هوالحکیم

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فصل پنجم

سیستم‌های می‌نیمم-فاز و فاز خطی تعمیم‌یافته

MINIMUM-PHASE SYSTEMS AND GENERALIZED LINEAR PHASE

◊ مسئله‌های تحلیلی ـ تشریحی

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1. **[Opp nheim/Schafer/Buck Problem #5.39]** For a minimum phase sequence, we must have $|h_{\min}[0]|^2 \geq |h[n]|^2$. Since it is specified that one of (a)-(h) is minimum phase, the minimum phase sequence has to be (f).

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2. **[Oppenheim/Schafer/Buck Problem #5.40]** A zero phase sequence has all its poles and zeors in conjugate reciprocal pairs. Positive or negative integer delays can only add poles/zeros to 0 and/or ∞. Therefore, for the p rpose of determining if the system is generalized linear phase poles and zeros at $0/\infty$ can be ignored. Then the systems shown in (a) and (b) are not generalized linear phase, while the systems shown in (c) and (d) are. All the systems can have stable inverses except (d).

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3. **[Oppenheim/Schafer/Buck Problem #5.41]** Convolving two symmetric sequences yields a symmetric sequence, while convolving symmetric and anti-symmetric sequences yields an anti-symmetric sequence. Therefore, the imp lse respose of system A is anti-symmetric, and so is generalized linear phase. For system B, the convolution $h_1[n] * h_2[n]$ is symmetric, but $h_1[n] * h_2[n] + h_3[n]$ need neither be symmetric nor anti-symmetric. Hence in general, this system need not have generalized linear phase.

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4. **[Oppenheim/Schafer/Buck Problem #5.43]** See the derivation presented in class or the class notes.

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5. **[Oppenheim/Schafer/Buck Problem #5.51]** The statement is false. The response shown in Figure 5.35(c) of text (page 294) is a co nterexample.

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6. **[Oppenheim/Schafer/Buck Problem #5.54]** Since $x[n]$ is zero outside a finite interval, $X(z)$ has to be rational. Beca se $x[n]$ is real valued and minimum phase, it has to have additional zeros a $\frac{1}{2}z^{-j\pi/4}$ and a $\frac{1}{2}e^{-j3\pi/4}$. Since $x[n]$ has only 5 non-zero terms, this is the complete list of zeros $X(z)$ can have. It will have 4 poles a the origin, and so the ROC of $X(z)$ is $|z| > 0$. The ROC of $Y(z) = \frac{1}{X(z)}$ is $|z| > \frac{1}{2}$.

. **[Oppenheim/Schafer/Buck Problem #5.65]** Suppose that $H(z) = H_{\min}(z) \prod_{k=1}^{N} \frac{z^{-1}-a_k}{1-a_k z^{-1}}$, where $|a_k| < 1$. Then

$$h_{\min}[0] = \lim_{z\to\infty} H_{\min}(z) = \lim_{z\to\infty} H(z) \cdot \prod_{k=1}^{N} \frac{1}{-a_k} = h[0] \cdot \prod_{k=1}^{N} \frac{1}{-a_k}$$

Therefore, $|h_{\min}[0]| = \frac{|h[0]|}{\prod_{k=1}^{N} |a_k|} > |h[0]|$.

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8. **[Oppenheim/Schafer/Buck Problem #5.66]** We have

$$H(z) = H_{\min}(z) \cdot \frac{z^{-1} - z_k^*}{1 - z_k z^{-1}} = Q(z)(z^{-1} - z_k^*) = Q(z)z^{-1} - z_k^* Q(z)$$

Therefore, $h_{\min}[n] = q[n] = z_k q[n-1]$. So,

$$|h_{\min}[n]|^2 = h_{\min}[n] \cdot h_{\min}[n]^* = (q[n] - z_k q[n-1]) \cdot (q[n]^* - z_k^* q[n-1]^*)$$

Simiplifying yields $\epsilon = (1-|z_k|^2)\,|q[n]|^2$. Then since $\epsilon > 0$, we must have $\sum_{m=0}^{n} |h_{\min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 > 0$. The result follows.

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Causal Generalized Linear-Phase Systems

Solution for #4 (5.43)

Now we look at causal systems whose impulse response $h[n]$ is real.

If

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \tag{15.3}$$

then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

where $A_e(e^{j\omega})$ is a real, even and periodic function of ω.

If

$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \tag{15.4}$$

then

$$H(e^{j\omega}) = A_o(e^{j\omega})e^{-j\omega M/2})$$

where $A_o(e^{j\omega})$ is a real, odd and periodic function of ω.

Equations 15.3 and 15.4 are sufficient conditions for generalized linear-phase but they are not necessary conditions.

Type I FIR Linear-Phase Systems

Type I FIR's are symmetric about an integer.

A Type I FIR is characterized by

$$h[n] = h[M-n], \qquad 0 \le n \le M$$

and

$$M \text{ is an even integer}$$

.

We can show Type I FIR's have linear-phase by checking its Fourier Transform.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{k=M/2+1}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (k = M-n) \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{k=M/2+1}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k] = h[M-k]) \\
&= \sum_{n=0}^{M/2-1} h[n](e^{-j\omega n} + e^{-j\omega(M-n)}) + h[M/2]e^{-j\omega M/2} \\
&= e^{-j\omega M/2}\left[\sum_{n=0}^{M/2-1} h[n]2\cos(n-\frac{M}{2})\omega + h[M/2]\right]
\end{aligned}
$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contribute a phase of 0 or π to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2} \quad \text{or} \quad -\omega\frac{M}{2}+\pi$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = \frac{M}{2}, \quad \beta = 0 \text{ or } \pi$$

Type II FIR Linear-Phase Systems

Type II FIR's are symmetric about the half of an integer.

A Type II FIR is characterized by

$$h[n] = h[M-n], \qquad 0 \le n \le M$$

and

$$M \text{ is an odd integer}$$

.

We can show Type II FIR's have linear-phase by checking its Fourier Transform.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{k=(M+1)/2}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (k = M-n) \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{k=(M+1)/2}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k] = h[M-k]) \\
&= \sum_{n=0}^{(M-1)/2} h[n](e^{-j\omega n} + e^{-j\omega(M-n)}) \\
&= e^{-j\omega M/2}\left[\sum_{n=0}^{(M-1)/2} h[n]2\cos(n-\frac{M}{2})\omega\right]
\end{aligned}
$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contributes a phase of 0 or π to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2} \quad \text{or} \quad -\omega\frac{M}{2}+\pi$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = \frac{M}{2}, \quad \beta = 0 \text{ or } \pi$$

Type III FIR Linear-Phase Systems

Type III FIR's are anti-symmetric about an integer.

A Type III FIR is characterized by

$$h[n] = -h[M-n], \qquad 0 \le n \le M$$

and

$$M \text{ is an even integer}$$

Note that at $n = M/2$,

$$h[M/2] = -h[M-(M/2=\ )]\ -h[M/2]$$

So we have $h[M/2] = 0$.

.

We can show Type III FIR's have linear-phase by checking its Fourier Transform.

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + \sum_{k=M/2+1}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (h[M/2]=0, k=M-n) \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} - \sum_{k=M/2+1}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k]=-h[M-k]) \\
&= \sum_{n=0}^{M/2-1} h[n](e^{-j\omega n} - e^{-j\omega(M-n)}) \\
&= e^{-j\omega M/2}\left[(-j)\sum_{n=0}^{M/2-1} h[n]2\sin(n-\frac{M}{2})\omega\right]
\end{aligned}$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contribute a phase of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2}+\frac{\pi}{2} \quad \text{or} \quad -\omega\frac{M}{2}+\frac{3\pi}{2}$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha=\frac{M}{2}, \quad \beta = 0 \text{ or } \pi$$

Type IV FIR Linear-Phase Systems

Type IV FIR's are anti-symmetric about the half of an integer.

A Type IV FIR is characterized by

$$h[n] = -h[M-n], \qquad 0 \le n \le M$$

and

$$M \text{ is an odd integer}$$

.

We can show Type IV FIR's have linear-phase by checking its Fourier Transform.

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{k=(M+1)/2}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (k = M-n) \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} - \sum_{k=(M+1)/2}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k] = -h[M-k]) \\
&= \sum_{n=0}^{(M-1)/2} h[n](e^{-j\omega n} - e^{-j\omega(M-n)}) \\
&= e^{-j\omega M/2}\left[(-j)\sum_{n=0}^{(M-1)/2} h[n]2\sin(n-\frac{M}{2})\omega\right]
\end{aligned}$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contributes a phase of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2}+\frac{\pi}{2} \quad \text{or} \quad -\omega\frac{M}{2}+\frac{3\pi}{2}$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

The four types of FIR filters all have linear-phase, which is a desirable characteristic in many situations since we can concentrate on the magnitude response when designing filters. What is the difference among the different types?

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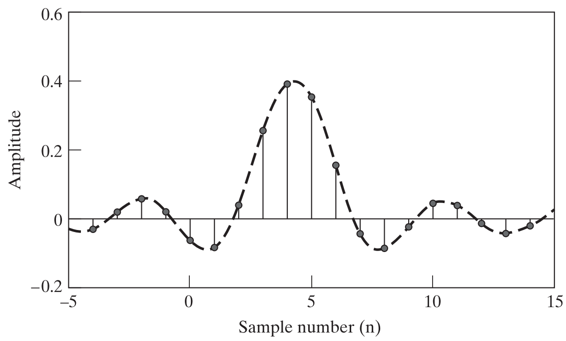
Solution for #5 (5.51)



Ideal lowpass filter impulse responses, with $\omega_C = 0.4\pi$
Delay $= \alpha = 4.3$.