پردازش سیگنال دیجیتال

درس ۴

تبدیل فوریه گسسته–زمان

Discrete-Time Fourier Transform

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Frequency Response

- The **frequency response** defines a systems output
  - for complex exponential at all frequencies
- If input signals can be represented as a sum of complex exponentials

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}$$

  - we can determine the output of the system

$$y[n] = \sum_k \alpha_k H\left(e^{j\omega_k}\right) e^{j\omega_k n}$$

- Different from continuous-time frequency response
  - Discrete-time frequency response is periodic with 2π

$$H\left(e^{j(\omega+2\pi r)}\right) = \sum_{k=-\infty}^{\infty} h[k] e^{-j(\omega+2\pi r)k} = \sum_{k=-\infty}^{\infty} h[k] e^{-j2\pi rk} e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

$$H\left(e^{j(\omega+2\pi r)}\right) = H\left(e^{j\omega}\right)$$

Discrete-Time Fourier Transform

- Many sequences can be expressed as a weighted sum of complex exponentials as

$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega \quad \text{(inverse transform)}}$$

- Where the weighting is determined as

$$\boxed{X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{(forward transform)}}$$

- $X(e^{j\omega})$ is the Fourier spectrum of the sequence $x[n]$
  - It specifies the magnitude and phase of the sequence
  - The phase wraps at 2π hence is not uniquely specified
- The frequency response of a LTI system is the DTFT of the impulse response

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \quad \text{and} \quad h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

Discrete-Time Fourier Transform Pair

$$\boxed{X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{and} \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega}$$

- Let's show that they constitute a transform pair
  - Substitute first equation into second to get

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{j\omega n}d\omega = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega\right)$$

  - Evaluate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega = \begin{cases} \dfrac{\sin[\pi(n-m)]}{\pi(n-m)} = 0 & \text{for } n \neq m \\ 1 & \text{for } n = m \end{cases}$$

$$= \delta[n-m]$$

  - Substitute the integral with this result to get

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] = x[n]$$

Existence of DTFT

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

- For a given sequence the DTFT exist if the infinite sum convergence

$$\left|X\left(e^{j\omega}\right)\right|<\infty \quad \text{for all } \omega$$

- Or

$$\left|X\left(e^{j\omega}\right)\right|=\left|\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}\right|\leq\sum_{n=-\infty}^{\infty}\left|x[n]\right|\left|e^{-j\omega n}\right|=\sum_{n=-\infty}^{\infty}\left|x[n]\right|<\infty$$

- So the DTFT exists if a given sequence is absolute summable
- All stable systems are absolute summable and have DTFTs

DTFT Demo

Square Wave

Triangular Wave

From *Fundamentals of Signals and Systems Using the Web and Matlab*
by Edward W. Kamen and Bonnie S. Heck