پردازش سیگنال دیجیتال

درس ۹

نمونه‌برداری از سیگنال‌های پیوسته–زمان

Sampling of Continuous-Time Signals

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Sampling of Continuous-Time Signals

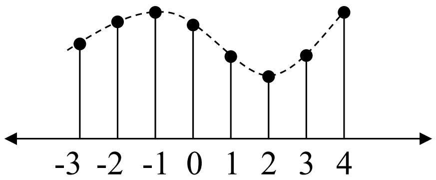
Signal Types

- Analog signals: continuous in time and amplitude
  - Example: voltage, current, temperature,…
- Digital signals: discrete both in time and amplitude
  - Example: attendance of this class, digitizes analog signals,…
- Discrete-time signal: discrete in time, continuous in amplitude
  - Example: hourly change of temperature in Tehran

- Theory for digital signals would be too complicated
  - Requires inclusion of nonlinearities into theory
- Theory is based on discrete-time continuous-amplitude signals
  - Most convenient to develop theory
  - Good enough approximation to practice with some care
- In practice we mostly process digital signals on processors
  - Need to take into account finite precision effects
- Our text book is about the theory hence its title
  - *Discrete-Time Signal Processing*

Periodic (Uniform) Sampling

- **Sampling** is a continuous to discrete-time conversion



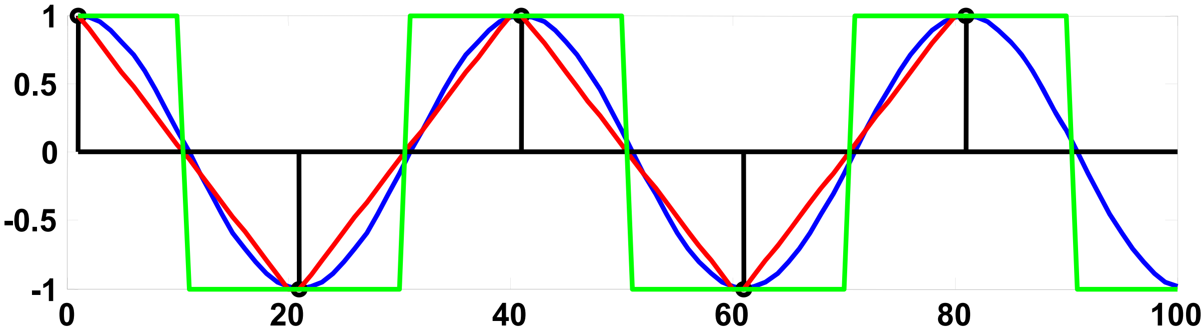
- Most common sampling is periodic

$$x[n] = x_c(nT) \quad -\infty < n < \infty$$

- T is the sampling period in second
- $f_s = 1/T$ is the sampling frequency in Hz
- Sampling frequency in radian-per-second $\Omega_s = 2\pi f_s$ rad/sec
- Use [.] for discrete-time and (.) for continuous time signals
- This is the ideal case not the practical but close enough
  - In practice it is implement with an analog-to-digital converters
  - We get digital signals that are quantized in amplitude and time

Periodic Sampling

- Sampling is, in general, not reversible
- Given a sampled signal one could fit infinite continuous signals through the samples



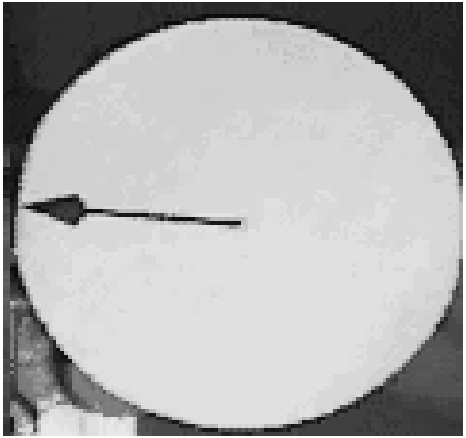
- Fundamental issue in digital signal processing
  - If we loose information during sampling we cannot recover it
- *Under certain conditions an analog signal can be sampled without loss so that it can be reconstructed perfectly*

Sampling Demo

- In this movie the video camera is sampling at a fixed rate of 30 frames/second.
- Observe how the rotating phasor **aliases** to different speeds as it spins faster.

$$p(t) = e^{-j2\pi f_o t}$$

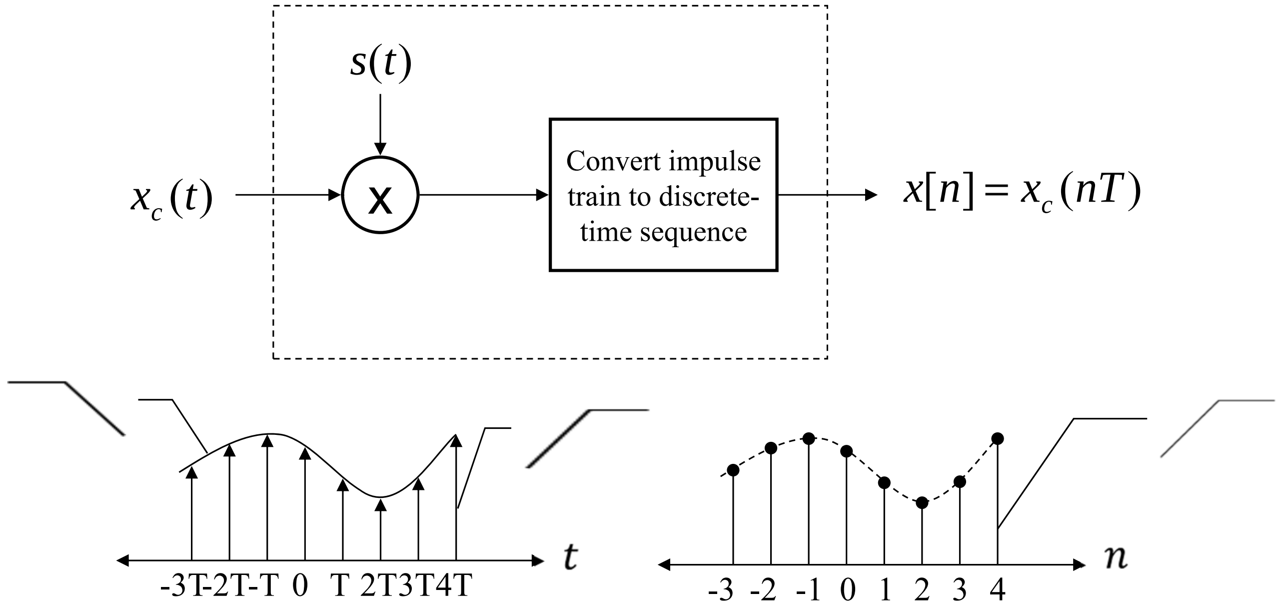
$$p[n] = p(nT) = p(n/f_s) = e^{-j2\pi \frac{f_o}{f_s} n}$$



Demo from *DSP First: A Multimedia Approach* by McClellan, Schafer, Yoder

Representation of Sampling

- Mathematically convenient to represent in two stages
  - Impulse train modulator
  - Conversion of impulse train to a sequence



Continuous-Time Fourier Transform

- Continuous-Time Fourier transform pair is defined as

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt$$

$$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega) e^{j\Omega t} d\Omega$$

- We write $x_c(t)$ as a weighted sum of complex exponentials
- Remember some Fourier Transform properties
  - Time Convolution (frequency domain multiplication)

$$x(t) * y(t) \leftrightarrow X(j\Omega) Y(j\Omega)$$

  - Frequency Convolution (time domain multiplication)

$$x(t) y(t) \leftrightarrow \tfrac{1}{2\pi} X(j\Omega) * Y(j\Omega)$$

  - Modulation (Frequency shift)

$$x(t) e^{j\Omega_o t} \leftrightarrow X\left(j\left(\Omega - \Omega_o\right)\right)$$

Frequency Domain Representation of Sampling

- Modulate (multiply) continuous-time signal with impulse train:

$$x_s(t) = x_c(t)s(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

- Let's take the Fourier Transform of $x_s(t)$ and $s(t)$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

- Fourier transform of impulse train is again a impulse train
- Note that multiplication in time is convolution in frequency
- We represent frequency with $\Omega = 2\pi f$ hence $\Omega_s = 2\pi f_s$

$$\Omega_s = \frac{2\pi}{T}$$

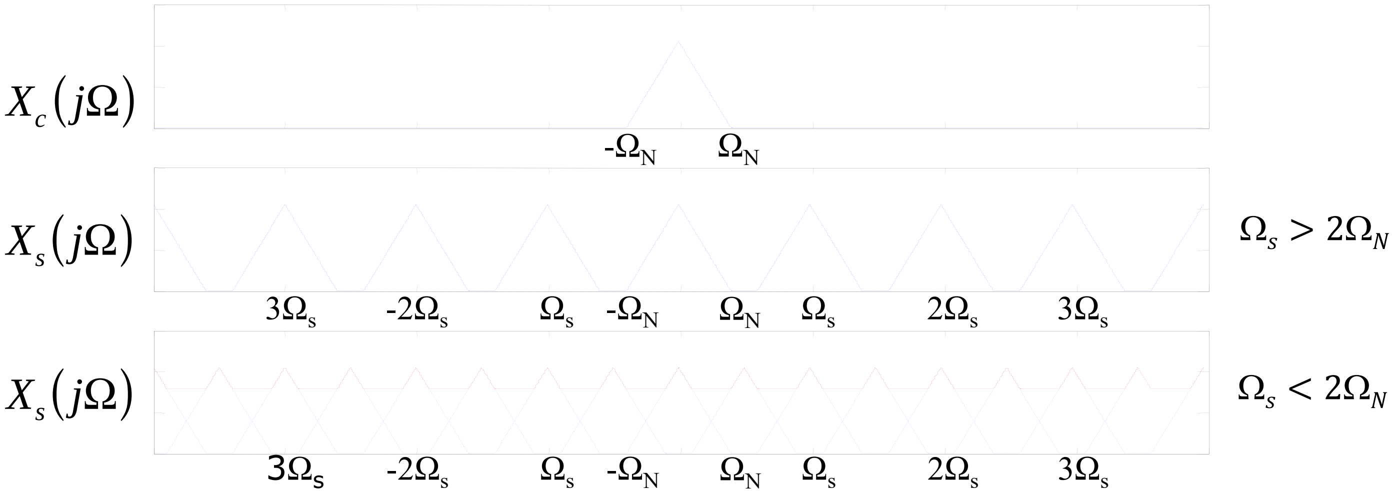
$$X_s(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\Theta) S(\Theta - \Omega) d\Theta = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

Frequency Domain Representation of Sampling

- **Convolution with impulse** creates replicas at impulse location:

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\big(j(\Omega - k\Omega_s)\big)$$

- This tells us that the impulse train modulator
  - Creates images of the Fourier transform of the input signal
  - Images are periodic with sampling frequency
  - If $\Omega_s < \Omega_N$ sampling maybe irreversible due to aliasing of images



Nyquist Sampling Theorem

- Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\Omega) = 0 \qquad \text{for } |\Omega| \geq \Omega_N$$

- Then $x_c(t)$ is uniquely determined by its samples $x[n]= x_c(nT)$ if

$$\Omega_s = \frac{2\pi}{T} = 2\pi f_s \geq 2\Omega_N$$

- Ω_N is generally known as the Nyquist Frequency
- The minimum sampling rate that must be exceeded is known as the Nyquist Rate