بسم الله الرحمن الرحیم

دانشگاه تهران

پردازش سیگنال دیجیتال

درس ۵

خصوصیات تبدیل فوریه گسسته–زمان

Discrete-Time Fourier Transform Properties

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Absolute and Square Summability

- Absolute summability is sufficient condition for DTFT
- Some sequences may not be absolute summable but only square summable

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- To represent square summable sequences with DTFT
  - We can relax the uniform convergence condition
  - Convergence is in mean-squared sense

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) - X_M(e^{j\omega}) \right|^2 = 0$$

  - Error does not converge to zero for every value of ω
  - The mean-squared value of the error over all ω does

Example: Ideal Lowpass Filter

- The periodic DTFT of the ideal lowpass filter is

$$H_{lp}\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

- The inverse can be written as

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lp}\left(e^{j\omega}\right) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi j n}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi j n}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right) = \frac{\sin \omega_c n}{\pi n}$$

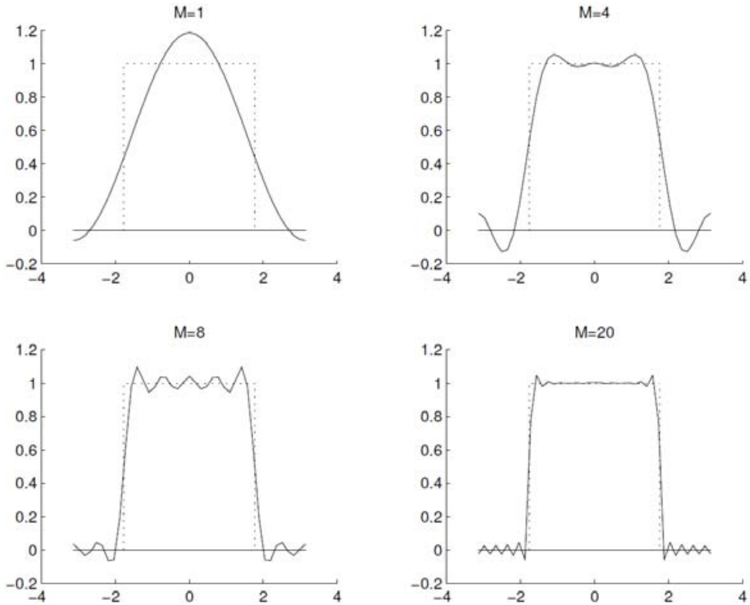
- Not causal
- Not absolute summable but it has a DTFT?
- The DTFT converges in the mean-squared sense
- Role of Gibbs phenomenon

Gibbs phenomenon

$$h[n] = \frac{\sin \omega_c n}{\pi n}$$

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n} = FT\left\{ \frac{\sin \omega_c n}{\pi n} y[n] \right\} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{\sin[(2M+1)(\omega-\theta)/2]}{\sin[(\omega-\theta)/2]} d\theta \qquad y[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{else} \end{cases}$$

$H_M(e^{j\omega})$



$$H\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

$$\lim_{M \to \infty} \int_{-\pi}^{\pi} \left| H\left(e^{j\omega}\right) - H_M\left(e^{j\omega}\right) \right|^2 = 0$$

$$\lim_{M \to \infty} \int_{-\pi}^{\pi} \left| H\left(e^{j\omega}\right) - H_M\left(e^{j\omega}\right) \right| \neq 0$$

The oscillation at $\omega = \omega_c$ is called the Gibbs phenomenon.

Example: Generalized DTFT

- DTFT of $x[n]=1$
- Not absolute summable
- Not even square summable
- But we define its DTFT as a pulse train

$$X(e^{j\omega})=\sum_{r=-\infty}^{\infty}2\pi\,\delta(\omega+2\pi r)$$

- Let's place into inverse DTFT equation

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{r=-\infty}^{\infty}2\pi\delta(\omega+2\pi r)\right]e^{j\omega n}d\omega$$

$$=\int_{-\pi}^{\pi}\delta(\omega)e^{j\omega n}d\omega=e^{j0n}=1$$

Symmetric Sequence and Functions

	Conjugate-symmetric	Conjugate-antisymmetric
Sequence	$x_e[n] = x_e^*[-n]$	$x_o[n] = -x_o^*[-n]$
$x[n] = x_e[n] + x_o[n]$	$x_e[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right)$	$x_o[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right)$
Function	$X_e\left(e^{j\omega}\right) = X_e^*\left(e^{-j\omega}\right)$	$X_o\left(e^{j\omega}\right) = -X_o^*\left(e^{-j\omega}\right)$
$X\left(e^{j\omega}\right) = X_o\left(e^{j\omega}\right) + X_e\left(e^{j\omega}\right)$	$X_e\left(e^{j\omega}\right) = \frac{1}{2}\left[X\left(e^{j\omega}\right) + X^*\left(e^{-j\omega}\right)\right]$	$X_o\left(e^{j\omega}\right) = \frac{1}{2}\left[X\left(e^{j\omega}\right) - X^*\left(e^{-j\omega}\right)\right]$

Symmetry Properties of DTFT

Sequence $x[n]$	Discrete-Time Fourier Transform $X(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part)
$j\mathrm{Im}\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part)
$x_e[n]$	$X_R(e^{j\omega}) = \mathrm{Re}\{X(e^{j\omega})\}$
$x_o[n]$	$jX_I(e^{j\omega}) = j\mathrm{Im}\{X(e^{j\omega})\}$
Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (conjugate symmetric)
Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
$x_e[n]$	$X_R(e^{j\omega})$
$x_o[n]$	$jX_I(e^{j\omega})$

Example: Symmetry Properties

- DTFT of the real sequence $x[n] = a^n u[n]$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |a| < 1$$

- Some properties are

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = X^*(e^{-j\omega})$$

$$X_R(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R(e^{-j\omega})$$

$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I(e^{-j\omega})$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}} = |X(e^{-j\omega})|$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = -\angle X(e^{-j\omega})$$

Fourier Transform Theorems

Sequence	DTFT
$x[n]$ $y[n]$	$X(e^{j\omega})$ $Y(e^{j\omega})$
$ax[n] + by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
$x[n-n_d]$	$e^{-j\omega n_d}X\left(e^{j\omega}\right)$
$x[-n]$	$X(e^{-j\omega})$
$nx[n]$	$j\dfrac{dX\left(e^{j\omega}\right)}{d\omega}$
$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
$x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

Parseval's Theorem: $\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$

Parseval's Theorem: $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$

Fourier Transform Pairs

Sequence	DTFT
$\delta[n-n_o]$	$e^{-j\omega n_o}$
1	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
$a^n u[n] \quad \|a\|<1$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_o+2\pi k)$
$\cos(\omega_o n+\phi)$	$\sum_{k=-\infty}^{\infty} \left[\pi e^{j\varphi}\delta(\omega-\omega_o+2\pi k)+\pi e^{-j\varphi}\delta(\omega+\omega_o+2\pi k)\right]$

DTFTs of Sums of Complex Exponentials

Recall CT result: $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$

What about DT: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$

a) We expect an impulse (of area 2π) at $\omega = \omega_o$

b) But $X(e^{j\omega})$ must be periodic with period 2π

In fact

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)$$

Note: The integration in the synthesis equation is over 2π period, only need $X(e^{j\omega})$ in *one* 2π period. Thus,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_0 n}$$