بسم الله الرحمن الرحیم

پردازش سیگنال دیجیتال

درس ۸

تبدیل Z معکوس

The Inverse z-Transform

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The Inverse z-Transform

- Formal inverse z-transform is based on a Cauchy integral $x[n] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$

- Less formal ways sufficient most of the time
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

The Inverse z-Transform by Inspection Method

Inspection Method

- Make use of known z-transform pairs such as

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

- Example: The inverse z-transform of

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \quad \rightarrow \quad x[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse z-Transform by Partial Fraction Expansion

- Assume that a given z-transform can be expressed as

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Apply partial fractional expansion

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - d_i z^{-1}\right)^m}$$

- First term exist only if $M > N$
  - B_r is obtained by long division
- Second term represents all first order poles
- Third term represents an order s pole
  - There will be a similar term for every high-order pole
- Each term can be inverse transformed by inspection

Partial Fractional Expression

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1-d_i z^{-1}\right)^m}$$

- Coefficients are given as

$$A_k = \left(1-d_k z^{-1}\right)X(z)\Big|_{z=d_k}$$

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[\left(1-d_i w\right)^s X\left(w^{-1}\right)\right] \right\}_{w=d_i^{-1}}$$

- Easier to understand with examples

Example: 2nd Order z-Transform

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \qquad \text{ROC}: |z|>\frac{1}{2}$$

- Order of nominator is smaller than denominator (in terms of z^{-1})
- No higher order pole

$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$A_1=\left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}}=\frac{1}{\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^{-1}\right)}=-1$$

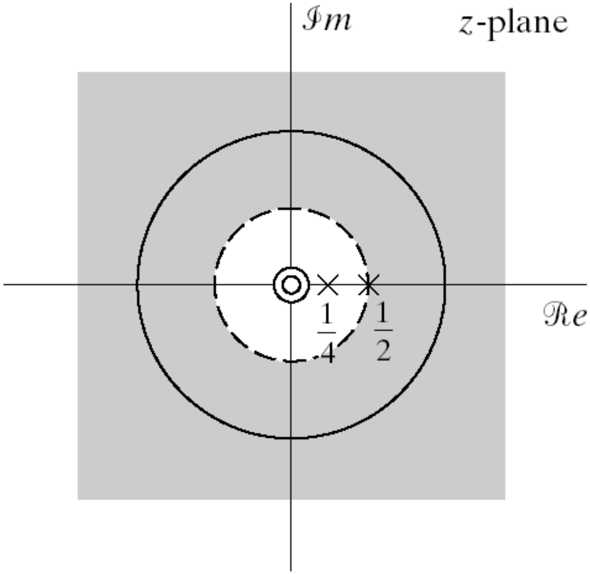
$$A_2=\left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}}=\frac{1}{\left(1-\frac{1}{4}\left(\frac{1}{2}\right)^{-1}\right)}=2$$

Example Continued

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)} \qquad |z| > \frac{1}{2}$$

- ROC extends to infinity
  - Indicates right sided sequence

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$



Example #2

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{\left(1 + z^{-1}\right)^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)} \qquad |z| > 1$$

- Long division to obtain B_o

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \overline{\big) z^{-2} + 2z^{-1} + 1} \\ \underline{z^{-2} - 3z^{-1} + 2} \\ 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}$$

$$X(z) = 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

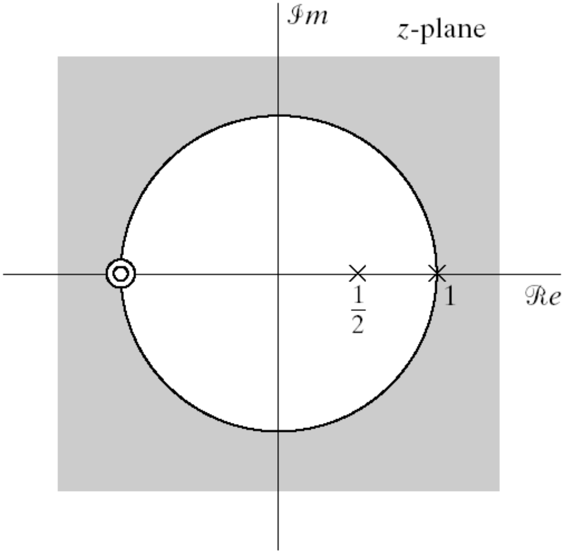
$$A_1 = \left(1 - \frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = -9 \qquad A_2 = \left(1 - z^{-1}\right)X(z)\Big|_{z=1} = 8$$

Example #2 Continued

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}} \qquad |z| > 1$$

- ROC extends to infinity
  - Indicates right-sides sequence

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] - 8u[n]$$



Inverse z-Transform by Power Series Expansion

- The z-transform is power series

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

- In expanded form

$$X(z)=\cdots+x[-2]z^{2}+x[-1]z^{1}+x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots$$

- z-transforms of this form can generally be inversed easily
- Especially useful for finite-length series
- Example

$$X(z)=z^{2}\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)\left(1-z^{-1}\right)$$

$$=z^{2}-\frac{1}{2}z-1+\frac{1}{2}z^{-1}$$

$$x[n]=\delta[n+2]-\frac{1}{2}\delta[n+1]-\delta[n]+\frac{1}{2}\delta[n-1]$$

$$x[n]=\begin{cases}1 & n=-2\\ -\frac{1}{2} & n=-1\\ -1 & n=0\\ \frac{1}{2} & n=1\\ 0 & \text{otherwise}\end{cases}$$

z-Transform Properties: Linearity

- Notation

$$x[n] \xleftrightarrow{Z} X(z) \qquad \text{ROC} = R_x$$

- Linearity

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z) \qquad ROC = R_{x_1} \cap R_{x_2}$$

  – Note that the ROC of combined sequence may be larger than either ROC
  – This would happen if some pole/zero cancellation occurs

  – Example:

$$x[n] = a^n u[n] - a^n u[n-N]$$

    - Both sequences are right-sided
    - Both sequences have a pole $z = a$
    - Both have a ROC defined as $|z| > |a|$
    - In the combined sequence the pole at $z = a$ cancels with a zero at $z = a$
    - The combined ROC is the entire z plane except $z = 0$

- We did make use of this property already, where?

z-Transform Properties: Time Shifting

$$x[n-n_o] \overset{Z}{\longleftrightarrow} z^{-n_o}X(z) \qquad \text{ROC} = R_x$$

- Here n_o is an integer
  - If **positive** the sequence is shifted right
  - If **negative** the sequence is shifted left
- The ROC can change the new term may
  - Add or remove poles at $z = 0$ or $z = \infty$

- Example

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) \qquad |z| > \frac{1}{4}$$

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

z-Transform Properties: Multiplication by Exponential

$$z_o^n x[n] \overset{Z}{\longleftrightarrow} X(z/z_o) \qquad \text{ROC} = |z_o| R_x$$

- ROC is scaled by $|z_o|$
- All pole/zero locations are scaled
- If z_o is a positive real number: z-plane shrinks or expands
- If z_o is a complex number with unit magnitude it rotates
- Example: We know the z-transform pair

$$u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-z^{-1}} \qquad \text{ROC}: \ |z| > 1$$

- Let's find the z-transform of

$$x[n] = r^n \cos(\omega_o n) u[n] = \frac{1}{2}\left(re^{j\omega_o}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_o}\right)^n u[n]$$

$$X(z) = \frac{1/2}{1-re^{j\omega_o}z^{-1}} + \frac{1/2}{1-re^{-j\omega_o}z^{-1}} \qquad |z| > r$$

z-Transform Properties: Differentiation

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz} \qquad \text{ROC} = R_x$$

- Example: We want the inverse z-transform of

$$X(z) = \log\left(1 + az^{-1}\right) \qquad |z| > |a|$$

- Let's differentiate to obtain rational expression

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}} \Rightarrow -z\frac{dX(z)}{dz} = az^{-1}\frac{1}{1+az^{-1}}$$

- Making use of z-transform properties and ROC

$$nx[n] = a(-a)^{n-1}u[n-1]$$

$$x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

z-Transform Properties: Conjugation

$$x^*[n] \overset{Z}{\longleftrightarrow} X^*(z^*) \qquad \text{ROC} = R_x$$

- Proof

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X^*(z) = \left(\sum_{n=-\infty}^{\infty} x[n] z^{-n} \right)^* = \sum_{n=-\infty}^{\infty} x^*[n] z^n$$

$$X^*(z^*) = \sum_{n=-\infty}^{\infty} x^*[n] (z^n)^* = \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = Z\{x^*[n]\}$$

z-Transform Properties: Time Reversal

$$x[-n] \stackrel{z}{\longleftrightarrow} X(1/z) \qquad \text{ROC} = \frac{1}{R_x}$$

- ROC is inverted

- Example:

$$x[n] = a^{-n}u[-n]$$

- Time reversed version of $a^n u[n]$

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

z-Transform Properties: Convolution

$$x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z)X_2(z) \qquad \text{ROC}: R_{x1} \cap R_{x_2}$$

- **Convolution in time domain is multiplication in z-domain**
- Example: Let's calculate the convolution of

$$x_1[n] = a^n u[n] \quad \text{and} \quad x_2[n] = u[n]$$

$$X_1(z) = \frac{1}{1 - az^{-1}} \quad \text{ROC}: |z| > |a| \qquad X_2(z) = \frac{1}{1 - z^{-1}} \quad \text{ROC}: |z| > 1$$

- Multiplications of z-transforms is

$$Y(z) = X_1(z)X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

- ROC: if $|a| < 1$ ROC is $|z| > 1$, if $|a| > 1$ ROC is $|z| > |a|$
- Partial fractional expansion of $Y(z)$

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - az^{-1}}\right) \quad \text{assume ROC}: |z| > 1$$

$$y[n] = \frac{1}{1-a}\left(u[n] - a^{n+1}u[n]\right)$$