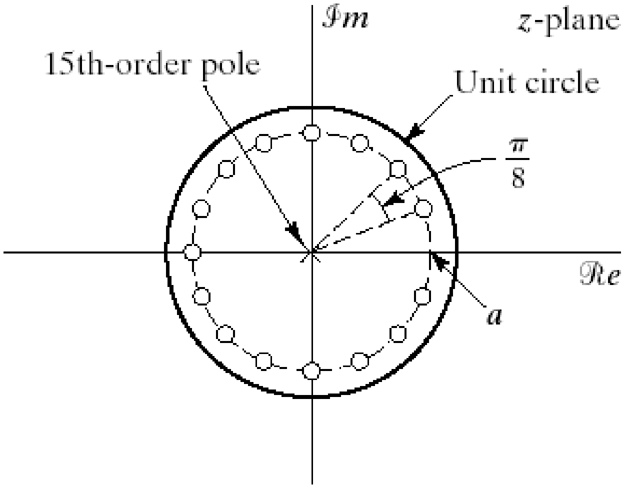
$$\frac{1}{2} > |z|$$



Finite Length Sequence

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$



Properties of The ROC of z-Transform

- The ROC is a ring or disk centered at the origin
- DTFT exists if and only if the ROC includes the **unit circle**
- The ROC cannot contain any poles
- The ROC for finite-length sequence is the entire z-plane
  - except possibly $z = 0$ and $z = \infty$
- The ROC for a right-handed sequence extends outward from the outermost pole possibly including $z = \infty$
- The ROC for a left-handed sequence extends inward from the innermost pole possibly including $z = 0$
- The ROC of a two-sided sequence is a ring bounded by poles
- The ROC must be **a connected region**
- A z-transform does not uniquely determine a sequence without specifying the ROC

Stability, Causality, and the ROC

- Consider a system with impulse response $h[n]$
- The z-transform $H(z)$ and the pole-zero plot shown below
- Without any other information $h[n]$ is not uniquely determined
  - $|z| > 2$ or $|z| < ½$ or $½ < |z| < 2$
- If system **stable** ROC must include unit-circle: $½ < |z| < 2$
- If system is **causal** must be right sided: $|z| > 2$