پردازش سیگنال دیجیتال

درس ۲۳

سری فوریه‌ی گسسته

The Discrete Fourier Series

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The Discrete Fourier Series

Discrete Fourier Series (DFS)

- Given a periodic sequence $\tilde{x}[n]$ with period N so that

$$\tilde{x}[n] = \tilde{x}[n + rN]$$

- The Fourier series representation can be written as

$$\tilde{x}[n] = \frac{1}{N} \sum_{k} \tilde{X}[k] e^{j(2\pi/N)kn}$$

- The Fourier series representation of **continuous-time** periodic signals require infinite many complex exponentials
- Note that for **discrete-time** periodic signals we have

$$e^{j(2\pi/N)(k+mN)n} = e^{j(2\pi/N)kn} e^{j(2\pi mn)} = e^{j(2\pi/N)kn}$$

- Due to the periodicity of the complex exponential we **only** need N exponentials for discrete time Fourier series

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

Discrete Fourier Series Pair

- A periodic sequence in terms of Fourier series coefficients

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

- The Fourier series coefficients can be obtained via

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn}$$

- For convenience we sometimes use

$$\boxed{W_N = e^{-j\frac{2\pi}{N}}}$$

- **Analysis equation**

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

- **Synthesis equation**

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

Example 1

- DFS of a periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1 & n = rN \\ 0 & \text{otherwise} \end{cases}$$

- Since the period of the signal is N

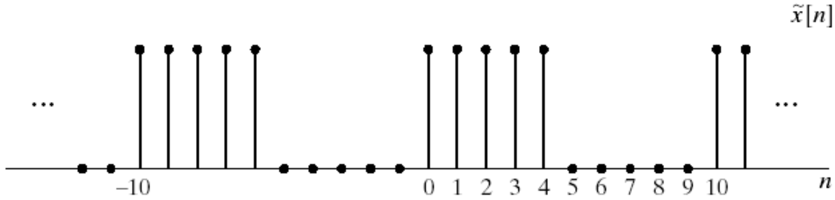
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn} = \sum_{n=0}^{N-1} \delta[n] e^{-j(2\pi/N)kn} = e^{-j(2\pi/N)k0} = 1$$

- We can represent the signal with the DFS coefficients as

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}$$

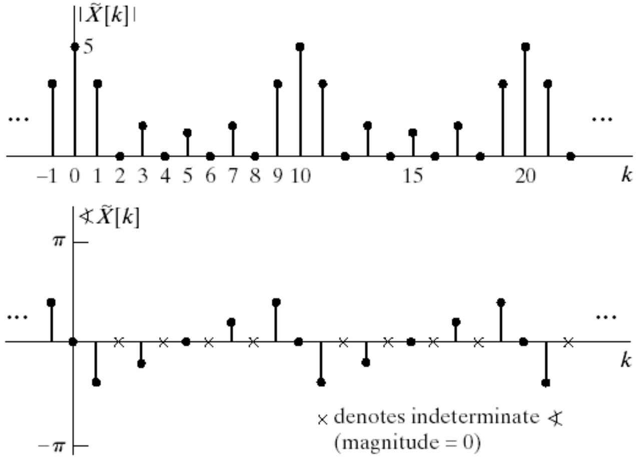
Example 2

- DFS of a periodic rectangular pulse train



- The DFS coefficients

$$\tilde{X}[k] = \sum_{n=0}^{4} e^{-j(2\pi/10)kn} = \frac{1-e^{-j(2\pi/10)k5}}{1-e^{-j(2\pi/10)k}} = e^{-j(4\pi k/10)}\frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$



Properties of DFS

- Linearity

$$\tilde{x}_1[n] \xleftrightarrow{DFS} \tilde{X}_1[k]$$
$$\tilde{x}_2[n] \xleftrightarrow{DFS} \tilde{X}_2[k]$$
$$a\tilde{x}_1[n]+b\tilde{x}_2[n] \xleftrightarrow{DFS} a\tilde{X}_1[k]+b\tilde{X}_2[k]$$

- Shift of a Sequence

$$\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k]$$
$$\tilde{x}[n-m] \xleftrightarrow{DFS} e^{-j2\pi km/N}\tilde{X}[k]$$
$$e^{j2\pi nm/N}\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k-m]$$

- Duality

$$\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k]$$
$$\tilde{X}[n] \xleftrightarrow{DFS} N\tilde{x}[-k]$$

Properties of DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n]$, $\tilde{x}_2[n]$	$\tilde{X}_1[k]$, $\tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$

Properties of DFS Cont'd

Periodic Sequence (Period N)	DFS Coefficients (Period N)
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$

Periodic Convolution

- Take two periodic sequences

$$\tilde{x}_1[n] \xleftrightarrow{DFS} \tilde{X}_1[k]$$
$$\tilde{x}_2[n] \xleftrightarrow{DFS} \tilde{X}_2[k]$$

- Let's form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

- The periodic sequence with given DFS can be written as

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

- **Periodic convolution** is commutative

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]$$

Periodic Convolution Cont'd

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

- Substitute periodic convolution into the DFS equation

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \right) W_N^{kn}$$

- Interchange summations

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} \right)$$

- The inner sum is the DFS of shifted sequence

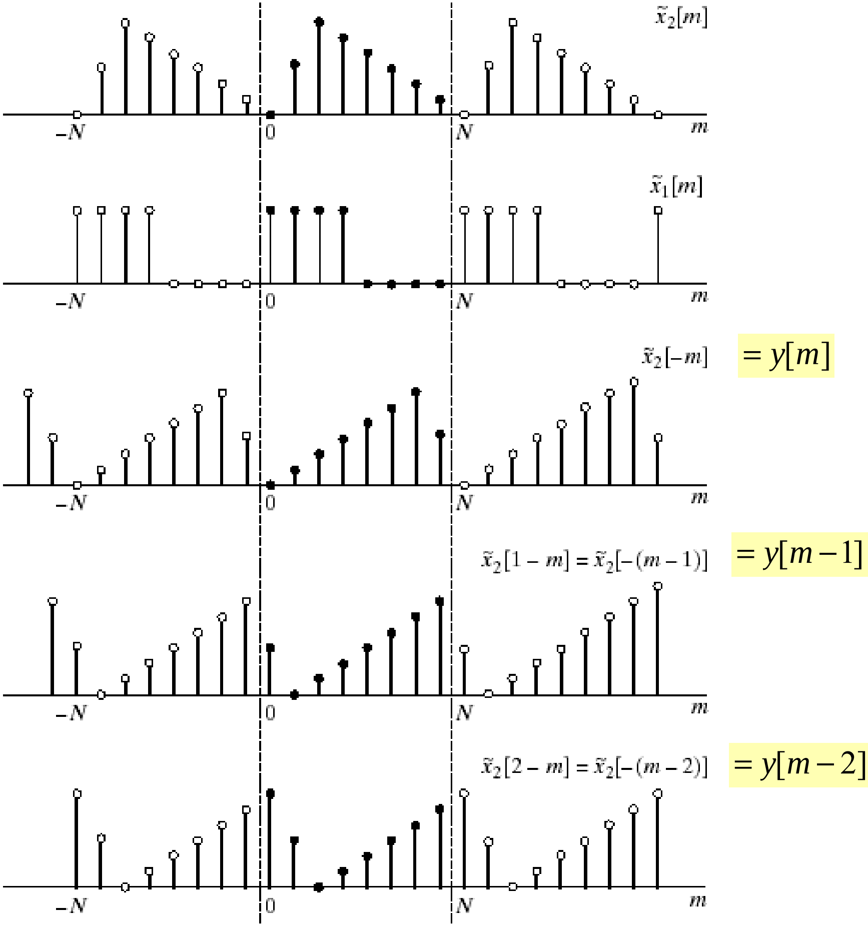
$$\boxed{\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} = W_N^{km} \tilde{X}_2[k]}$$

- Substituting

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} \right) = \sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \tilde{X}_2[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

Graphical Periodic Convolution

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$



The Fourier Transform of Periodic Signals

- Periodic sequences are not absolute or square summable
  - $\Rightarrow$ they **don't** have a Fourier Transform
- We can represent them as sums of complex exponentials: **DFS**
- We can combine **DFS** and **Fourier transform**
- Fourier transform of periodic sequences
  - Periodic impulse train with values proportional to DFS coefficients

$$\tilde{X}\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega-\frac{2\pi k}{N}\right)$$

  - This is periodic with 2π since DFS is periodic
- The inverse transform can be written as

$$\frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon}\tilde{X}\left(e^{j\omega}\right)e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon}\sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega-\frac{2\pi k}{N}\right)e^{j\omega n}d\omega$$

$$\frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\int_{0-\varepsilon}^{2\pi-\varepsilon}\delta\left(\omega-\frac{2\pi k}{N}\right)e^{j\omega n}d\omega=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi k}{N}n}$$

Example

$$\tilde{X}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k] \delta\left(\omega - \frac{2\pi k}{N}\right)$$

- Consider the periodic impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

- The DFS was calculated previously to be

$$\tilde{P}[k] = 1 \quad \text{for all k}$$

- Therefore the Fourier transform is

$$\tilde{P}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Relation between Finite-length and Periodic Signals

- Consider finite length signal $x[n]$ spanning from 0 to $N-1$
- Convolve with periodic impulse train

$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

- The Fourier transform of the periodic sequence is

$$\tilde{X}\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)\tilde{P}\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)\sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$\tilde{X}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X\left(e^{j\frac{2\pi k}{N}}\right)\delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$\boxed{\tilde{X}\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)}$$

- This implies that

$$\tilde{X}[k] = X\left(e^{j\frac{2\pi k}{N}}\right) = X\left(e^{j\omega}\right)\Big|_{\omega = \frac{2\pi k}{N}}$$

- DFS coefficients of a periodic signal can be thought as equally spaced samples of the Fourier transform of **one** period

Example

- Consider the following sequence

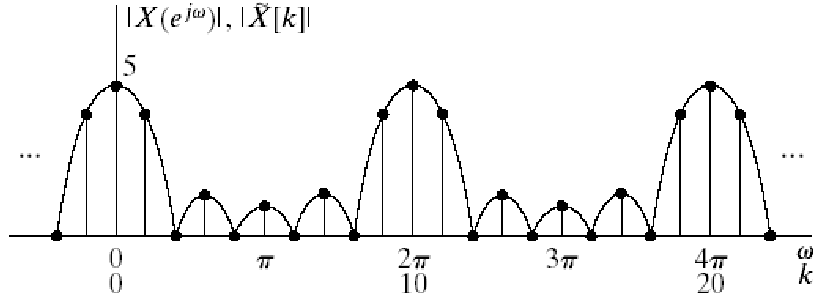
$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

- The Fourier transform

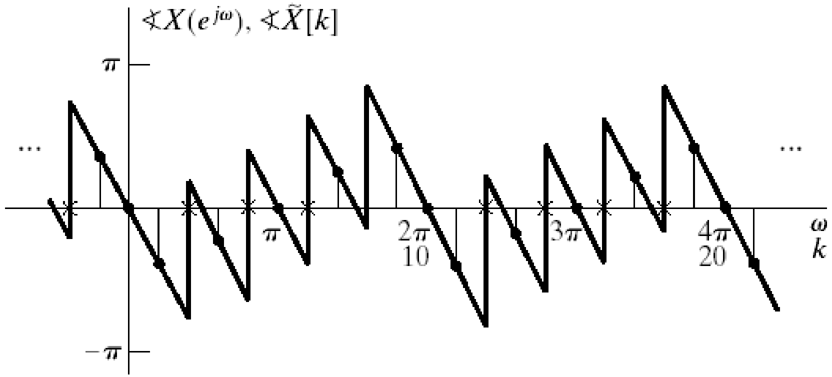
$$X\left(e^{j\omega}\right) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

- The DFS coefficients

$$\tilde{X}[k] = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$



(a)



$$\boxed{\tilde{X}[k] = X\left(e^{j\frac{2\pi k}{N}}\right) = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}}}$$