پردازش سیگنال دیجیتال

درس ۲۱

طراحی فیلترهای گسسته–زمان با پنجره‌زنی

Discrete-Time Filter Design by Windowing

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Discrete-Time Filter Design by Windowing

Filter Design by Windowing

- Simplest way of designing **FIR filters**
- Method is all discrete-time no continuous-time involved

- **Start** with ideal frequency response

$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n} \qquad h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

- Choose ideal frequency response as desired response
  - Most ideal impulse responses are of infinite length
- The easiest way to obtain a causal FIR filter from ideal is

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- More generally

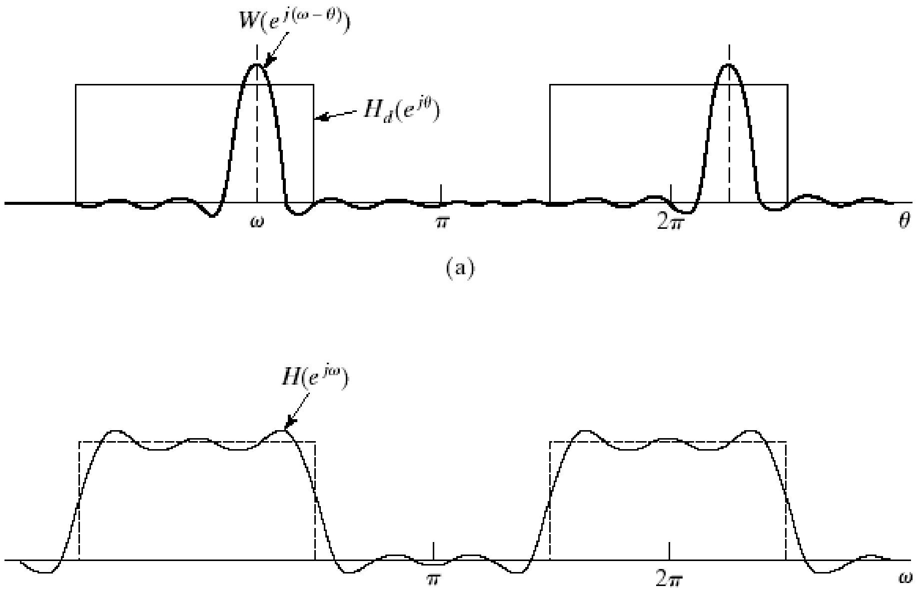
$$h[n] = h_d[n] w[n] \quad \text{where} \quad w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

Windowing in Frequency Domain

- Windowed frequency response (Periodic Convolution)

$$H\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d\left(e^{j\theta}\right)W\left(e^{j(\omega-\theta)}\right)d\theta$$

- The windowed version is smeared version of desired response



- If $w[n] = 1$ for all n, then $W(e^{j\omega})$ is impulse train with 2π period (ideal case)

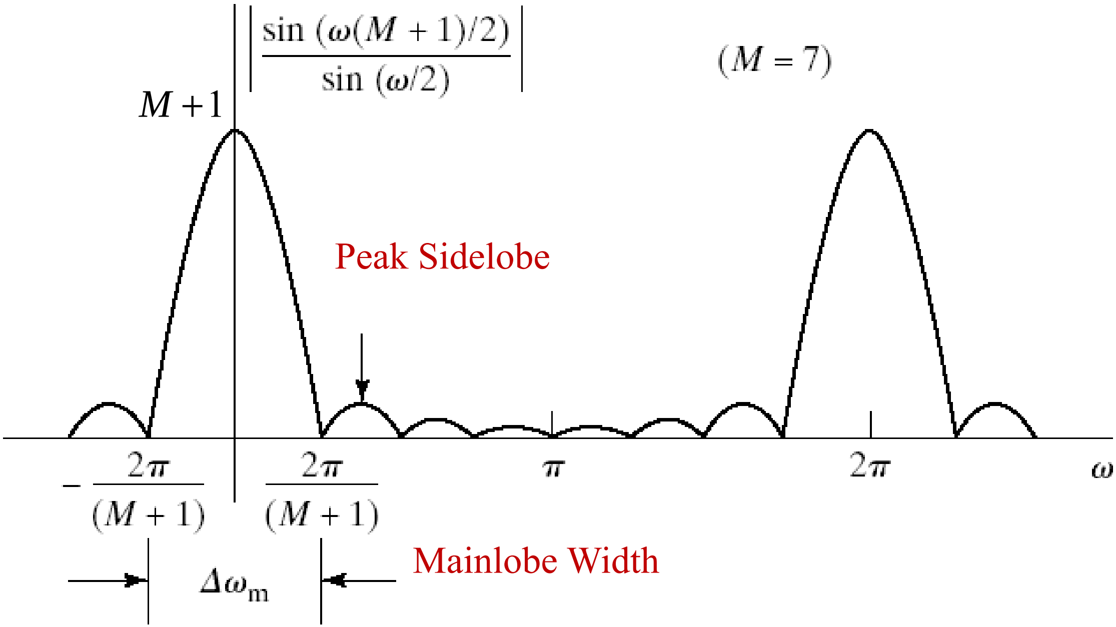
Properties of Windows

- Prefer windows that concentrate around DC in frequency
  - (More similar to impulse ⇒) Less smearing, closer approximation
- Prefer window that has minimal span in time
  - Less coefficient in designed filter, computationally efficient
- **So we want concentration in time and in frequency**
  - *Contradictory requirements!*

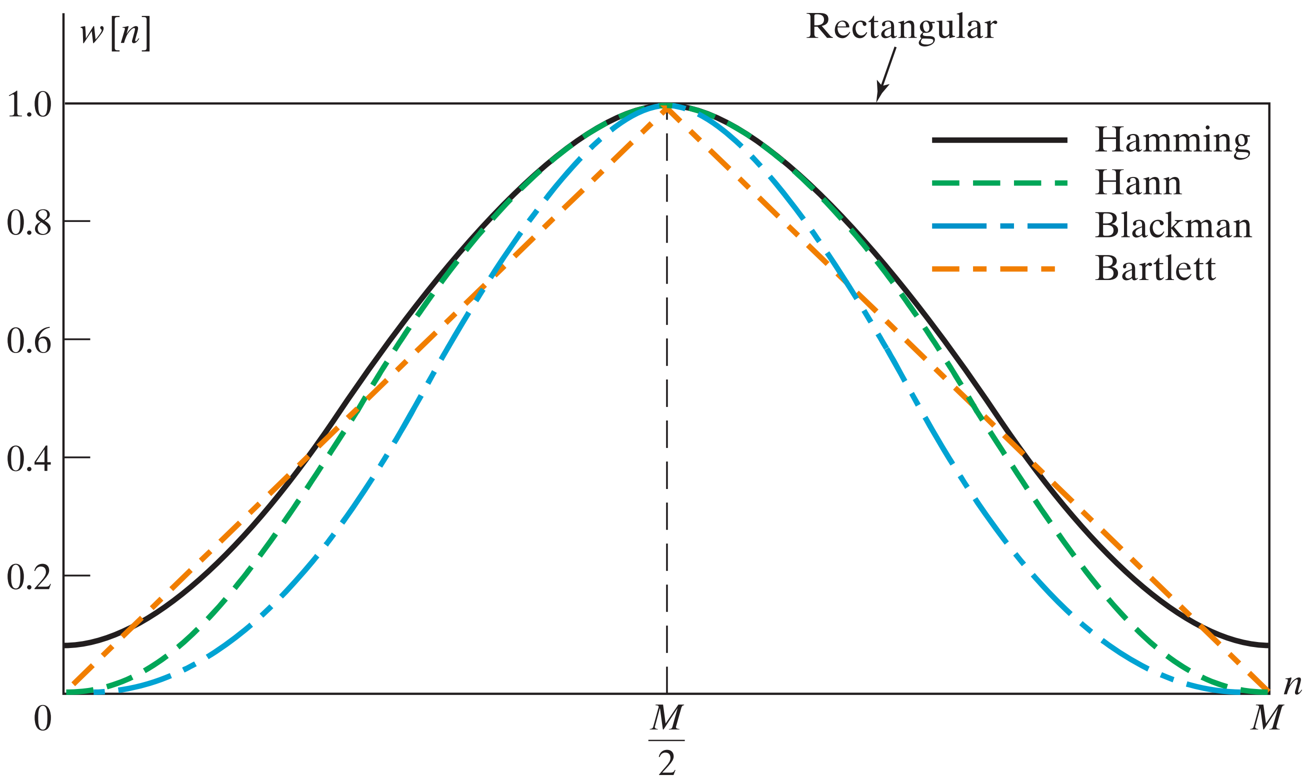
Example: Rectangular window

- Example: **Rectangular window**

$$w[n]=\begin{cases}1 & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}\Rightarrow \qquad W\left(e^{j\omega}\right)=\sum_{n=0}^{M}e^{-j\omega n}=\frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}=e^{-j\omega M/2}\,\frac{\sin[\omega(M+1)/2]}{\sin[\omega/2]}$$



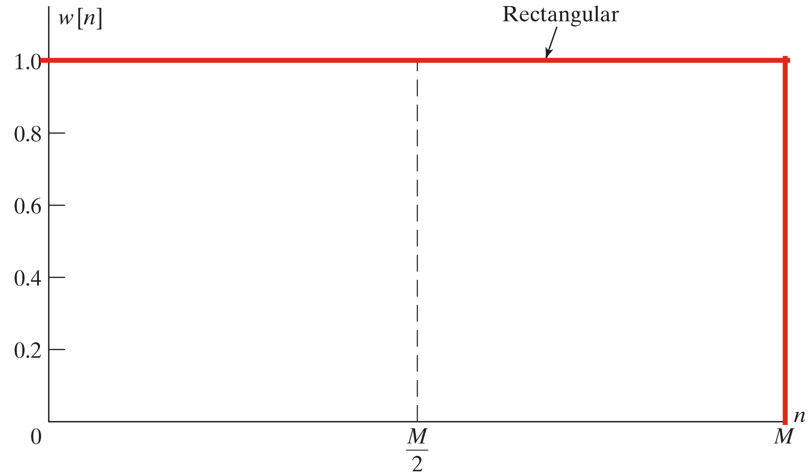
Commonly Used Windows

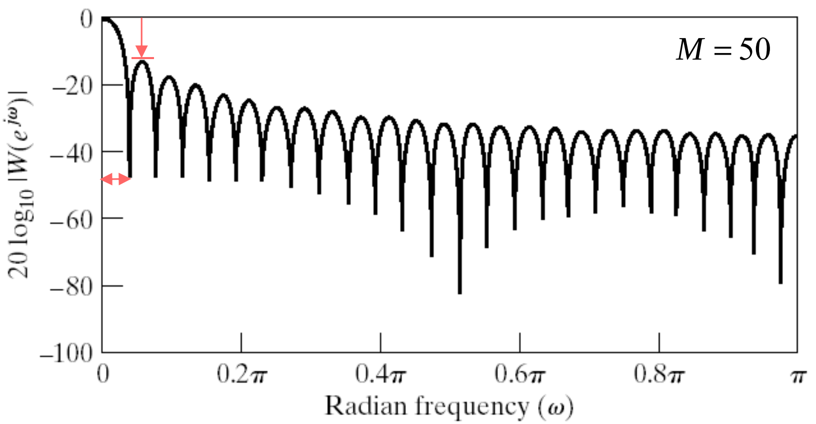


Rectangular Window

- **Narrowest main lob**
  - $4\pi/(M+1)$
  - Sharpest transitions at discontinuities in frequency response $H_d(e^{jw})$
- **Large side lobs**
  - -13 dB
  - Large oscillation around discontinuities
- Simplest possible window

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

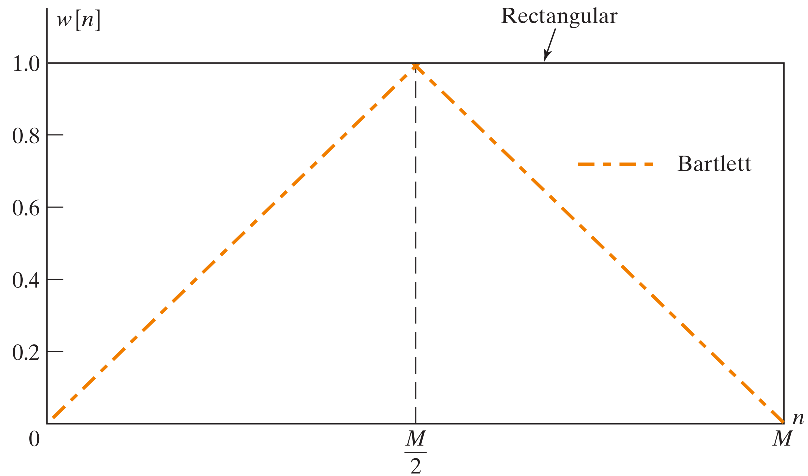


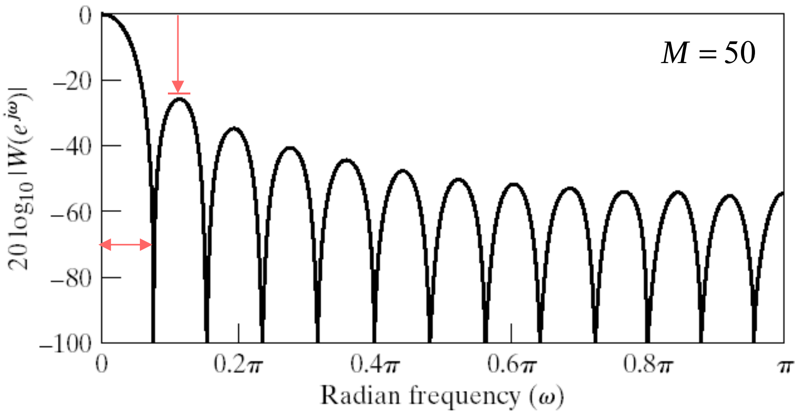


Bartlett (Triangular) Window

- **Medium main lob**
  - $8\pi/M$
- **Side lobs**
  - -25 dB
- Simple equation:

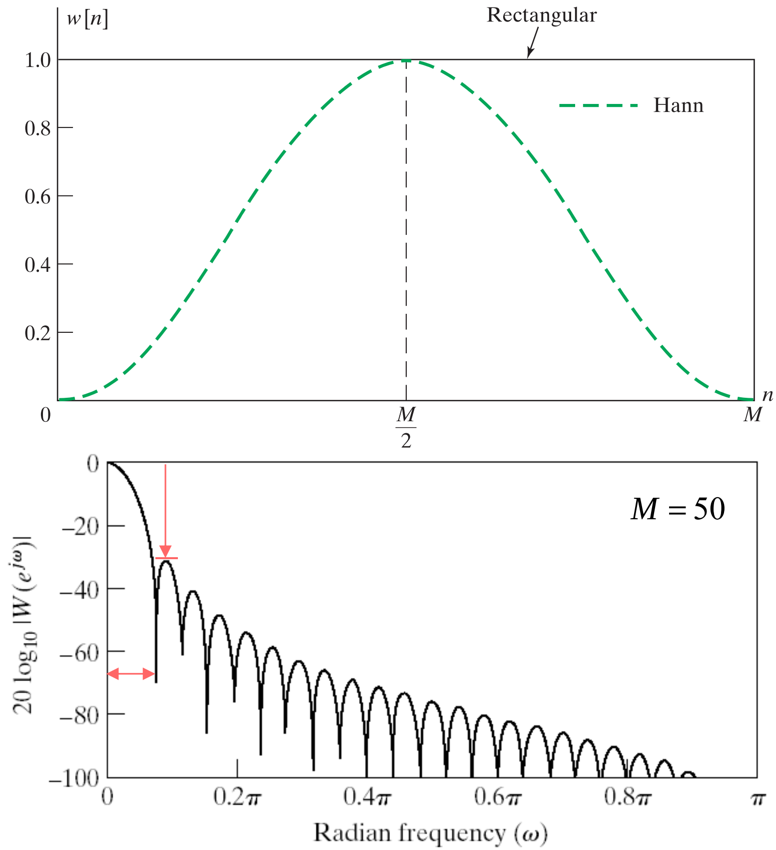
$$w[n]=\begin{cases} 2n/M & 0\le n\le M/2 \\ 2-2n/M & M/2\le n\le M \\ 0 & \text{otherwise} \end{cases}$$





- **Medium main lob**
  - $8\pi/M$
- **Side lobs**
  - -31 dB
- Hamming window performs better
- Same complexity as Hamming

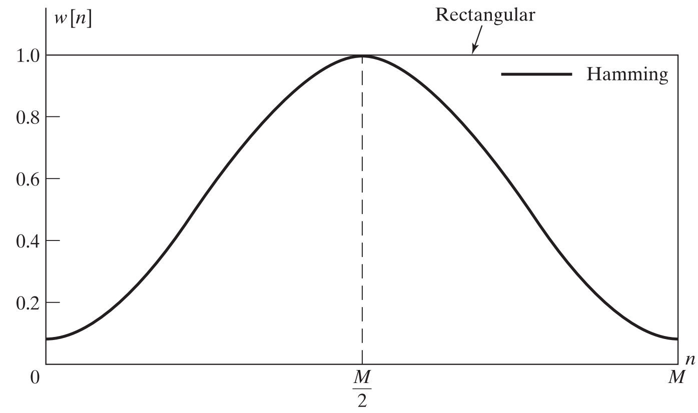
$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

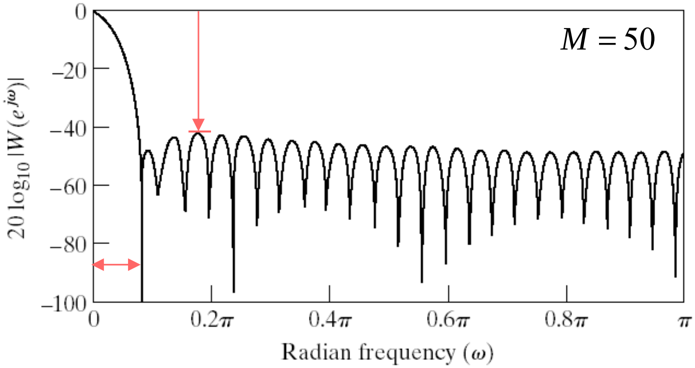


Hamming Window

- **Medium main lob**
  - $8\pi/M$
- **Good side lobs**
  - -41 dB
- Simpler than Blackman

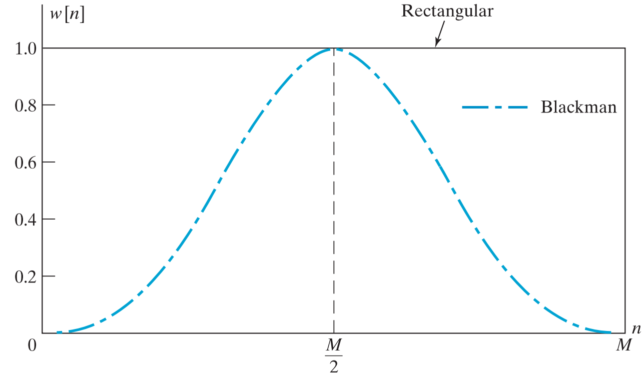
$$w[n]=\begin{cases}0.54-0.46\cos\left(\dfrac{2\pi n}{M}\right) & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}$$



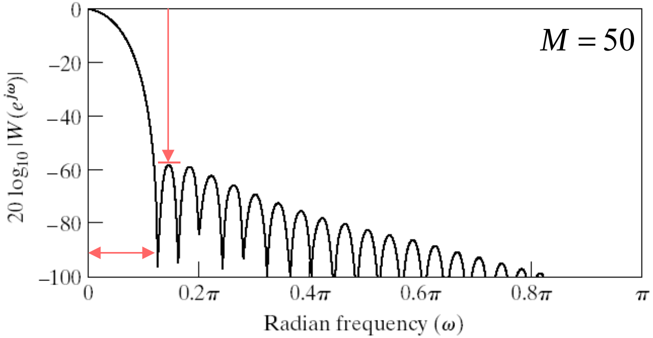


Blackman Window

- **Large main lob**
  - $12\pi/M$
- **Very good side lobs**
  - -57 dB
- Complex equation



$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{4\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$



Incorporation of Generalized Linear Phase

- Windows are designed with **linear phase** in mind
  - Symmetric around $M/2$

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- So their Fourier transform are of the form

$$W\left(e^{j\omega}\right) = W_e\left(e^{j\omega}\right)e^{-j\omega M/2} \qquad \text{where } W_e\left(e^{j\omega}\right) \text{ is a real and even}$$

- *Will keep symmetry properties of the desired impulse response*
- Assume symmetric desired response:

$$H_d\left(e^{j\omega}\right) = H_e\left(e^{j\omega}\right)e^{-j\omega M/2}$$

- With symmetric window

$$\boxed{H\left(e^{j\omega}\right) = A_e\left(e^{j\omega}\right)e^{-j\omega M/2}} \qquad A_e\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right)W_e\left(e^{j(\omega-\theta)}\right)d\theta$$

  - Periodic convolution of real functions

Linear-Phase Lowpass filter

$$H_{lp}\left(e^{j\omega}\right) = A_e\left(e^{j\omega}\right)e^{-j\omega M/2}$$

- Desired frequency response
  (with generalized linear phase):

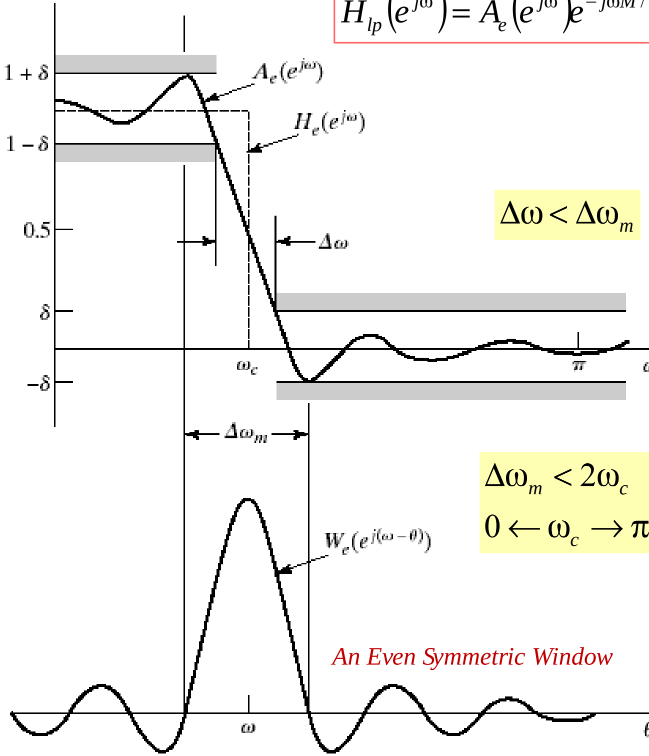
$$H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

- Corresponding impulse response
  (is also symmetric):

$$h_{lp}[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}$$

- Desired response is **even symmetric**, use symmetric window

$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n]$$



$$\Delta\omega < \Delta\omega_m$$

$$\Delta\omega_m < 2\omega_c$$

$$0 \leftarrow \omega_c \rightarrow \pi$$

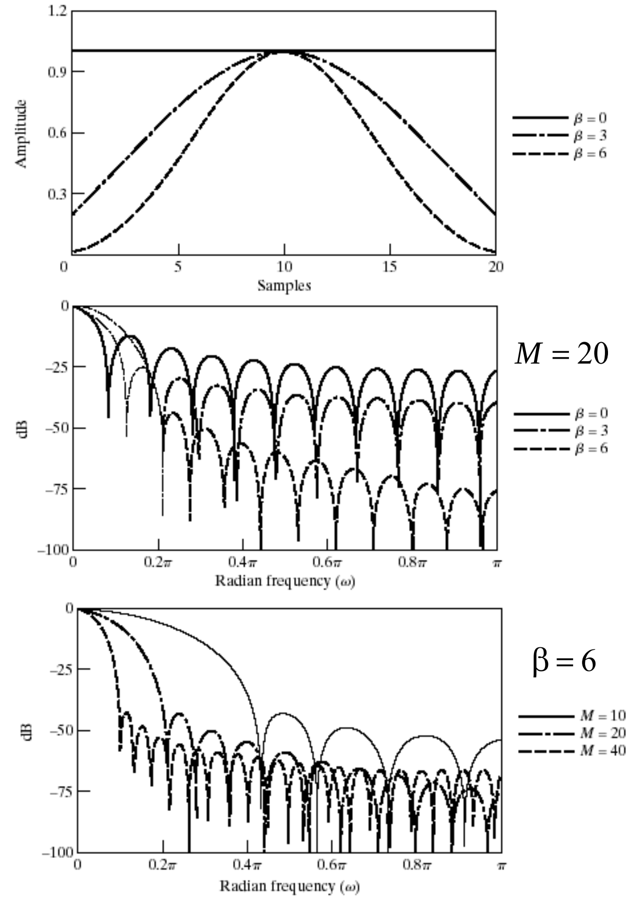
$$A_e\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right)W_e\left(e^{j(\omega-\theta)}\right)d\theta$$

Kaiser Window Filter Design Method

- Parameterized equation
  forming a set of windows
  - Has parameter to change
    main-lob **width** and side-lob **area**
    trade-off

$$w[n]=\begin{cases}\dfrac{I_0\left(\beta\sqrt{1-\left(\dfrac{n-M/2}{M/2}\right)^2}\right)}{I_0(\beta)} & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}$$

  - $I_0(.)$ represents **zeroth-order modified Bessel function of 1st kind**



Determining Kaiser Window Parameters

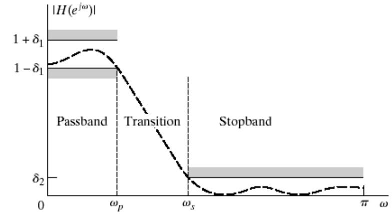
- Given filter specifications Kaiser developed empirical equations
  - Given the peak approximation error δ or in dB as $A = -20\log_{10}\delta$
  - and **transition band** width $\Delta\omega = \omega_s - \omega_p$
- The shape parameter β should be

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases}$$

- The filter order M is determined approximately by

$$M = \frac{A-8}{2.285\Delta\omega}$$

Example: Kaiser Window Design of a Lowpass Filter



- Specifications $\omega_p = 0.4\pi, \omega_p = 0.6\pi, \delta_1 = 0.01, \delta_2 = 0.001$
- Window design methods assume $\delta_1 = \delta_2 = 0.001$
- Determine cut-off frequency
  – Due to the symmetry we can choose it to be $\omega_c = 0.5\pi$
- Compute

$$\Delta\omega = \omega_s - \omega_p = 0.2\pi \qquad A = -20\log_{10}\delta = 60$$

- And Kaiser window parameters

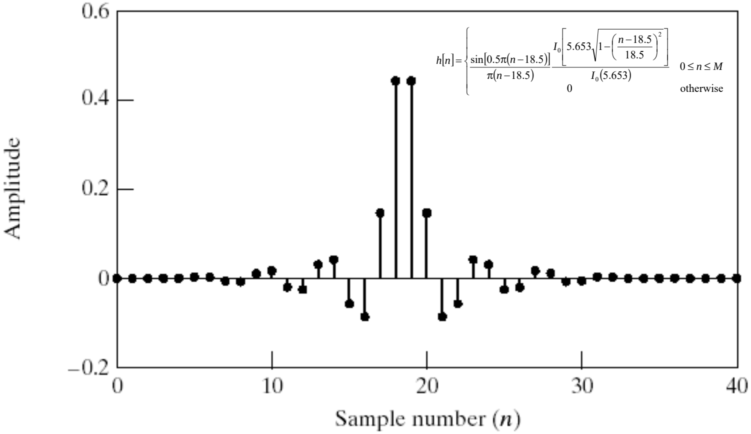
$$\beta = 5.653 \qquad M = 37$$

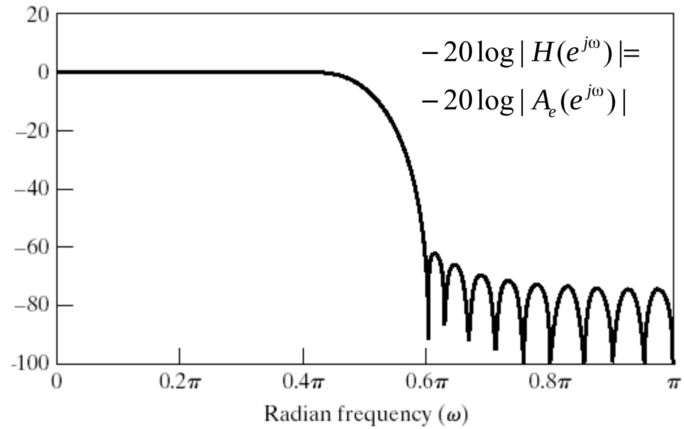
Odd (Type II FIR with Lin. Phase)

- Then the impulse response is given as

$$h[n] = \begin{cases} \dfrac{\sin[0.5\pi(n-18.5)]}{\pi(n-18.5)} \dfrac{I_0\left[5.653\sqrt{1-\left(\dfrac{n-18.5}{18.5}\right)^2}\right]}{I_0(5.653)} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

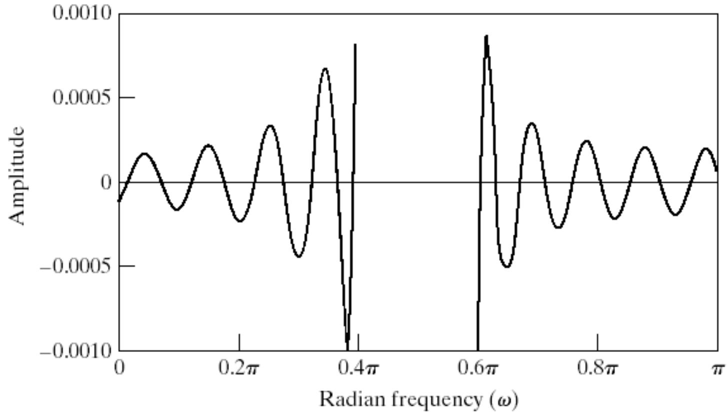
Example Cont'd





$M = 37$

Approximation Error



$$E_A(\omega) = \begin{cases} 1 - A_e(e^{j\omega}), & 0 \le \omega \le \omega_p, \\ 0 - A_e(e^{j\omega}), & \omega_s \le \omega \le \pi. \end{cases}$$

General Frequency Selective Filters

- A general **multiband** impulse response can be written as

$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}$$

$$G_{N_{mb}+1} = 0$$

- Window methods can be applied to multiband filters

- Example multiband frequency response
  - Special cases of
    - Bandpass
    - Highpass
    - Bandstop