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دانشگاه تهران
پردیس البرز
گروه فنی

راه‌حل تکلیف شماره‌ی ۲

فصل دوم

سیستم‌های خطی تغییرناپذیر با زمان

LINEAR TIME-INVARIANT SYSTEMS

◊ مسئله‌های تحلیلی ـ تشریحی

1. **[Oppenheim/Schafer/Buck Problem #2.47]**
$y[n] = x[n] + 2x[n-1] + x[n-2] = x[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2])$. Therefore, $h[n]$, the impulse response is $\delta[n] + 2\delta[n-1] + \delta[n-2]$. The system is stable because $\sum_{k\in\mathbb{Z}} |h[n]| = 4 < \infty$.

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= 1 + 2e^{-j\omega} + e^{-2j\omega} \\
&= 2e^{-j\omega}(0.5e^{j\omega} + 1 + 0.5e^{-j\omega}) \\
& 2e^{-j\omega}(1 + \cos(\omega))
\end{aligned}
$$

$|H(e^{j\omega})| = 2(\cos(\omega) + 1)$, and $\arg H(e^{j\omega}) = -\omega$.

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2. **[Oppenheim/Schafer/Buck Problem #2.59]**

$$
\begin{aligned}
R_x[n] &= \sum_{k=-\infty}^{\infty} x^*[k]x[n+k] \\
&= \sum_{r=-\infty}^{\infty} x^*[-r]x[n-r] \qquad \text{Substitute } r = -k \\
&= x^*[-n] * x[n]
\end{aligned}
$$

Therefore, $g[n] = x^*[-n]$.

For part (b), note that $x^*[-n] \overset{\mathcal{F}}{\leftrightarrow} X^*(e^{j\omega})$. Hence $R_x(e^{j\omega}) = X(e^{j\omega})\, x^*(e^{j\omega}) = |X(e^{j\omega})|^2$.

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3. **[Oppenheim/Schafer/Buck Problem #2.60]**
$x_2[n] = -\sum_{k=0}^{4} x[n-k]$. Hence $y_2[n] = -\sum_{k=0}^{4} y[n-k]$.
For part (b), note that $\delta[n] = \sum_{k=0}^{\infty} x[n-k]$. Therefore by linearity, $h[n] = \sum_{k=0}^{\infty} y[n-k]$.
One thing that you should note however is that the solution is not unique, even though it seems to be. The reason is that we could have potentially formed $\delta[n]$ by other combinations of the $x[n]$ as well. In general, $h[n]$ satisfies the relation

$$h[n-1] - h[n] = y[n]$$

This leads to a unique solution if we make additional assumptions (such as causality, FIR'ness etc.). Otherwise, we get a solution that is unique upto the addition of a constant.

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4. **[Oppenheim/Schafer/Buck Problem #2.62]**
In this question, you were asked to use the definition of causality to show that $h[n] \neq 0$ for some $n < 0$ if and only if the system is causal. A system is causal if whenever $\{x_1[n]\} \stackrel{T}{\to} \{y_1[n]\}$ and $\{x_2[n]\} \stackrel{T}{\to} \{y_1[n]\}$ satisfy $x_1[n] = x_2[n]$ for all $n \leq n_0$, then $y_1[n] = y_2[n]$ for all $n \leq n_0$.

For the forward direction, assume that $h[n_0] \neq 0$ for some $n_0 < 0$. Let $x_1[n] = \delta[n+n_0]$. Then
$$y_1[0] = \sum_{k=-\infty}^{\infty} x_1[k]h[0-k] = x[-n_0]h[n_0] = h[n_0] \neq 0$$
Let $x_2[n] = 0$ for all $n \in \mathbb{Z}$. Then by a previous homework problem, $y_2[n] = 0$ for all $n \in \mathbb{Z}$. Therefore, we have $x_1[n] = x_2[n]$ for all $n < -n_0 - 1$. Now, $-n_0 - 1 \geq 0$. However, $h[n_0] = y_1[0] \neq y_2[0] = 0$. Therefore, the system is not causal.

For the converse, note that $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$. So, if $x_1[n] = x_2[n]$ for all $n \leq n_0$, then $x_1[n-k] = x_2[n-k]$ for $n \leq n_0, k \geq 0$. Hence
$$y_1[n] = \sum_{k=0}^{\infty} h[k]x_1[n-k] = \sum_{k=0}^{\infty} h[k]x_2[n-k]$$
for $n \leq n_0$. Therefore the system is causal.

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5. **[Oppenheim/Schafer/Buck Problem #2.66]**

$$\begin{aligned} E\left(e^{j\omega}\right) &= H_1\left(e^{j\omega}\right) X\left(e^{j\omega}\right) \\ F\left(e^{j\omega}\right) &= E\left(e^{-j\omega}\right) = H_1\left(e^{-j\omega}\right) X\left(e^{-j\omega}\right) \\ G\left(e^{j\omega}\right) &= H_1\left(e^{j\omega}\right) F\left(e^{j\omega}\right) = H_1\left(e^{j\omega}\right) H_1\left(e^{-j\omega}\right) X\left(e^{-j\omega}\right) \\ Y\left(e^{j\omega}\right) &= G\left(e^{-j\omega}\right) = H_1\left(e^{-j\omega}\right) H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) \end{aligned}$$

Therefore, $H\left(e^{j\omega}\right) = H_1\left(e^{-j\omega}\right) H\left(e^{j\omega}\right)$. Therefore $h[n] = h_1[-n] * h_1[n]$.

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6. **[Oppenheim/Schafer/Buck Problem #2.81]**
Because s and e are uncorrelated, we have $E\{s[n]e[m]\} = 0$ for all n, m. Hence
$$E\{y[n]y[n+m]\} = E\{s[n]e[n]s[n+m]e[n+m]\} = E\{s[n]s[n+m]e[n]e[n+m]\}$$
Using the fact that s, e are uncorrelated, we get
$$\begin{aligned} &= E\{s[n]s[n+m]\}\, E\{e[n]e[n+m]\} \\ &= \sigma_s^2\sigma_e^2\delta[m] \end{aligned}$$
because $f[m]\delta[m] = f[0]\delta[m]$.