پردازش سیگنال دیجیتال

درس ۲۰

طراحی فیلترهای IIR گسسته–زمان از روی فیلترهای پیوسته–زمان

Discrete-Time IIR Filter Design from Continuous-Time Filters

کاظم فولادی قلعه
دانشکده مهندسی، پردیس فارابی
دانشگاه تهران

http://courses.fouladi.ir/dsp

Discrete-Time IIR Filter Design from Continuous-Time Filters

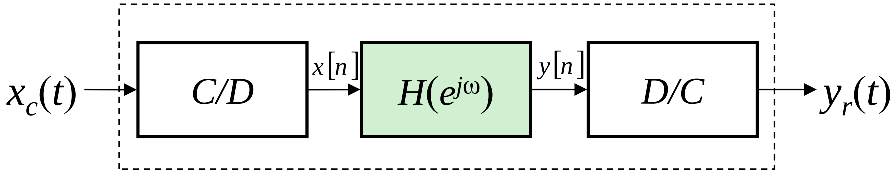
Filter Design Techniques

- **Filter:** Any discrete-time system that modifies certain frequencies
- Frequency-selective filters pass only certain frequencies

- **Filter Design Steps**
  - **Specification**
    - Problem or application specific
  - **Approximation** of specification with a discrete-time system
    - Our focus is to go from spec to discrete-time system
  - **Implementation**
    - Realization of discrete-time systems depends on target technology

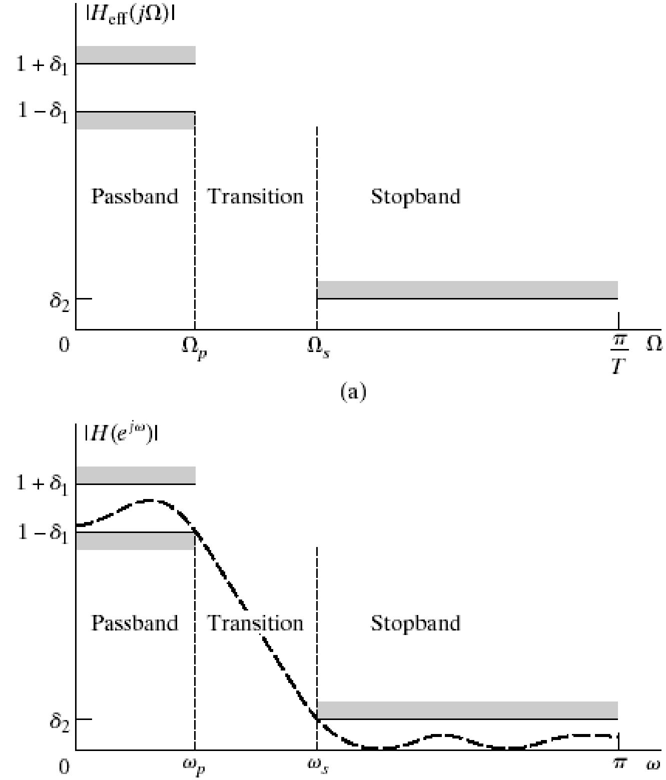
Filter Design Techniques

- We already studied the use of discrete-time systems to implement a continuous-time system
  – If our specifications are given in continuous time we can use:



$$H\left(e^{j\omega}\right) = H_c\left(j\omega / T\right) \qquad |\omega| < \pi$$

Filter Specifications



- **Specifications**
  - Passband

$$0.99 \le \left|H_{eff}(j\Omega)\right| = 1.01 \quad 0 \le \Omega \le 2\pi(2000)$$

  - Stopband

$$\left|H_{eff}(j\Omega)\right| \le 0.001 \quad 2\pi(3000) \le \Omega$$

- **Parameters**

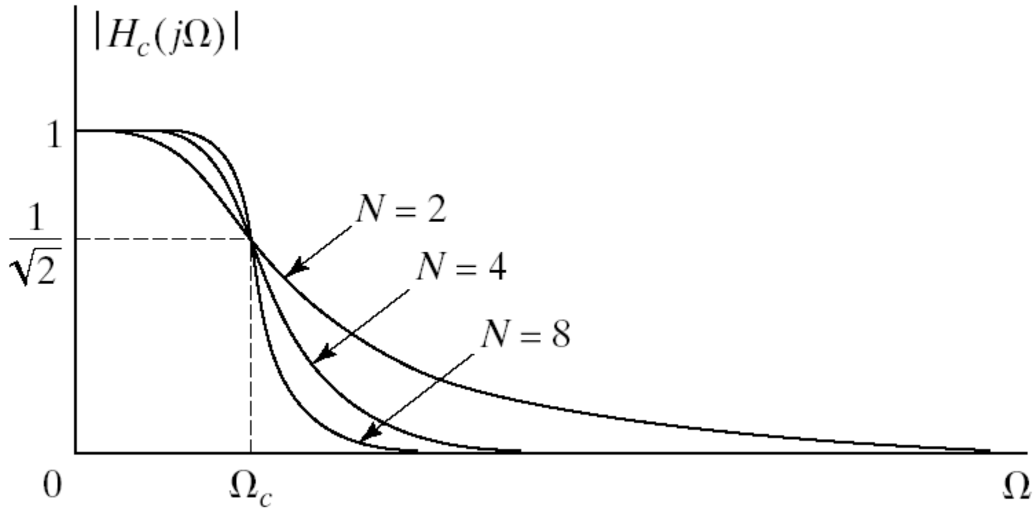
$$\delta_1 = 0.01$$
$$\delta_2 = 0.001$$
$$\Omega_p = 2\pi(2000)$$
$$\Omega_s = 2\pi(3000)$$

- **Specs in dB**
  - Ideal passband gain =20log(1) = 0 dB
  - Max passband gain = 20log(1.01) = 0.086dB
  - Max stopband gain = 20log(0.001) = -60 dB

Butterworth Lowpass Filters

- **Passband** is designed to be maximally flat
- The magnitude-squared function is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega / j\Omega_c)^{2N}} \qquad |H_c(s)|^2 = \frac{1}{1+(s / j\Omega_c)^{2N}}$$

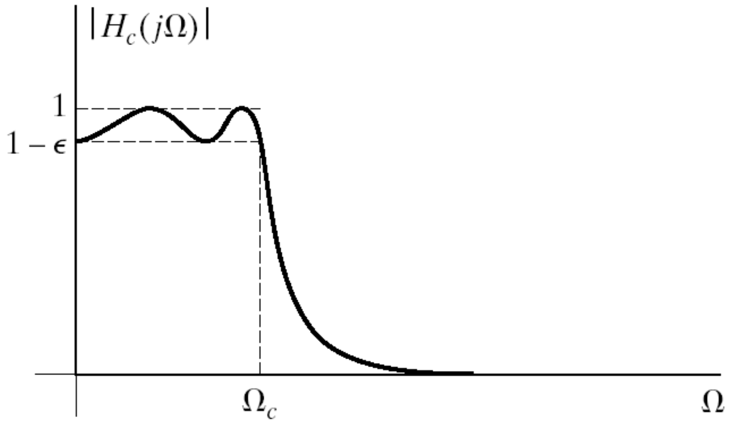


Poles: $s_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{(j\pi/2N)(2k+N-1)}$ for $k = 0,1,\ldots,2N\text{-}1$

Chebyshev Filters

- Equiripple in the passband and monotonic in the stopband (Type I)
- Or equiripple in the stopband and monotonic in the passband (Type II)

Type I: $$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)} \qquad V_N(x) = \cos\left(N\cos^{-1}x\right)$$



Type II: $$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\left(\varepsilon^2 V_N^2(\Omega/\Omega_c)\right)^{-1}} \qquad V_N(x) = \cos\left(N\cos^{-1}x\right)$$

Filter Design by Impulse Invariance

- Remember **impulse invariance**
  - Mapping a continuous-time impulse response to discrete-time
  - Mapping a continuous-time frequency response to discrete-time

$$h[n] = T_d h_c(nT_d)$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$

- If the continuous-time filter is bandlimited to

$$H_c(j\Omega) = 0 \qquad |\Omega| \geq \pi / T_d$$

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \qquad |\omega| \leq \pi$$

- If we start from discrete-time specifications T_d cancels out
  - Start with discrete-time spec in terms of ω
  - Go to continuous-time $\Omega = \omega / T$ and design continuous-time filter
  - Use impulse invariance to map it back to discrete-time $\omega = \Omega T$
- Works best for **bandlimited filters** due to possible aliasing

Impulse Invariance of System Functions

- Develop impulse invariance relation between system functions (disc. vs. cont.)
- Partial fraction expansion of transfer function

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

- Corresponding impulse response

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Impulse response of discrete-time filter

$$h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] = \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n u[n]$$

- System function

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

- Pole $s = s_k$ in s-domain transform into pole at $e^{s_k T_d}$

Example

- Impulse invariance applied to Butterworth

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

- Since sampling rate T_d cancels out we can assume $T_d = 1$
- Map spec to continuous time

$$T_d = 1 \;\Rightarrow\; \Omega = \omega / T_d = \omega$$

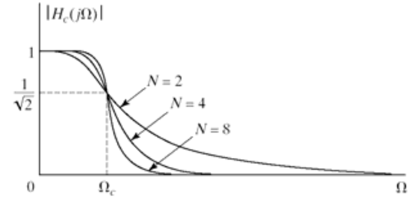
$$0.89125 \le \left|H\left(j\Omega\right)\right| \le 1 \qquad 0 \le |\Omega| \le 0.2\pi$$

$$\left|H\left(j\Omega\right)\right| \le 0.17783 \qquad 0.3\pi \le |\Omega| \le \pi$$

- Butterworth filter is monotonic so spec will be satisfied if

$$\left|H_c\left(j0.2\pi\right)\right| \ge 0.89125 \text{ and } \left|H_c\left(j0.3\pi\right)\right| \le 0.17783$$

$$\left|H_c\left(j\Omega\right)\right|^2 = \frac{1}{1 + \left(j\Omega / j\Omega_c\right)^{2N}}$$



- Determine N and Ω_c to satisfy these conditions

Example Cont'd

- Satisfy both constrains

$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.17783}\right)^2$$

- Solve these equations to get

$$N=5.8858\cong 6 \quad \text{and} \quad \Omega_c=0.70474$$

- N must be an integer so we round it up to meet the spec
- Poles of transfer function

$$s_k=(-1)^{1/12}(j\Omega_c)=\Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k=0,1,\ldots,11$$

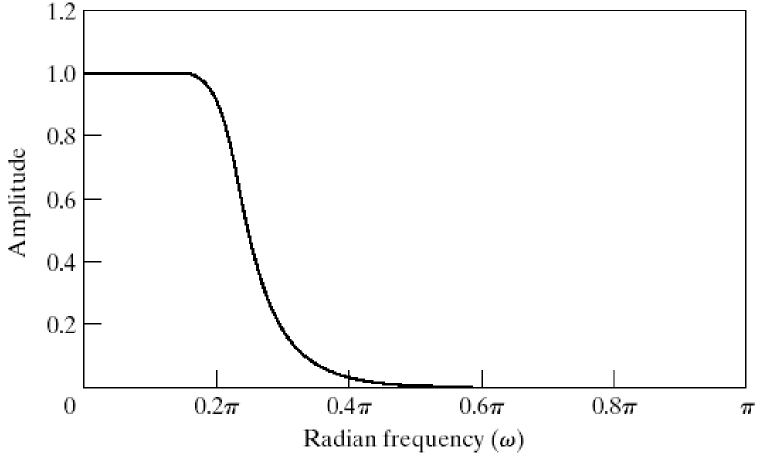
- The transfer function

$$H(s)=\frac{0.12093}{(s^2+0.364s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)}$$

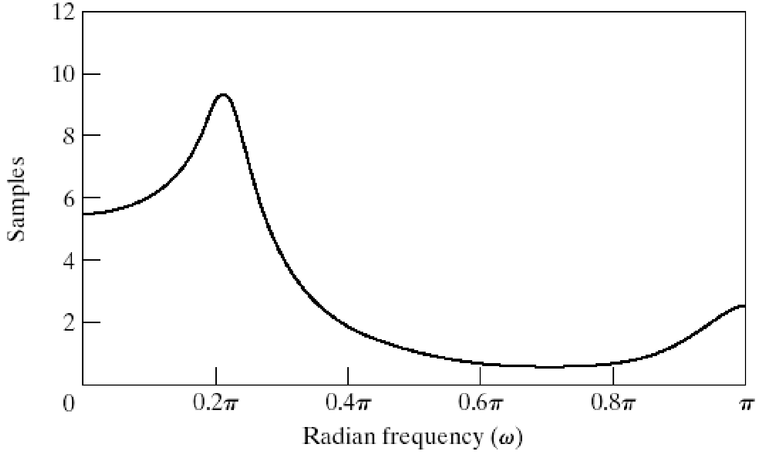
- Mapping to z-domain

$$H(z)=\frac{0.2871-0.4466z^{-1}}{1-1.2971z^{-1}+0.6949z^{-2}}+\frac{-2.1428+1.1455z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}}$$
$$+\frac{1.8557-0.6303z^{-1}}{1-0.9972z^{-1}+0.257z^{-2}}$$

Example Cont'd

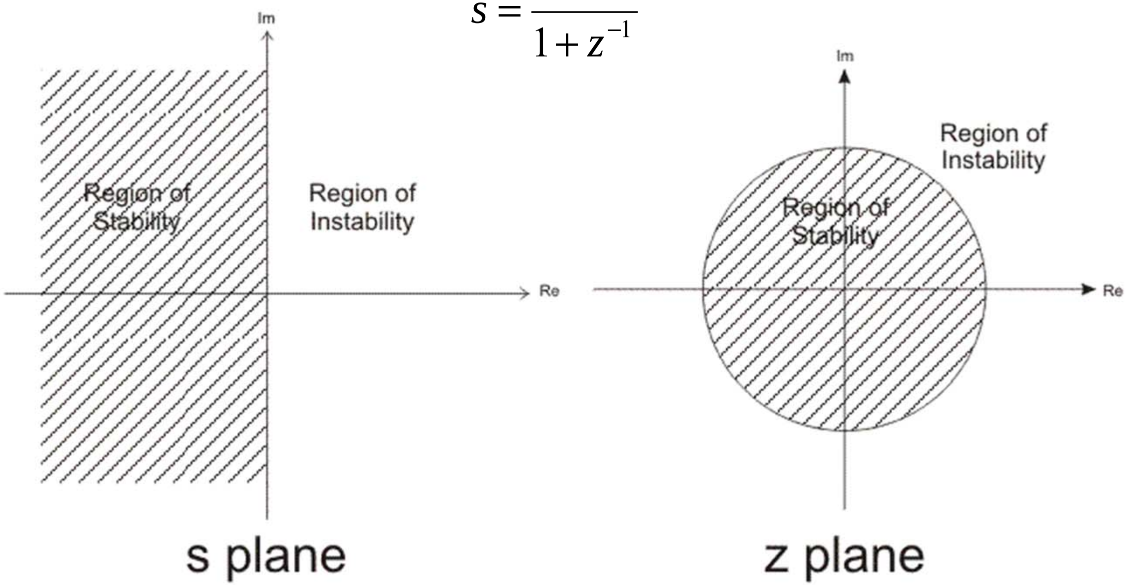


Group Delay:



Bilinear Transformation

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$



Filter Design by Bilinear Transformation

- Get around the **aliasing problem** of impulse invariance
- Map the entire s-plane onto the unit-circle in the z-plane
  - Nonlinear transformation
  - Frequency response subject to warping
- **Bilinear transformation**

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

- Transformed system function

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

- Again T_d cancels out so we can ignore it
- We can solve the transformation for z as

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s} = \frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2} \qquad s = \sigma + j\Omega$$

- Maps the **left-half** s-plane into the **inside of** the unit-circle in z
  - Stable in one domain would stay in the other

Bilinear Transformation

- On the unit circle the transform becomes

$$z = \frac{1 + j\Omega T_d / 2}{1 - j\Omega T_d / 2} = e^{j\omega}$$

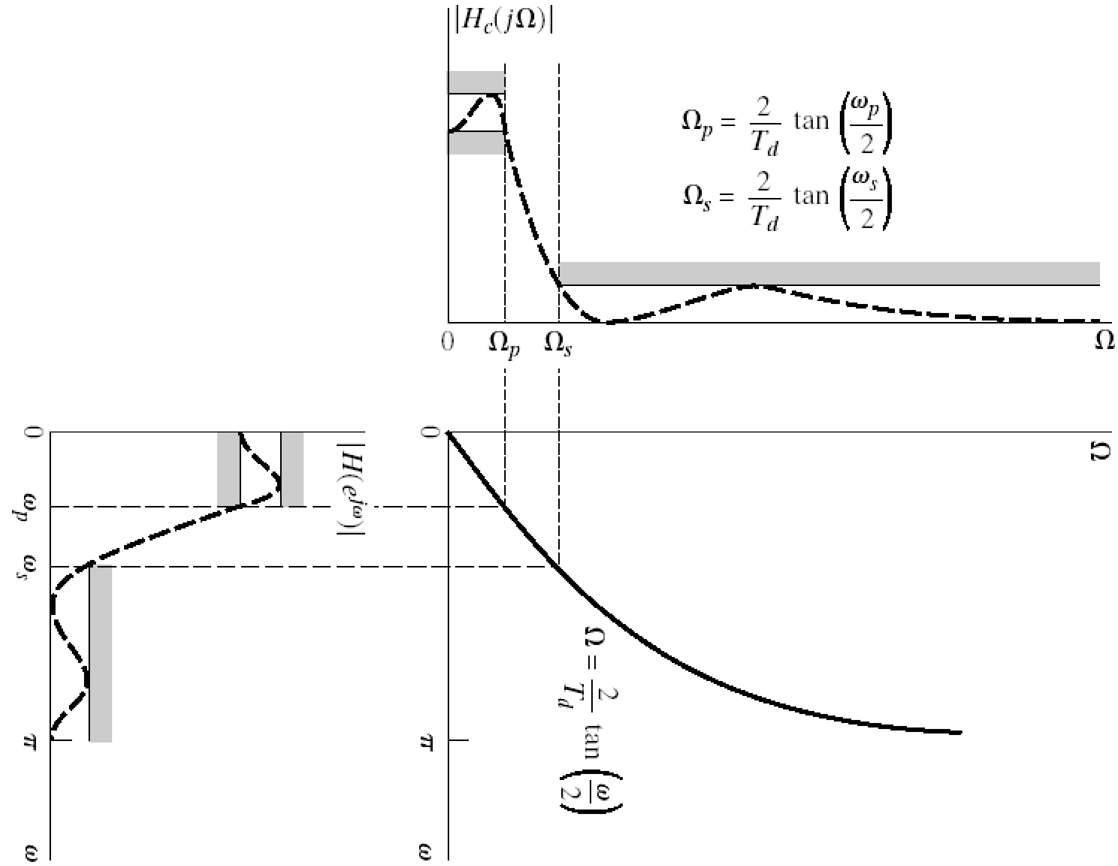
- To derive the relation between ω and Ω

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2} j \sin(\omega/2)}{2e^{-j\omega/2} \cos(\omega/2)}\right] = \frac{2j}{T_d}\tan\left(\frac{\omega}{2}\right)$$

- Which yields

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$$

Bilinear Transformation



Example

- Bilinear transform applied to Butterworth

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \quad 0.3\pi \le |\omega| \le \pi$$

- Apply bilinear transformation to specifications

$$0.89125 \le |H(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$$

$$|H(j\Omega)| \le 0.17783 \quad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| < \infty$$

- We can assume T_d =1 and apply the specifications to

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

- To get

$$1+\left(\frac{2\tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \text{ and } 1+\left(\frac{2\tan 0.15\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

Example Cont'd

- Solve N and Ω_c

$$N = \frac{\log\left[\left(\left(\frac{1}{0.17783}\right)^2 - 1\right) \Big/ \left(\left(\frac{1}{0.89125}\right)^2 - 1\right)\right]}{2\log\left[\tan(0.15\pi)/\tan(0.1\pi)\right]} = 5.305 \cong 6 \qquad \Omega_c = 0.766$$

- The resulting transfer function has the following poles

$$s_k = (-1)^{1/12}(j\Omega_c) = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\ldots,11$$

- Resulting in

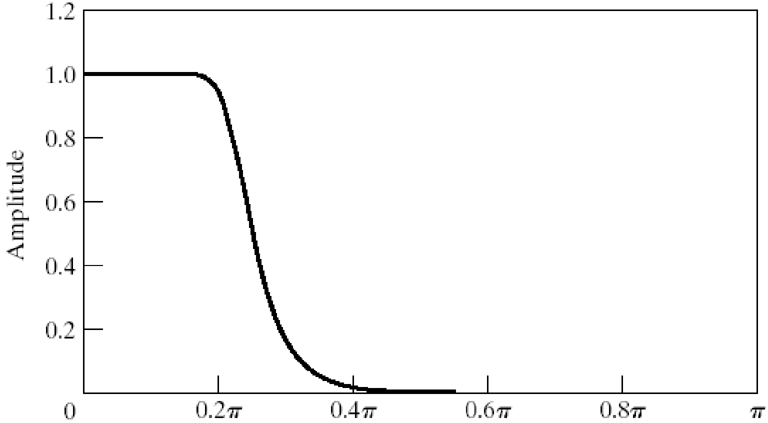
$$H_c(s) = \frac{0.20238}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

- Applying the bilinear transform yields

$$H(z) = \frac{0.0007378(1 + z^{-1})^6}{(1 - 1.2686z^{-1} + 0.7051z^{-2})(1 - 1.0106z^{-1} + 0.3583z^{-2})}$$

$$\times \frac{1}{(1 - 0.9044z^{-1} + 0.2155z^{-2})}$$

Example Cont'd



Group Delay: