هوالحکیم

راه‌حل تکلیف شماره‌ی ۶

فصل پنجم

تحلیل تبدیل سیستم‌های خطی تغییرناپذیر با زمان

TRANSFORM ANALYSIS OF LINEAR TIME-INVARIANT SYSTEMS

◊ مسئله‌های تحلیلی ـ تشریحی

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1. **[Oppenheim/Schafer/Buck Problem #5.4]** $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$. Therefore, $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{z}{z-2}$, with ROC $\frac{1}{2} < |z| < 2$. Further, $y[n] = 6\frac{1}{2^n}u[n] = 6\left(\frac{3}{4}\right)^n u[n]$, and so $Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}$, with ROC $\frac{3}{4} < |z|$. Hence

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

So, there is a signle pole at $\frac{3}{4}$, a zero at 2, and the region of convergence extends outwards from the circle $|z| = \frac{3}{4}$. It is clear that this system is stable because the ROC includes the unit circle. Now, $H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}$. Therefore, $h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$. Hence this system is also causal. One difference equation represanting the system may be obtained by noting that $Y(z) - \frac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$. Hence $y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$.

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2. **[Oppenheim/Schafer/Buck Problem #5.10]** Since the number of poles equals the number of zeros, there is an additional pole at $z = \infty$. Since the system is causal, the ROC is everything outside the outermost pole, and so includes the unit circle. Hence the system is stable. The inverse of the system has the poles and zeros switched. Therefore, it has a pole at $z = \infty$. Therefore, it cannot simultaneously be causal and stable.

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3. **[Oppenheim/Schafer/Buck Problem #5.21]** For part (a), note that $H(e^{j\omega}) = 1 - h_{\text{lp}}(e^{j\omega})$. Therefore, it is a highpass filter. For part (b), note that $x[n]$ is first modulated by π (which switches the high and low frequencies), and then low pass filtered (which removes the frequencies which were formerly high frequencies) and then demodulated (switching the high and low frequencies again). Therefore, it is a high pass filter. For part (c), note that $h_{\text{lp}}[2n]$ is a downsampled version of the filter. Therefore, it is a low pass filter with gain $\frac{1}{2}$, and cutoff frequency $\pi/2 = 2w_c$. For part (d), the upsampling operation makes this a bandstop filter, cutting off frequencies in the range $\frac{\pi}{8} < |\omega| < \frac{7\pi}{8}$. Finally for part (e), the input is upsampled before it is sent to the filter. This effectively doubles the cutoff frequency of the filter, and so it is a low pass filter with cutoff $2\omega_c = \pi/2$.

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4. **[Oppenheim/Schafer/Buck Problem #5.25]** This statement is false. For example, note that the system with impulse response $h[n] = \delta[n+1] + \delta[n] + 4\delta[n-1] + \delta[n-1] + \delta[n-2] + \delta[n-3]$ is clearly not causal. However, $H(e^{j\omega}) = e^{j\omega} + 1 + 4e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} = e^{-j\omega}(4 + 2\cos\omega + 2\cos(2\omega))$. Hence $\text{arg}H(e^{j\omega}) = -\omega$, and so $\text{grd}H(e^{j\omega}) = 1$.

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5. **[Oppenheim/Schafer/Buck Problem #5.28]** $H(z) = \frac{A}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}$. Since $H(1) = 6$, we get $A = 4$. Therefore, $H(z) = \frac{12/5}{1-\frac{1}{2}z^{-1}} + \frac{8/5}{1+\frac{1}{3}z^{-1}}$. The region of convergence is $|z| > \frac{1}{2}$. Therefore, $h[n] = \frac{12}{15}\left(\frac{1}{2}\right)^n u[n] + \frac{8}{5}\left(-\frac{1}{3}\right)^n u[n]$. For part (c, i), we have $Y(z) = X(z)H(z) = \frac{3}{1-z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$, with ROC containing $|z| > 1$. Hence $y[n] = 3u[n] + \left(\frac{-1}{3}\right)^n u[n]$. For part (c, ii), Since $x(t) = 50 + 10\cos(20\pi t) + 30\cos(40\pi t)$, we have

$$x[n] = 50 + 10\cos(\pi n/2) + 20\cos(\pi n) = 50 + 5e^{jn\pi/2} + 5e^{-jn\pi/2} + 15e^{jn\pi} + 15e^{-jn\pi}$$

Therefore,

$$y[n] = 50H(e^{j0}) + 5e^{jn\pi/2}H(e^{j\pi/2}) + 5e^{-jn\pi/2}H(e^{-j\pi/2}) + 15e^{jn\pi}H(e^{j\pi}) + 15e^{-jn\pi}H(e^{-j\pi})$$

By substituting, we see that $H(e^{j0}) = 6$, $H(e^{\pm j\pi/2}) = \frac{7\cdot 12}{25} \mp j\frac{12}{25}$ and $H(e^{\pm j\pi}) = 4$. Hence

$$y[n] = 300 + 24\sqrt{2}\cos(\frac{\pi}{2}n - \tan^{-1}(1/7)) + 120\cos\pi n$$

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6. **[McClellan et al. #1.1]** Note that we can write $H(e^{j\omega}) = |H(e^{j\omega})| \cdot e^{j\theta(\omega)}$, and so

$$\theta(\omega) = \Re\left\{-j\ln\frac{H(e^{j\omega})}{|H(e^{j\omega})|}\right\} = \Re\left\{-j\ln H(e^{j\omega})\right\} + \Re\left\{j\left|H(e^{j\omega})\right|\right\} = \Re\left\{-j\ln H(e^{j\omega})\right\}$$

Therefore,

$$-\frac{d\theta(\omega)}{d\omega} = \Re\left\{\frac{-j\frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})}\right\}$$