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رابطه‌ی میان اندازه و فاز

Relationship between Magnitude and Phase

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Relationship between Magnitude and Phase

Relation between Magnitude and Phase

- For **general LTI system**
  - Knowledge about magnitude doesn't provide any information about phase
  - Knowledge about phase doesn't provide any information about magnitude

- For **linear constant-coefficient difference equations** however
  - There is some constraint between **magnitude** and **phase**
  - If magnitude and number of pole-zeros are known
    - Only a finite number of choices for phase
  - If phase and number of pole-zeros are known
    - Only a finite number of choices for magnitude (ignoring scale)

- **A class of systems called minimum-phase**
  - Magnitude specifies phase uniquely
  - Phase specifies magnitude uniquely

Square Magnitude System Function

- Explore possible choices of system function of the form

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)^* H\left(e^{j\omega}\right) = H^*\left(1/z^*\right)H(z)\Big|_{z=e^{j\omega}}$$

- Restricting the system to be rational

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)} \qquad H^*\left(1/z^*\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k^* z\right)}$$

- The square system function

$$C(z) = H(z)H^*\left(1/z^*\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)\left(1-d_k^* z\right)}$$

- Given $\left|H\left(e^{j\omega}\right)\right|^2$ we can get $C(z)$
- What information on $H(z)$ can we get from $C(z)$?

Poles and Zeros of Magnitude Square System Function

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

- For every pole d_k in $H(z)$ there is a pole of $C(z)$ at d_k and $(1/d_k)^*$
- For every zero c_k in $H(z)$ there is a zero of $C(z)$ at c_k and $(1/c_k)^*$
- Poles and zeros of $C(z)$ occur in conjugate reciprocal pairs
- If one of the pole/zero is **inside** the unit circle the reciprocal will be **outside**
  - Unless there are both on the unit circle
- If $H(z)$ is **stable** all poles have to be inside the unit circle
  - We can infer which poles of $C(z)$ belong to $H(z)$
- However, zeros cannot be uniquely determined
  - Example to follow

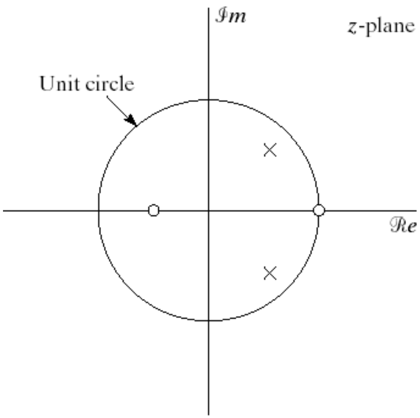
Example

- Two systems with

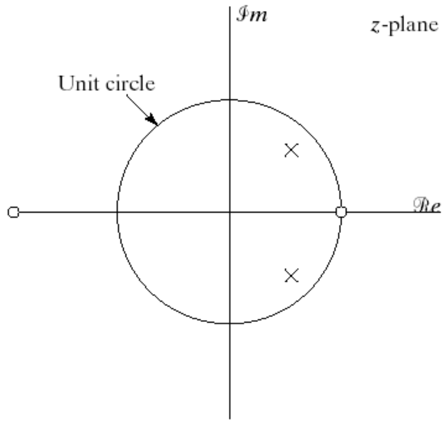
$$H_1(z)=\frac{2\left(1-z^{-1}\right)\left(1-0.5z^{-1}\right)}{\left(1-0.8e^{j\pi/4}z^{-1}\right)\left(1-0.8e^{-j\pi/4}z^{-1}\right)}$$

$$H_2(z)=\frac{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)}{\left(1-0.8e^{j\pi/4}z^{-1}\right)\left(1-0.8e^{-j\pi/4}z^{-1}\right)}$$

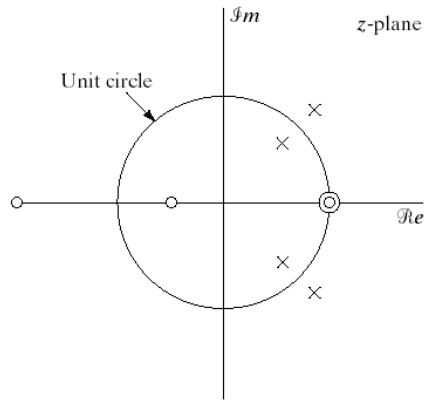
- Both share the same magnitude square system function



$H_1(z)$



$H_2(z)$

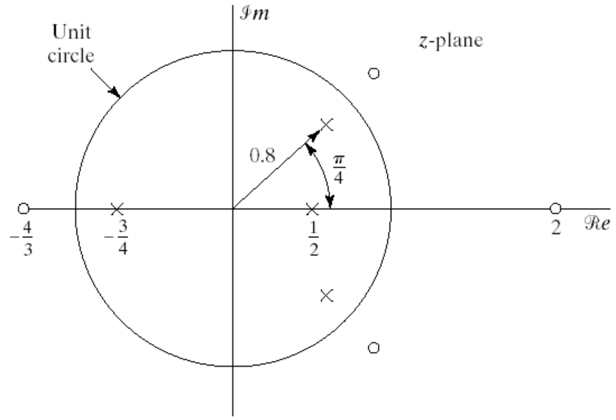


$$C(z)=H_1(z)H_1^*\left(1/z^*\right)=H_2(z)H_2^*\left(1/z^*\right)$$

All-Pass System

- A system with <u>frequency response magnitude</u> constant
- Important uses such as compensating for phase distortion
- Simple all-pass system

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$



- Magnitude response constant

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- Most general form with real impulse response

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1} - d_k}{1 - d_k z^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1} - c_k^*)(z^{-1} - c_k)}{(1 - c_k z^{-1})(1 - c_k^* z^{-1})}$$

- A: positive constant, d_k: real poles, c_k: complex poles

Phase of All-Pass Systems

$$H_{ap}\left(e^{j\omega}\right)=\frac{e^{-j\omega}-a^*}{1-ae^{-j\omega}}=e^{-j\omega}\frac{1-a^*e^{j\omega}}{1-ae^{-j\omega}}$$

- Let's write the phase with *a* represented in polar form

$$\angle\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right]=-\omega-2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

- The group delay of this system can be written as

$$\mathrm{grd}\left[\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right]=\frac{1-r^2}{1-2r\cos(\omega-\theta)+r^2}=\frac{1-r^2}{\left|1-re^{j\theta}e^{-j\omega}\right|^2}$$

- For stable and causal system $|r| < 1$
  - Group delay of all-pass systems is always positive
- Phase between 0 and π is always negative

$$\arg\left[H_{ap}\left(e^{j\omega}\right)\right]\leq 0 \quad \text{for} \quad 0\leq\omega<\pi$$