

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ



پردازش سیگنال دیجیتال

درس ۷

تبدیل Z

The z-Transform

کاظم فولادی قلعه
دانشکده مهندسی، پردیس فارابی
دانشگاه تهران

<http://courses.fouladi.ir/dsp>

The z-Transform

- Counterpart of the Laplace transform for discrete-time signals
- Generalization of the Fourier Transform
 - Fourier Transform does not exist for all signals
- The z-Transform is often time more convenient to use
- Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

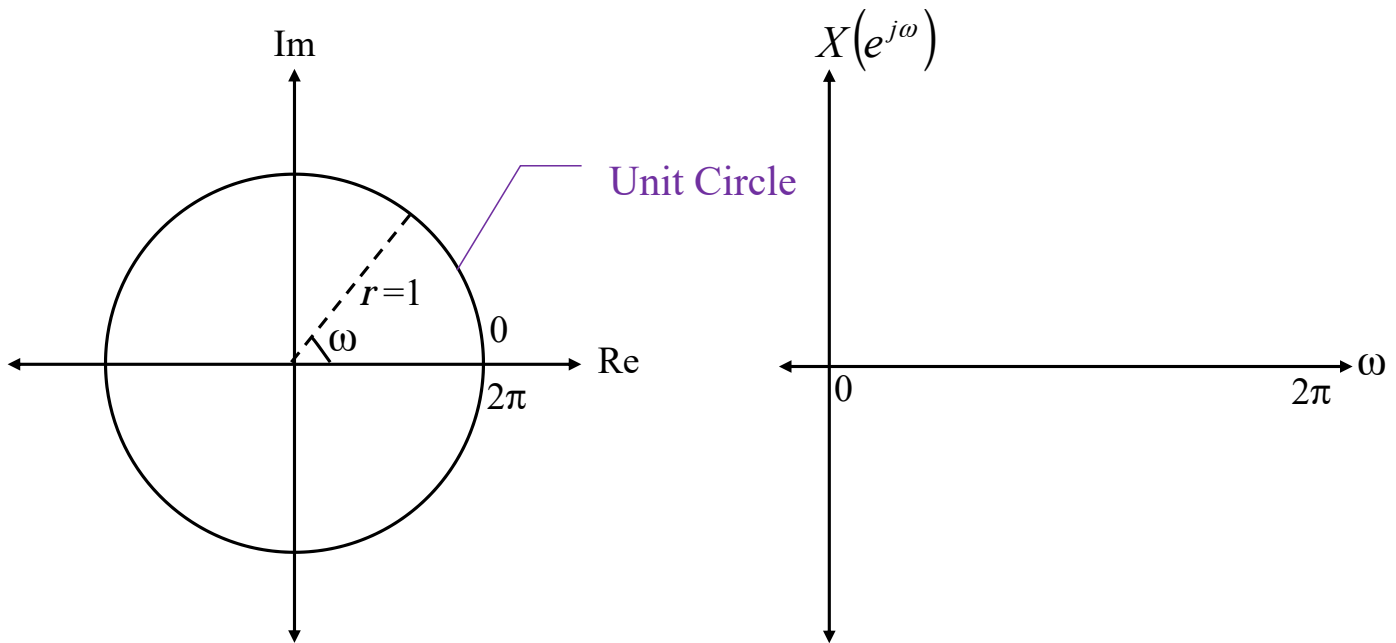
- Compare to DTFT definition:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z is a complex variable that can be represented as $z = r e^{j\omega}$
- Substituting $z = e^{j\omega}$ will reduce the z-transform to DTFT

The z-transform and the DTFT

- The z-transform is a function of the complex z variable
- Convenient to describe on the complex z -plane
- If we plot $z = e^{j\omega}$ for $\omega = 0$ to 2π we get the unit circle



Convergence of the z-Transform

- DTFT does not always converge

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- Infinite sum not always finite if $x[n]$ no absolute summable
- **Example:** $x[n] = a^n u[n]$ for $|a| > 1$ does not have a DTFT

- Complex variable z can be written as $re^{j\omega}$ so the z-transform

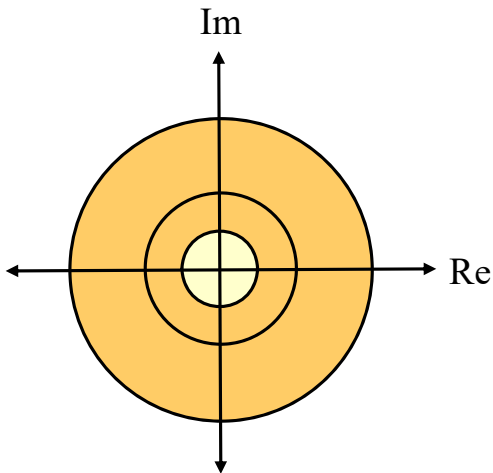
$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] (re^{-j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-j\omega n}$$

- DTFT of $x[n]$ multiplied with exponential sequence r^n
 - For certain choices of r the sum maybe made finite

$$\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$$

Region of Convergence

- The set of values of z for which the z -transform converges
- Each value of r represents a circle of radius r
- The region of convergence is made of circles



- **Example:** z -transform converges for values of $0.5 < r < 2$
 - ROC is shown on the left
 - In this example the ROC includes the unit circle, so DTFT exists
- **Not all sequence have a z-transform**
- **Example:** $x[n] = \cos(\omega_0 n)$
 - Does not converge for any r
 - No ROC, No z -transform
 - But DTFT exists?!
 - Sequence has finite energy
 - DTFT converges in the mean-squared sense

Right-Sided Exponential Sequence Example

$$x[n] = a^n u[n] \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- For Convergence we require

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

- Hence the ROC is defined as

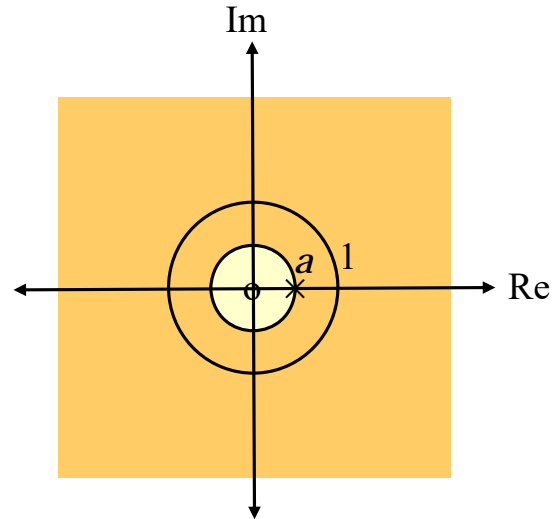
$$|az^{-1}|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

- Geometric series formula

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1 - a}$$



- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

Same Example Alternative Way

$$x[n] = a^n u[n] \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$\sum_{n=N_1}^{N_2} \alpha^n = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1 - \alpha}$$

$$\sum_{n=0}^{\infty} (az^{-1})^n = \frac{(az^{-1})^0 - (az^{-1})^{\infty}}{1 - az^{-1}} \quad |z| > |a|$$

- For the term with infinite exponential to vanish we need

$$|az^{-1}| < 1 \Rightarrow |a| < |z|$$

- Determines the ROC (same as the previous approach)
- In the ROC the sum converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}$$

Two-Sided Exponential Sequence Example

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^{\infty}}{1 + \frac{1}{3}z^{-1}} = \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$\sum_{n=-\infty}^{-1} \left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^0}{1 - \frac{1}{2}z^{-1}} = \frac{-1}{1 - \frac{1}{2}z^{-1}}$$

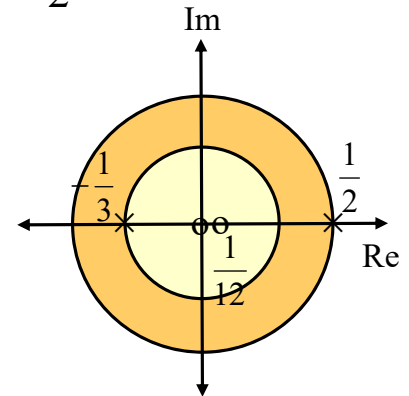
$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{2z\left(z - \frac{1}{2}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$

$$\text{ROC: } \left| -\frac{1}{3}z^{-1} \right| < 1$$

$$\frac{1}{3} < |z|$$

$$\text{ROC: } \left| \frac{1}{2}z^{-1} \right| > 1$$

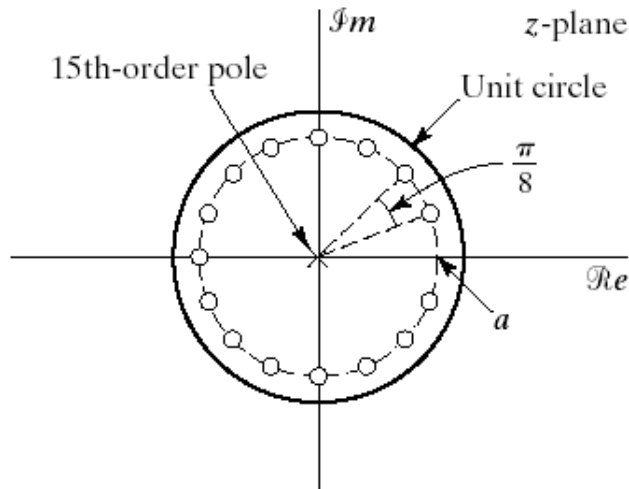
$$\frac{1}{2} > |z|$$



Finite Length Sequence

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$



Properties of The ROC of z-Transform

- The ROC is a ring or disk centered at the origin
- DTFT exists **if and only if** the ROC includes the **unit circle**
- The ROC cannot contain any poles
- The ROC for **finite-length sequence** is the entire z-plane
 - except possibly $z = 0$ and $z = \infty$
- The ROC for a **right-handed sequence** extends outward from the outermost pole possibly including $z = \infty$
- The ROC for a **left-handed sequence** extends inward from the innermost pole possibly including $z = 0$
- The ROC of a **two-sided sequence** is a ring bounded by poles
- The ROC must be a **connected region**
- A z-transform does not uniquely determine a sequence without specifying the ROC

Stability, Causality, and the ROC

- Consider a system with impulse response $h[n]$
- The z-transform $H(z)$ and the pole-zero plot shown below
- Without any other information $h[n]$ is not uniquely determined
 - $|z| > 2$ or $|z| < \frac{1}{2}$ or $\frac{1}{2} < |z| < 2$
- If system **stable** ROC must include unit-circle: $\frac{1}{2} < |z| < 2$
- If system is **causal** must be right sided: $|z| > 2$

