# پردازش سیگنال دیجیتال

## درس ۱۶

# سیستم‌های می‌نیمم–فاز

# Minimum-Phase Systems

کاظم فولادی قلعه
دانشکده مهندسی، پردیس فارابی
دانشگاه تهران

http://courses.fouladi.ir/dsp

# Minimum-Phase Systems

# Minimum-Phase System

- A system with **all** poles and zeros **inside** the unit circle

- Both the system function and the inverse is **causal** and **stable**

- Name "**minimum-phase**" comes from the *property of the phase*
  - Not obvious to see with the given definition
  - Will look into it

- Given a magnitude square system function that is minimum phase,
  The original system is **uniquely** determined

- **Minimum-phase** and **All-pass** decomposition
  - Any rational system function can be decomposed as

$$H(z) = H_{\min}(z) H_{ap}(z)$$

# Example 1: Minimum-Phase System

- Consider the following system

$$H_1(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$$

- One pole inside the unit circle:
  - Make part of minimum-phase system
- One zero outside the unit circle:
  - Add an all-pass system to reflect this zero inside the unit circle

$$H_1(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}} = 3\frac{1}{1+\frac{1}{2}z^{-1}}\left(z^{-1}+\frac{1}{3}\right) = 3\frac{1}{1+\frac{1}{2}z^{-1}}\left(z^{-1}+\frac{1}{3}\right)\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}$$

$$H_1(z) = \left(3\frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{z^{-1}+\frac{1}{3}}{1-\frac{1}{3}z^{-1}}\right) = H_{\min}(z)H_{ap}(z)$$

# Example 2: Minimum-Phase System

- Consider the following system

$$H_2(z) = \frac{\left(1 + \frac{3}{2} e^{j\pi/4} z^{-1}\right)\left(1 + \frac{3}{2} e^{-j\pi/4} z^{-1}\right)}{1 - \frac{1}{3} z^{-1}}$$

- One pole inside the unit circle:
- Complex conjugate zero pair outside the unit circle

$$H_2(z) = \frac{\left(1 + \frac{3}{2} e^{j\pi/4} z^{-1}\right)\left(1 + \frac{3}{2} e^{-j\pi/4} z^{-1}\right)}{1 - \frac{1}{3} z^{-1}}$$

$$= \frac{\frac{3}{2} e^{j\pi/4} \frac{3}{2} e^{-j\pi/4} \left(\frac{2}{3} e^{-j\pi/4} + z^{-1}\right)\left(\frac{2}{3} e^{j\pi/4} + z^{-1}\right)}{1 - \frac{1}{3} z^{-1}}$$

# Example 2 Cont'd

$$H_2(z)=\frac{\frac{9}{4}\left(\frac{2}{3}e^{-j\pi/4}+z^{-1}\right)\left(\frac{2}{3}e^{j\pi/4}+z^{-1}\right)}{1-\frac{1}{3}z^{-1}}\cdot\frac{\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)}{\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)}$$

$$H_2(z)=\frac{\frac{9}{4}\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}\cdot\frac{\left(\frac{2}{3}e^{-j\pi/4}+z^{-1}\right)\left(\frac{2}{3}e^{j\pi/4}+z^{-1}\right)}{\left(1+\frac{2}{3}e^{-j\pi/4}z^{-1}\right)\left(1+\frac{2}{3}e^{j\pi/4}z^{-1}\right)}$$

$$H_2(z)=H_{\min}(z)H_{ap}(z)$$

# Frequency-Response Compensation

- In some applications a signal is distorted by an LTI system
- Could filter with inverse filter to recover input signal
  - Would work only with **minimum-phase systems**
- Make use of minimum-phase all-pass decomposition
  - Invert minimum phase part
- Assume a distorting system $H_d(z)$
- Decompose it into

$$H_d(z) = H_{d,\min}(z) H_{d,ap}(z)$$

- Define compensating system as

$$H_c(z) = \frac{1}{H_{d,\min}(z)}$$

- Cascade of the distorting system and compensating system

$$G(z) = H_c(z) H_d(z) = H_{d,\min}(z) H_{d,ap}(z) \frac{1}{H_{d,\min}(z)} = H_{d,ap}(z)$$

# Properties of Minimum-Phase Systems

- Minimum Phase-Lag Property
  - Continuous phase of a non-minimum-phase system

$$\arg\left[H_d\left(e^{j\omega}\right)\right] = \arg\left[H_{\min}\left(e^{j\omega}\right)\right] + \arg\left[H_{ap}\left(e^{j\omega}\right)\right]$$

  - All-pass systems have negative phase between 0 and π
  - So any non-minimum phase system will have a more negative phase compared to the minimum-phase system
  - The **negative** of the phase is called the **phase-lag function**
  - **The name** minimum-phase comes from **minimum phase-lag**

- Minimum Group-Delay Property

$$\mathrm{grd}\left[H_d\left(e^{j\omega}\right)\right] = \mathrm{grd}\left[H_{\min}\left(e^{j\omega}\right)\right] + \mathrm{grd}\left[H_{ap}\left(e^{j\omega}\right)\right]$$

  - Group-delay of all-pass systems is positive
  - Any **non-minimum-phase** system will always have **greater group delay**

# Properties of Minimum-Phase System

- Minimum Energy-Delay Property

$$\sum_{k=0}^{n} |h[k]|^2 \leq \sum_{k=0}^{n} |h_{\min}[k]|^2$$

  - Minimum-phase system concentrates energy in the early part
- Consider a minimum-phase system $H_{\min}(z)$
- Any $H(z)$ that has the same magnitude response as $H_{\min}(z)$
  - has the same poles as $H_{\min}(z)$
  - any number of zeros of $H_{\min}(z)$ are flipped outside the unit-circle
- Decompose one of the zeros of $H_{\min}(z)$

$$H_{\min}(z) = Q(z)\left(1 - z_k z^{-1}\right)$$

- Write $H(z)$ that has the same magnitude response as

$$H(z) = Q(z)\left(z^{-1} - z_k^*\right)$$

- We can write these in time domain

$$h_{\min}[n] = q[n] - z_k q[n-1] \qquad\qquad h[n] = q[n-1] - z_k^* q[n]$$

# Derivation Cont'd

- Evaluate each sum

$$\sum_{k=0}^{n}\left|h[k]\right|^2=\sum_{k=0}^{n}\left(\left|q[k-1]\right|^2-z_k^*q^*[k-1]q[k]-z_kq[k-1]q^*[k]+\left|z_k\right|^2\left|q[k]\right|^2\right)$$

$$\sum_{k=0}^{n}\left|h_{\min}[k]\right|^2=\sum_{k=0}^{n}\left(\left|q[k]\right|^2-z_k^*q^*[k-1]q[k]-z_kq[k-1]q^*[k]+\left|z_k\right|^2\left|q[k-1]\right|^2\right)$$

- And the difference is

$$\sum_{k=0}^{n}\left|h[k]\right|^2-\sum_{k=0}^{n}\left|h_{\min}[k]\right|^2=\left(\left|z_k\right|^2-1\right)\sum_{k=0}^{n}\left(\left|q[k]\right|^2-\left|q[k-1]\right|^2\right)=\left(\left|z_k\right|^2-1\right)\left|q[n]\right|^2$$

- Since $|z_k| < 1$

$$\sum_{k=0}^{n}\left|h[k]\right|^2\le\sum_{k=0}^{n}\left|h_{\min}[k]\right|^2$$