هوالحکیم

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گروه فنی

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راه‌حل تکلیف شماره‌ی ٤

فصل چهارم

نمونه‌برداری از سیگنال‌های پیوسته-زمان

Sampling of Continuous-Time Signals

◊ مسئله‌های تحلیلی ـ تشریحی

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1. **[Oppenheim/Schafer/Buck Problem #4.15]** We note that $x_r[n] = x[n]$ (upto multiplication by a constant) iff $X\left(e^{j\omega}\right) = 0$ for $\pi/3 \leq |\omega| \leq \pi$. For (a), $X\left(e^{j\omega}\right) = 0$ for $|\omega| > \pi/4$, and therefore there is no aliasing. For part (b), $X\left(e^{j\omega}\right)$ has impluses at $\pm\pi/2$, and so aliasing occurs. For part (c), note that $X(e^{j\omega})$ can be obtained by convolving $\mathsf{rect}(-\pi/8, \pi/8)$ with itself. (multiplication in time domain = convolution in frequency domain). Therefore, $X(e^{j\omega}) = 0$ for $|\omega| > \pi/4$. Therefore, no aliasing occurs.

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2. **[Oppenheim/Schafer/Buck Problem #4.19]** We need $T \leq \pi/\Omega_0$.

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3. **[Oppenheim/Schafer/Buck Problem #4.21]** Note that the $X_c(j\Omega)$ has frequency components only in the range $[\Omega_1, \Omega_2]$. Therefore, to recover the signal, we can use a (complex) ideal bandpass filter to filter out the rest of the frequency spectrum, and hence we do not care what happens there.

   We now show that $\Omega_s = \Delta\Omega$ is the smallest frequency at which one can sample the signal, with no aliasing. We consider 2 cases.

   (a) Suppose that the sampling frequency $\Omega_s < \Delta\Omega$. Then

   $$\begin{aligned} TX_s(\Omega_2) &= \sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega_2 - k\Omega_s\right)\right) \\ &= X_c(\Omega_2) + X_c(\Omega_2 - \Omega_s) + \sum_{k\neq 0,1} X_c\left(j\left(\Omega_2 - k\Omega_s\right)\right) \end{aligned}$$

   Now, for $\Omega_s < \Delta\Omega$ we have $\Omega_1 < \Omega_2 - \Omega_s < \Omega_2$. Therefore, both $X_c(j(\Omega_2 - \Omega_s))$ and $X_c(j\Omega_2)$ are non-zero, and hence aliasing takes place. Therefore, if $\Omega_s < \Delta\Omega$, then there will be aliasing.

   (b) Now, suppose that $\Omega_s > \Delta\Omega$. Then, we have

   $$TX_s(\Omega_2) = \sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega_2 - k\Omega_s\right)\right)$$

   Now, note that for any $k > 0$, and $\Omega \in [\Omega_1, \Omega_2]$, we have

   $$\Omega - k\Omega_s < \Omega_2 - k\Omega_s < \Omega_2 - k\Delta\Omega < \Omega_2 - \Delta\Omega < \Omega_1$$

   and so $X_c\left(j\left(\Omega - k\Omega_s\right)\right) = 0$. Similarly, for any $k < 0$ and $\Omega \in [\Omega_1, \Omega_2]$, we have

   $$\Omega - k\Omega_s > \Omega_1 - k\Omega_s > \Omega_1 - k\Delta\Omega > \Omega_1 - \Delta\Omega > \Omega_2$$

   and so $X_c\left(j\left(\Omega - k\Omega_s\right)\right) = 0$.
   Therefore for all $k \neq 0$, and $\Omega \in [\Omega_1, \Omega_2]$, we have $X_c\left(j\left(\Omega - k\Omega_s\right)\right) = 0$. Therefore, $TX_s(\Omega) = X_c(\Omega)$, and so the signal can be recovered by using a (complex) band pass filter, which filters the frequency range $[+\Omega_1, +\Omega_2]$

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Combining both parts, we see that if $\Omega_s < \Delta\Omega$, then aliasing occurs, while if $\Omega_s > \Delta\Omega$, no aliasing occurs. Therefore, $\Delta\Omega$ is the smallest sampling frequency that causes no aliasing.

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4. **[Oppenheim/Schafer/Buck Problem #4.23]** We have $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T_1$. Therefore, we are sampling at the Nyquist rate or higher. So there will be no aliasing. In particular, we man reconstruct the signal by sinc interpolation

$$x_c(t) = \sum_{n\in\mathbb{Z}} x[n] \frac{\sin\left[\pi(t - nT_1)/T_1\right]}{\pi(t - nT_1)/T_1}$$

where $x[n] = x_c(nT_1)$. Now, the output $y_c(t)$ is given by

$$y_c(t) = \sum_{n\in\mathbb{Z}} x[n] \frac{\sin\left[\pi(t - nT_2)/T_2\right]}{\pi(t - nT_2)/T_2}$$

Therefore,

$$\begin{aligned} x_c\left(\frac{tT_1}{T_2}\right) &= \sum_{n\in\mathbb{Z}} x[n] \frac{\sin\left[\pi(\frac{tT_1}{T_2} - nT_1)/T_1\right]}{\pi(\frac{tT_1}{T_2} - nT_1)/T_1} \\ &= \sum_{n\in\mathbb{Z}} x[n] \frac{\sin\left[\pi(t - nT_2)/T_2\right]}{\pi(t - nT_2)/T_2} \\ &= y_c(t) \end{aligned}$$

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5. **[Oppenheim/Schafer/Buck Problem #4.26]** If $x[n] \overset{\mathcal{F}}{\leftrightarrow} X\left(e^{j\omega}\right)$, then since $x_d[n]$ is obtained by downsampling $x[n]$, we have

$$x_d[n] \overset{\mathcal{F}}{\leftrightarrow} X_d\left(e^{j\omega}\right) = \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right)$$

Further, since we can consider $x_s[n]$ the result of upsamling $x_d[n]$, we have

$$x_s[n] \overset{\mathcal{F}}{\leftrightarrow} X_s\left(e^{j\omega}\right) = X_d\left(e^{j\omega M}\right)$$

Therefore, if the upsampling/downsamling factors are $M = 3$, we have

$$X_d\left(e^{j\omega}\right) = \frac{X\left(e^{j\frac{\omega}{3}}\right) + X\left(e^{j\frac{\omega}{3} - \frac{2\pi j}{3}}\right) + X\left(e^{j\frac{\omega}{3} - \frac{4\pi j}{3}}\right)}{3}$$

(a) Figures 1 and 2 show $X_d\left(e^{j\omega}\right)$ and $X_s\left(e^{j\omega}\right)$ for $M = 3$ and $\omega_H = \pi/2$. The dotted lines show the 3 components, while the solid line shows the sum.

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(b) Figures 3 and 4 show $X_d\left(e^{j\omega}\right)$ and $X_s\left(e^{j\omega}\right)$ for $M = 3$ and $\omega_H = \pi/4$.

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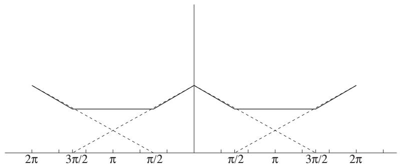

Figure 1: $X_d\left(e^{j\omega}\right)$ for $M=3$ and $\omega_H=\pi/2$

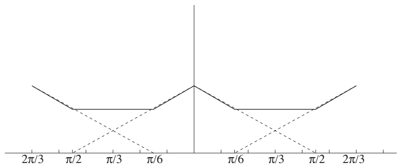

Figure 2: $X_s\left(e^{j\omega}\right)$ for $M=3$ and $\omega_H=\pi/2$

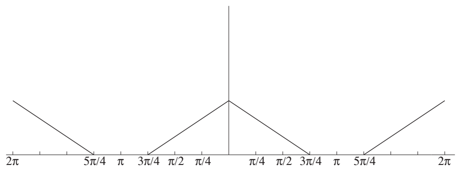

Figure 3: $X_d\left(e^{j\omega}\right)$ for $M=3$ and $\omega_H=\pi/4$

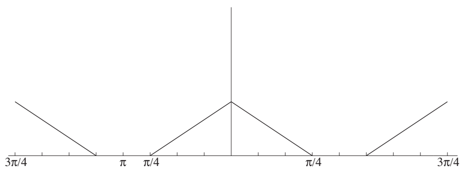

Figure 4: $X_s\left(e^{j\omega}\right)$ for $M=3$ and $\omega_H=\pi/4$