پردازش سیگنال دیجیتال

درس ۱۱

تغییر نرخ نمونه‌برداری

Changing the Sampling Rate

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Changing the Sampling Rate

Changing the Sampling Rate

- A continuous-time signal can be represented by its samples as

$$x[n] = x_c(nT)$$

- We can use bandlimited interpolation to go back to the continuous-time signal from its samples
- **Some applications require us to change the sampling rate**
  - Or to obtain a new discrete-time representation of the same continuous-time signal of the form

$$x'[n] = x_c(nT') \qquad \text{where } T \neq T'$$

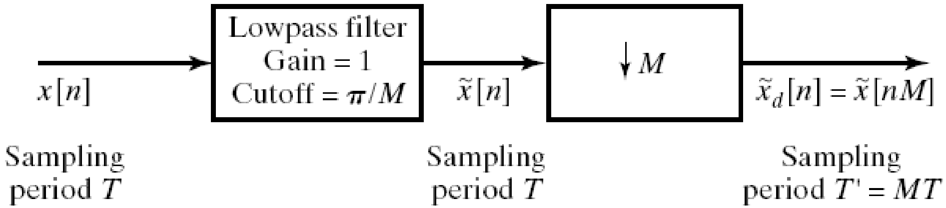
- **The problem is to get $x'[n]$ given $x[n]$**
- One way of accomplishing this is to
  - **Reconstruct** the continuous-time signal from $x[n]$
  - **Resample** the continuous-time signal using new rate to get $x'[n]$
  - This requires analog processing which is often undesired

Sampling Rate Reduction by an Integer Factor: Downsampling

- We reduce the sampling rate of a sequence by "sampling" it

$$x_d[n] = x[nM] = x_c(nMT)$$

- This is accomplished with a sampling rate **compressor**



- We obtain $x_d[n]$ that is identical to what we would get by reconstructing the signal and resampling it with $T' = MT$
- There will be no aliasing if

$$\frac{\pi}{T'} = \frac{\pi}{MT} > \Omega_N$$

Frequency Domain Representation of Downsampling

- Recall the DTFT of $x[n]=x_c(nT)$

$$X\left(e^{j\omega}\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{T}-\frac{2\pi k}{T}\right)\right)$$

- The DTFT of the downsampled signal can similarly written as

$$X_d\left(e^{j\omega}\right)=\frac{1}{T'}\sum_{r=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{T'}-\frac{2\pi r}{T'}\right)\right)=\frac{1}{MT}\sum_{r=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi r}{MT}\right)\right) \qquad T'=MT$$

- Let's represent the summation index as

$$r=i+kM \quad \text{where} \quad -\infty<k<\infty \quad \text{and} \quad 0\le i<\text{M}$$

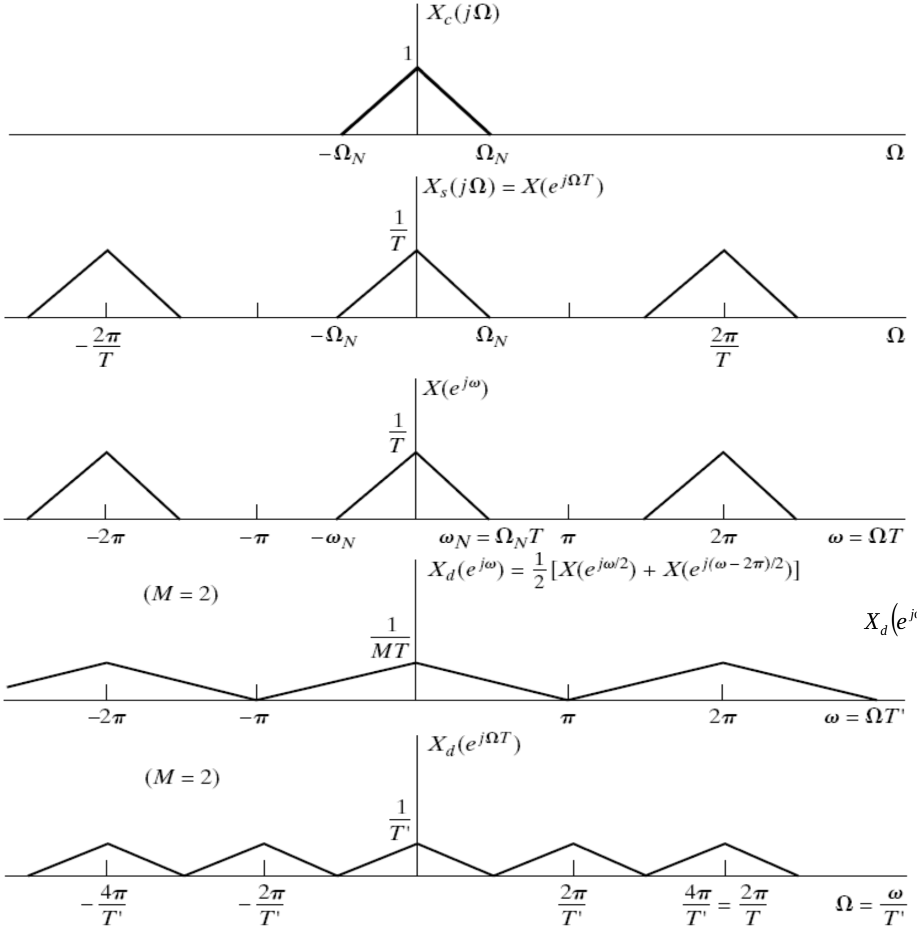
$$X_d\left(e^{j\omega}\right)=\frac{1}{M}\sum_{i=0}^{M-1}\left[\underbrace{\boxed{\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi k}{T}-\frac{2\pi i}{MT}\right)\right)}}_{X\left(e^{j(\omega-2\pi i)/M}\right)}\right]$$

- And finally

$$X_d\left(e^{j\omega}\right)=\frac{1}{M}\sum_{i=0}^{M-1}X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)$$

M shifted replicates of $X(e^{j\omega})$

Frequency Domain Representation of Downsampling: No Aliasing



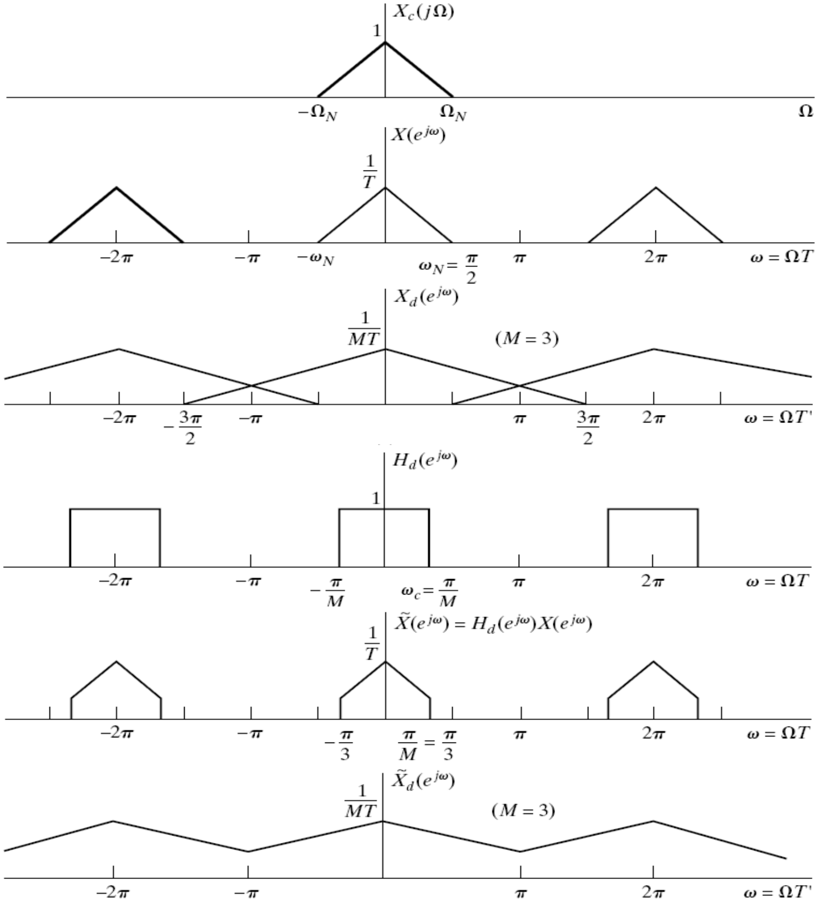
$$X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)$$

$\omega = \Omega T'$

$T' = MT$

$M = 2$

Frequency Domain Representation of Downsampling w/ Prefilter



$$X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)$$

$\omega = \Omega T'$

$T' = MT$

$M = 3$

$\omega = \Omega T'$

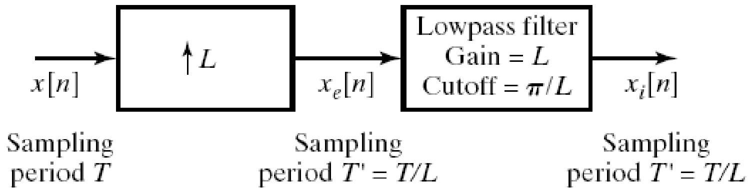
$T' = MT$

$M = 3$

- We increase the sampling rate of a sequence interpolating it

$$x_i[n] = x[n/L] = x_c(nT/L)$$

- This is accomplished with a sampling rate **expander**



- We obtain $x_i[n]$ that is identical to what we would get by reconstructing the signal and resampling it with $T' = T/L$
- Upsampling consists of two steps
  - Expanding

$$x_e[n] = \begin{cases} x[n/L] & n = 0, \mp L, \mp 2L, \ldots \\ 0 & \text{otherwise} \end{cases} = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

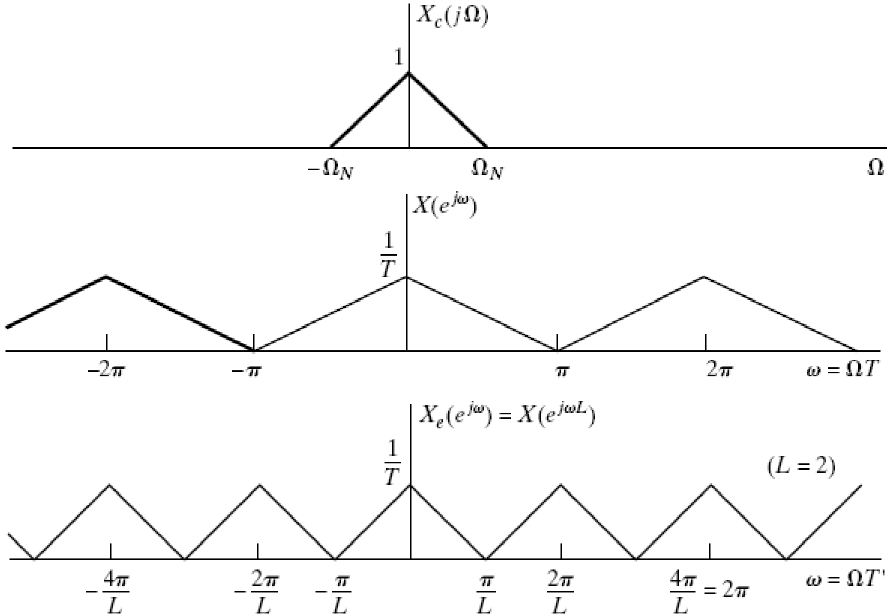
  - Interpolating

Frequency Domain Representation of Expander

- The DTFT of $x_e[n]$ can be written as

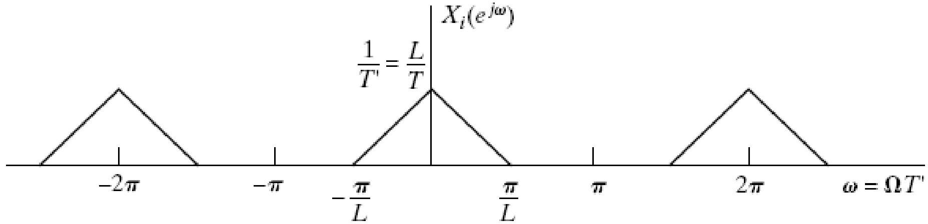
$$X_e\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}x[k]\delta[n-kL]\right)e^{-j\omega n}=\sum_{k=-\infty}^{\infty}x[k]e^{-j\omega Lk}=X\left(e^{j\omega L}\right)$$

- The output of the expander is frequency-scaled

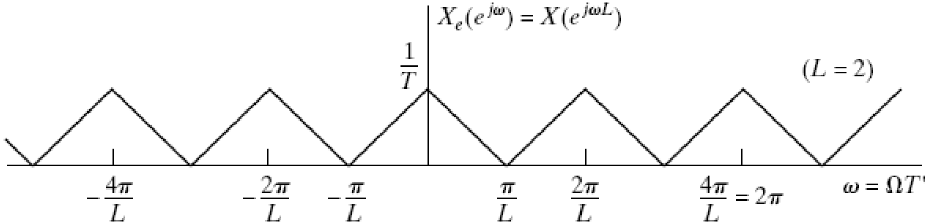


Frequency Domain Representation of Interpolator

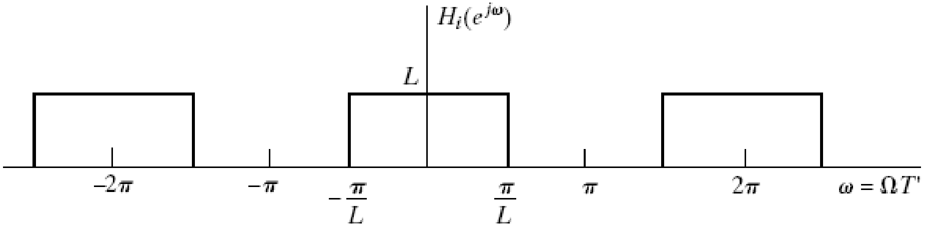
- The DTFT of the desired interpolated signals is



- The extrapolator output is given as



- To get interpolated signal we apply the following LPF



Interpolator in Time Domain

- $x_i[n]$ in a low-pass filtered version of $x[n]$
- The low-pass filter impulse response is

$$h_i[n] = \frac{\sin(\pi n / L)}{\pi n / L}$$

- Hence the interpolated signal is written as

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin(\pi(n-kL)/L)}{\pi(n-kL)/L}$$

- Note that

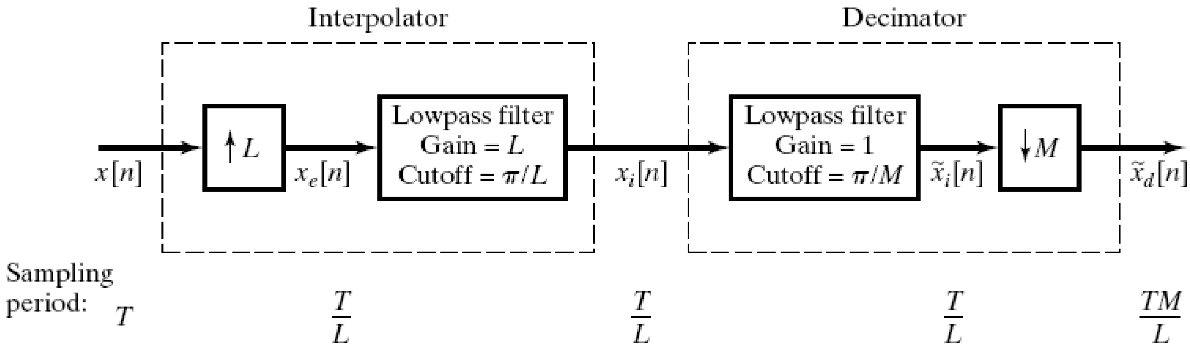
$$h_i[0] = 1$$
$$h_i[n] = 0 \quad \mathrm{n} = \mp\mathrm{L}, \mp 2\mathrm{L}, \ldots$$

- Therefore the filter output can be written as

$$x_i[n] = x[n/L] = x_c(nT/L) = x_c(nT') \quad \text{for} \quad n = 0, \mp\mathrm{L}, \mp 2\mathrm{L}, \ldots$$

Changing the Sampling Rate by Non-Integer Factor

- Combine decimation and interpolation for non-integer factors



- The two low-pass filters can be combined into a single one