پردازش سیگنال دیجیتال

درس ۶

سیگنال‌های تصادفی گسسته–زمان

Discrete-Time Random Signals

کاظم فولادی قلعه
دانشکده مهندسی، پردیس فارابی
دانشگاه تهران

http://courses.fouladi.ir/dsp

Discrete-Time Random Signals

In many situations, the processes that generate signals are so complex as to make precise description of a signal extremely difficult or undesirable, if not impossible.
In such cases, modeling the signal as a **random process** is analytically useful.

Random Signals

A **random signal** is considered to be a member of an ensemble of discrete-time signals that is characterized by a set of probability density functions.

More specifically, for a particular signal at a particular time, the amplitude of the signal sample at that time is assumed to have been determined by an underlying scheme of probabilities.

That is, each individual sample $x[n]$ of a particular signal is assumed to be an outcome of some underlying random variable x_n. The entire signal is represented by a collection of such random variables, one for each sample time, $-\infty < n < \infty$. This collection of random variables is referred to as a ***random process***, and we assume that a particular sequence of samples $x[n]$ for $-\infty < n < \infty$ has been generated by the random process that underlies the signal.

To completely describe the random process, we need to specify the individual and joint probability distributions of all the random variables.

Random Signals and LTI Systems

Although, for simplicity, we assume that $x[n]$ and $h[n]$ are real valued, the results can be generalized to the complex case.

Consider a stable LTI system with real impulse response $h[n]$.
Let $x[n]$ be a real-valued sequence that is a sample sequence of a wide-sense stationary discrete-time random process;
Then, the output of the linear system is also a sample sequence of a discrete-time random process related to the input process by the linear transformation

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

As we have shown, since the system is stable, $y[n]$ will be bounded if $x[n]$ is bounded. We will see shortly that if the input is stationary, then so is the output.

The input signal may be characterized by its mean m_x and its autocorrelation function $\varphi_{xx}[m]$, or we may also have additional information about 1st- or even 2nd-order probability distributions.

Characterization of Input and Output by Their Averages

For many applications, it is sufficient to characterize both the input and output in terms of simple averages, such as the mean, variance, and autocorrelation. Therefore, we will derive input–output relationships for these quantities.
The means of the input and output processes are, respectively,

$$m_{x_n} = \mathcal{E}\{\mathrm{x}_n\}, \qquad m_{y_n} = \mathcal{E}\{\mathrm{y}_n\},$$

where $\mathcal{E}\{\cdot\}$ denotes the expected value of a random variable.
In most of our discussion, it will not be necessary to carefully distinguish between the random variables x_n and y_n and their specific values $x[n]$ and $y[n]$.
This will simplify the mathematical notation significantly.
For example, above equations will alternatively be written

$$m_x[n] = \mathcal{E}\{x[n]\}, \qquad m_y[n] = \mathcal{E}\{y[n]\}.$$

… For Stationary Inputs

If $x[n]$ is stationary, then $m_x[n]$ is independent of n and will be written as m_x, with similar notation for $m_y[n]$ if $y[n]$ is stationary.
The mean of the output process is

$$m_y[n] = \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\},$$

where we have used the fact that the expected value of a sum is the sum of the expected values. Since the input is stationary, $m_x[n-k] = m_x$, and consequently,

$$m_y[n] = m_x \sum_{k=-\infty}^{\infty} h[k].$$

From the above equation, we see that the mean of the output is also constant. An equivalent expression to the above equation in terms of the frequency response is

$$m_y = H(e^{j0})m_x.$$

… Output is Also Stationary

Assuming temporarily that the output is nonstationary, the autocorrelation function of the output process for a real input is

$$\begin{aligned}\phi_{yy}[n, n+m] &= \mathcal{E}\{y[n]y[n+m]\} \\ &= \mathcal{E}\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right\} \\ &= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\mathcal{E}\{x[n-k]x[n+m-r]\}.\end{aligned}$$

Since $x[n]$ is assumed to be stationary, $\mathcal{E}\{x[n-k]x[n+m-r]\}$ depends only on the time difference $m+k-r$. Therefore,

$$\phi_{yy}[n, n+m] = \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] = \phi_{yy}[m]. \qquad (2.186)$$

That is, the output autocorrelation sequence also depends only on the time difference m. Thus, for an LTI system having a wide-sense stationary input, the output is also wide-sense stationary.

Autocorrelation Sequence of $h[n]$

By making the substitution $\ell = r - k$, we can express Eq. (2.186) as

$$\begin{aligned}\phi_{yy}[m] &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell] \sum_{k=-\infty}^{\infty} h[k]h[\ell+k] \\ &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell],\end{aligned} \qquad (2.187)$$

where we have defined

$$c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]h[\ell+k]. \qquad (2.188)$$

The sequence $c_{hh}[\ell]$ is referred to as the deterministic autocorrelation sequence or, simply, the autocorrelation sequence of $h[n]$. It should be emphasized that $c_{hh}[\ell]$ is the autocorrelation of an aperiodic—i.e., finite-energy—sequence and should not be confused with the autocorrelation of an infinite-energy random sequence. Indeed, it can be seen that $c_{hh}[\ell]$ is simply the discrete convolution of $h[n]$ with $h[-n]$. Equation (2.187), then, can be interpreted to mean that the autocorrelation of the output of a linear system is the convolution of the autocorrelation of the input with the aperiodic autocorrelation of the system impulse response.

Power Density Spectrum

Equation (2.187) suggests that Fourier transforms may be useful in characterizing the response of an LTI system to a random input. Assume, for convenience, that $m_x = 0$; i.e., the autocorrelation and autocovariance sequences are identical. Then, with $\Phi_{xx}(e^{j\omega})$, $\Phi_{yy}(e^{j\omega})$, and $C_{hh}(e^{j\omega})$ denoting the Fourier transforms of $\phi_{xx}[m]$, $\phi_{yy}[m]$, and $c_{hh}[\ell]$, respectively, from Eq. (2.187),

$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega}). \quad (2.189)$$

Also, from Eq. (2.188),

$$C_{hh}(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega})$$
$$= |H(e^{j\omega})|^2,$$

so

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega}). \quad (2.190)$$

Power Density Spectrum

Equation (2.190) provides the motivation for the term *power density spectrum*. Specifically,

$$\begin{aligned} \mathcal{E}\{y^2[n]\} = \phi_{yy}[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})\, d\omega \\ &= \text{total average power in output.} \end{aligned} \tag{2.191}$$

Substituting Eq. (2.190) into Eq. (2.191), we have

$$\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega})\, d\omega. \tag{2.192}$$

Suppose that $H(e^{j\omega})$ is an ideal bandpass filter, as shown in Figure 2.18(c). Since $\phi_{xx}[m]$ is a real, even sequence, its Fourier transform is also real and even, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \Phi_{xx}(e^{-j\omega}).$$

Power Density Spectrum

Likewise, $|H(e^{j\omega})|^2$ is an even function of ω. Therefore, we can write

$$\begin{aligned}\phi_{yy}[0] &= \text{average power in output} \\ &= \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \Phi_{xx}(e^{j\omega})\,d\omega + \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} \Phi_{xx}(e^{j\omega})\,d\omega.\end{aligned} \tag{2.193}$$

Thus, the area under $\Phi_{xx}(e^{j\omega})$ for $\omega_a \le |\omega| \le \omega_b$ can be taken to represent the mean-square value of the input in that frequency band. We observe that the output power must remain nonnegative, so

$$\lim_{(\omega_b-\omega_a)\to 0} \phi_{yy}[0] \ge 0.$$

This result, together with Eq. (2.193) and the fact that the band $\omega_a \le \omega \le \omega_b$ can be arbitrarily small, implies that

$$\Phi_{xx}(e^{j\omega}) \ge 0 \qquad \text{for all } \omega. \tag{2.194}$$

Hence, we note that the power density function of a real signal is real, even, and nonnegative.

White Noise

The concept of white noise is exceedingly useful in a wide variety of contexts in the design and analysis of signal processing and communications systems. A white-noise signal is a signal for which $\phi_{xx}[m] = \sigma_x^2\delta[m]$. We assume in this example that the signal has zero mean. The power spectrum of a white-noise signal is a constant, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2 \qquad \text{for all } \omega.$$

The average power of a white-noise signal is therefore

$$\phi_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_x^2\,d\omega = \sigma_x^2.$$

The concept of white noise is also useful in the representation of random signals whose power spectra are not constant with frequency. For example, a random signal $y[n]$ with power spectrum $\Phi_{yy}(e^{j\omega})$ can be assumed to be the output of an LTI system with a white-noise input. That is, we use Eq. (2.190) to define a system with frequency response $H(e^{j\omega})$ to satisfy the equation

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_x^2,$$

where σ_x^2 is the average power of the assumed white-noise input signal. We adjust the average power of this input signal to give the correct average power for $y[n]$. For example, suppose that $h[n] = a^n u[n]$. Then,

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}},$$

and we can represent all random signals whose power spectra are of the form

$$\Phi_{yy}(e^{j\omega}) = \left|\frac{1}{1 - ae^{-j\omega}}\right|^2 \sigma_x^2 = \frac{\sigma_x^2}{1 + a^2 - 2a\cos\omega}.$$

Cross-Correlation Between the Input and Output

Another important result concerns the cross-correlation between the input and output of an LTI system:

$$\begin{aligned}\phi_{yx}[m] &= \mathcal{E}\{x[n]y[n+m]\} \\ &= \mathcal{E}\left\{x[n]\sum_{k=-\infty}^{\infty} h[k]x[n+m-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} h[k]\phi_{xx}[m-k].\end{aligned} \tag{2.195}$$

In this case, we note that the cross-correlation between input and output is the convolution of the impulse response with the input autocorrelation sequence.

The Fourier transform of Eq. (2.195) is

$$\Phi_{yx}(e^{j\omega}) = H(e^{j\omega})\Phi_{xx}(e^{j\omega}). \tag{2.196}$$

This result has a useful application when the input is white noise, i.e., when $\phi_{xx}[m] = \sigma_x^2\delta[m]$. Substituting into Eq. (2.195), we note that

$$\phi_{yx}[m] = \sigma_x^2 h[m]. \tag{2.197}$$

… For a Zero-Mean White-Noise Input

That is, for a zero-mean white-noise input, the cross-correlation between input and output of a linear system is proportional to the impulse response of the system. Similarly, the power spectrum of a white-noise input is

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2, \qquad -\pi \le \omega \le \pi. \tag{2.198}$$

Thus, from Eq. (2.196),

$$\Phi_{yx}(e^{j\omega}) = \sigma_x^2 H(e^{j\omega}). \tag{2.199}$$

In other words, the cross power spectrum is in this case proportional to the frequency response of the system. Equations (2.197) and (2.199) may serve as the basis for estimating the impulse response or frequency response of an LTI system if it is possible to observe the output of the system in response to a white-noise input. An example application is in the measurement of the acoustic impulse response of a room or concert hall.

سیگنال‌های تصادفی (اتفاقی) گسسته-زمان

Discrete-Time Random (Stochastic) Signals

بسیاری از سیگنال‌های گسسته-زمان، ماهیت تصادفی (اتفاقی) دارند.

مثال: **سیگنال یک تراگذر** (transducer)

$$x[n] = x_d[n] + e[n]$$

$x_d[n]$: بخش قطعی — deterministic part

$e[n]$: بخش اتفاقی — stochatic part

سیگنال‌های تصادفی (اتفاقی) گسسته-زمان

DISCRETE-TIME RANDOM (STOCHASTIC) SIGNALS

برای تحلیل دقیق یک سیگنال تصادفی

می‌توان مقادیر سیگنال در هر لحظه را یک متغیر تصادفی x_n قلمداد کرد.

$$\text{signal}: \{x[n]\}_{n=-\infty}^{+\infty} = \{\ldots, x[-1], x[0], x[1], \ldots\}$$

برای بررسی دقیق به توابع توزیع/چگالی یا آماره‌های مرتبه بالای x_n نیاز داریم.

سیگنال‌های تصادفی (اتفاقی) گسسته–زمان

ایستان بودن

DISCRETE-TIME RANDOM (STOCHASTIC) SIGNALS

سیگنال ایستان *Stationary Signal*	سیگنالی که مشخصه‌های آماری آن با زمان تغییر نمی‌کند.
سیگنال ایستان به مفهوم وسیع *Wide-Sense Stationary (WSS) Signal*	سیگنالی که خودهمبستگی آن تنها تابعی از تفاضل زمان‌هاست.

$$E\{x[n]\} = m_x$$

$$\phi_{xx}[n, n+m] = \phi_{xx}[m]$$

خودهمبستگی (autocorrelation): همبستگی سیگنال با خودش را بیان می‌کند.

$$\phi_{xx}[n, n+m] = E\{x[n]x^*[n+m]\}$$

در حالت WSS خودهمبستگی $\phi_{xx}[n,n+m]$ فقط به اختلاف زمان‌ها وابسته است: $\phi_{xx}[m]$.

سیگنال‌های تصادفی (اتفاقی) گسسته–زمان

خروجی سیستم خطی تغییرناپذیر با زمان برای ورودی ایستان به مفهوم وسیع

DISCRETE-TIME RANDOM (STOCHASTIC) SIGNALS

خروجی یک سیستم LTI در هنگامی که ورودی WSS باشد، به‌صورت زیر محاسبه می‌شود:

$$E\{x[n]\} = m_x$$

$$\phi_{xx}[n, n+m] = \phi_{xx}[m]$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$\begin{aligned} m_y &= E\{y[n]\} = E\{h[n]*x[n]\} = E\{\sum_{k=-\infty}^{+\infty} h[k]x[n-k]\} \\ &= \sum_{k=-\infty}^{+\infty} h[k]E\{x[n-k]\} \\ &= m_x \sum_{k=-\infty}^{+\infty} h[k] \\ &= m_x H(e^{j0}) \end{aligned}$$

پس m_y ثابت است.

Prepared by Kazim Fouladi | Spring 2018 | 3rd Edition
کاظم فولادی

سیگنال‌های تصادفی (اتفاقی) گسسته-زمان

همبستگی خروجی با ورودی

Discrete-Time Random (Stochastic) Signals

همبستگی خروجی با ورودی، نیز فقط به اختلاف زمان‌ها وابسته است:

$$\phi_{yx}[n,n+m] = E\{y[n]x^*[n+m]\} = E\{\sum_k h[k]x[n-k]x^*[n+m]\}$$

$$= \sum_k h[k]E\{x[n-k]x^*[n+m]\}$$

$$= \sum_k h[k]\phi_{xx}[m+k]$$

$$= \phi_{xx}[m] * h[-m]$$

$$= \phi_{yx}[m]$$

سیگنال‌های تصادفی (اتفاقی) گسسته-زمان

خودهمبستگی خروجی

DISCRETE-TIME RANDOM (STOCHASTIC) SIGNALS

همبستگی خروجی با خودش، نیز فقط به اختلاف زمان‌ها وابسته است:

$$\phi_{yy}[n,n+m]=h^{*}[m]*\phi_{yx}[m]$$

$$\Downarrow$$

$$\phi_{yy}[m]=h^{*}[m]*h[-m]*\phi_{xx}[m]$$

سیگنال‌های تصادفی (اتفاقی) گسسته–زمان

خودهمبستگی دنباله‌ی پاسخ ضربه

DISCRETE-TIME RANDOM (STOCHASTIC) SIGNALS

تعریف می‌کنیم:

$$c_{hh}[m] = \sum_k h[k]h^*[m+k] = h[m] * h^*[-m]$$

Prepared by Kazim Fouladi | Spring 2018 | 3rd Edition

کاظم فولادی

چگالی طیف توان

برای سیگنال تصادفی $x[n]$

Power Density Spectrum

طیف توان (قدرت):

$$\phi_{xx}[m] \xleftrightarrow{FT} \Phi_{xx}(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} \phi_{xx}[k] e^{-j\omega k}$$

طیف توان $\Phi_{xx}(e^{j\omega})$ یک عدد حقیقی است و در حالت کلی همواره نامنفی است.

$$\phi^{*}_{xx}[-m] = \phi_{xx}[m] \quad \Rightarrow \quad \Phi^{*}_{xx}(e^{j\omega}) = \Phi_{xx}(e^{j\omega})$$

$$\Phi_{xx}(e^{j\omega}) = \lim_{N\to\infty} \frac{1}{2N+1} E\{| X_N(e^{j\omega}) |^2\}$$

$$\Phi_{yy}(e^{j\omega}) = H(e^{j\omega}) H^{*}(e^{j\omega}) \Phi_{xx}(e^{j\omega}) \Rightarrow$$

$$\Phi_{yy}(e^{j\omega}) = \left| H^{*}(e^{j\omega}) \right|^2 \Phi_{xx}(e^{j\omega})$$

واریانس خروجی

OUTPUT VARIANCE

واریانس خروجی:

$$\sigma^2_{y[n]} = E\{y^2[n]\} - m_y^2$$

$$E\{y^2[n]\} = \phi_{yy}[0] = \int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})d\omega = E\{y[n]y^*[n+m]\}\Big|_{m=0}$$