راه‌حل تکلیف شماره‌ی ۱

فصل دوم

سیگنال‌ها و سیستم‌های گسسته-زمان

DISCRETE-TIME SIGNALS AND SYSTEMS

◊ مسئله‌های تحلیلی ـ تشریحی

. **[Oppe e m c afer/Buck Problem #2.21]**
Consider a linear system with input $\{x[n]\}$ and output $\{y[n]\}$. If $x[n]$ is zero for all n, then $c \cdot x[n] = 0 = x[n]$ for every n. Therefore, the system response to $c \cdot x[n]$ should be the same as the response to $x[n]$ for all $c \in \mathbb{C}$. The system response to $c \cdot x[n]$ is $c \cdot y[n]$. Therefore, $c \cdot y[n] = y[n]$ for all $n \in \mathbb{Z}$, for every $c \in \mathbb{C}$. Therefore, $y[n] = 0$ for all n.

Alternatively, if $\{y[n]\}$ is the response to $\{x[n]\}$ $\{-y[n]\}$ should be the response to $\{-x[n]\} = \{x[n]\} = \{0\}$. Hence $\{y[n]\} = \{-y[n]\}$ and the result follows.

One common mistake that a number of people made was to assume that the output is the input convolved with some impulse response. However, this can be done only if the system is known to be LTI, not just linear.

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2. **[Oppenheim/Schafer/Buck Problem #2.23]**
Let $\mathcal{T}_d$ represant the ideal delay system (whose delay is d). Then if $y[n] = \mathcal{T}_d\{x[n]\}$, then we have

$$y[n] = x[n-d] \qquad -\infty < n < \infty$$

Then if $y_1[n] = \mathcal{T}\{x_1[n]\}$, and $y_2[n] = \mathcal{T}\{x_2[n]\}$, we have

$$\begin{aligned} y_1[n] &= x_1[n-n_d] \\ y_2[n] &= x_2[n-n_d] \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{T}\{ax_1[\cdot] + bx_2[\cdot]\} &= (ax_1 + bx_2)[n-n_d] \\ &= ax_1[n-n_d] + bx_2[n-n_d] \\ &= a\mathcal{T}\{x_1[n]\} + b\mathcal{T}\{x_2[n]\} \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Therefore, the system satisfies the definition of linearity (Equation 2.27), and so the delay system is linear.
Now consider the moving average system $\mathcal{M}$. If $y[n] = \mathcal{M}\{x[n]\}$, then we have

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

Set $y_1[n] = \mathcal{M}\{x_1[n]\}$, and $y_2[n] = \mathcal{M}\{x_2[n\}$. Then, we have

$$\begin{aligned}
\mathcal{T}\{ax_1[\cdot] + bx_2[\cdot\} \quad & \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}(ax_1+bx_2)[n-k] \\
& \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} ax_1[n-k] + bx_2[n-k] \\
& \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} ax_1[n-k] + \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} bx_2[n-k] \\
= \; & a\mathcal{M}\{x_1[n]\} + b\mathcal{M}\{x_2[n]\} \\
= \; & ay_1[n] + by_2[n]
\end{aligned}$$

Therefore, the moving average system is linear.

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3. **[Oppenheim/Schafer/Buck Problem #2.24]**
The impulse-response of the system can be written as

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] - 2\delta[n-4] - 2\delta[n-5$$

Now, we have

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=n-5}^{n} x[n]h[n-k]$$

since $h[n-k] = 0$ whenever $n-k < 0$, and whenever $n-k > 5$ (i.e., $h[n-k]$ is non-zero only when $n-5 \leq k \leq n$). Now,

(a) $x[n] = 0$ for all $n < 4$. Therefore, $y[n] = 0$ for all $n < 4$.

(b) $x[n] = 1$ for all $n \geq 4$. In particular, for all $n \geq 9$ we have

$$y[n] = \sum_{k=n-5}^{n} x[n]h[n-k] = \sum_{k=n-5}^{n} h[n-k] = \sum_{j=0}^{5} h[j] = 0$$

(c) By explicit calculation, we get $y[n] = n-3$ for $4 \leq n < 8$, and $y[8] = 2$.

Therefore, we have

$$y[n] = \begin{cases} 0 & \text{if } n < 4 \\ n-3 & \text{if } 4 \leq n < 8 \\ 2 & \text{if } n = 8 \\ 0 & \text{if } n > 8 \end{cases}$$

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4. **[Oppenheim/Schafer/Buck Problem #2.2]**

(a) $x[n] = 5^n u[n]$. Consider the LTI system given by the impulse response $h[n] = \delta[n] + \delta[n+1]$. Then $y[0] = 1 = x[0]$ and $y[1] = 6 \neq x[1]$. So $\frac{y[0]}{x[0]} \neq \frac{y[1]}{x[1]}$, and so it is not the case that $y[n] = c \cdot x[n]$ for all n. Therefore x is not an eigenfunction of this system, which is clearly stable.

(b) $x[n] = e^{j2\omega n}$. Let $h[n]$ be the impulse response for an arbitrary stable LTI system. Then we have

$$\begin{aligned}\mathcal{T}\{x[n]\} &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{j2\omega(n-k)} \\ &= e^{2\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-2j\omega k}\right)\end{aligned}$$

Therefore, $x[n]$ is an eigenfunction of the system.

(c) $x[n] = e^{j\omega n} + e^{j2\omega n}$. Let $h[n]$ be the impulse response for an arbitrary stable LTI system. Then we have

$$\begin{aligned}\mathcal{T}\{x[n]\} &= (h * x)[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k](e^{j2\omega(n-k)} + e^{j\omega(n-k)}) \\ &= e^{\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) + e^{2\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-2j\omega k}\right)\end{aligned}$$

Clearly, we have $\mathcal{T}\{x[n]\} = cx[n]$ for all n if and only if

$$\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-2j\omega k}\right)$$

For $x[n]$ to be an eigenfunction for all systems, we need the expression to hold for all choices of $h[n]$. Clearly this can only hold if $e^{j\omega k} = e^{-j2\omega k}$ for all k, which holds iff $\omega = 2\pi r$ for some integer r. Therefore, if $\omega = 2\pi r$, then $x[n]$ is the eigenfunction all LTI systems, else it is not.

(d) $x[n] = 5^n$. Let $h[n]$ be the impulse response for an arbitrary stable T system. Then we have

$$\begin{aligned} \mathcal{T}\{x[n]\} &= h[n] * x[n \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]5^{n-k} \\ &= 5^n \left(\sum_{k=-\infty}^{\infty} h[k]5^{-k} \right) \end{aligned}$$

Therefore, $x[n]$ is an eigenfunction of the system (see footnote)

(e) $x[n] = 5^n \cdot e^{j2\omega n}$. Let $h[n]$ be the impulse response for an arbitrary stable T system. Then we have

$$\begin{aligned} \mathcal{T}\{x[n]\} &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]5^{n-k}e^{j2\omega(n-k)} \\ &= 5^n e^{j2\omega n} \left(\sum_{k=-\infty}^{\infty} h[k]5^{-k}e^{-j2\omega k} \right) \end{aligned}$$

Therefore, $x[n]$ is an eigenfunction of the system.

In class, and in the text book, a function is called an eigenfunction even if the output blows up when the system is given the input, i.e., $\lambda = \infty$ is a valid eigenvalue. Note that in particular, Section 2.6.1 notes that complex exponentials $e^{j\omega n}$ are eigenfunctions of all LTI systems, with regard to the convergence of the FT of the impulse response.

Also note that if the system produces an unbounded output it does not necessarily mean that the system is unstable if the input is unbounded as well - therefore, we can have sequences/functions which grow exponentially, and are still eigenfunctions of stable systems.

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5. **[Oppenheim/Schafer/Buck Problem #2.30]**

(a) $\mathcal{T}(x[n]) = (\cos \pi n)x[n]$

i. The system is Stable. Suppose that $|x[n]| \leq B$ for all n. Then $|y[n]|$ $|(\cos \pi n)x[n]| \leq |x[n]| \leq B$.

ii. The system is Causal. Suppose that $x_1[n] = x_2[n] \qquad n \leq n_0$. Then $y_1[n] = (\cos \pi n)x_1[n] = (\cos \pi n)x_1[n] = y_2[n]$ for all $n \leq n_0$.

iii. The system is Linear. Clearly

$$\begin{aligned} \mathcal{T}(ax_1[n] + bx_2[n]) &= (\cos \pi n)(ax_1[n] + bx_2[n]) \\ &= = a(\cos \pi n)x_1[n] + a(\cos \pi n)x_1[n] \\ &= a\mathcal{T}(x_1[n]) + b\mathcal{T}(x_2[n]) \end{aligned}$$

iv. The system is not Time Invariant. Let $x_1[n] = \delta[n]$, and $x_2[n] = \delta[n-1]$. Then $\mathcal{T}(x_1[n]) = \delta[n]$, and $\mathcal{T}(x_2[n]) = -\delta[n-1]$.

(b) $\mathcal{T}(x[n]) = x[n^2]$

i. The system is Stable. Suppose that $|x[n]| \leq B$ for all n. Then $|y[n]| = |x[n^2]| \leq B$ for all n.

ii. The system is not Causal. Suppose that $x_1[n] = \delta[n-4]$ and $x_2[n] = 0$. Clearly $x_1[n] = x_2[n]$ for all $n \leq 3$. However, $y_2[2] = 0 \neq y_1[2] = 1$ where $\mathcal{T}(x_1[n]) = y_1[n]$ and $\mathcal{T}(x_2[n]) = y_2[n]$.

iii. The system is Linear.

$$\begin{aligned} \mathcal{T}(ax_1[n] + bx_2[n]) &= (ax_1 + bx_2)[n^2] \\ &= (ax_1[n^2] + bx_2[n^2]) \\ &= a\mathcal{T}(x_1[n]) + b\mathcal{T}(x_2[n]) \end{aligned}$$

iv. The system is not Time Invariant. Let $x_1[n] = \delta[n]$ and $x_2[n] = \delta[n-1]$. Then $\mathcal{T}(x_1[n]) = \delta[n]$ and $\mathcal{T}(x_2[n]) = \delta[n-1] + \delta[n+1]$.

(c) $\mathcal{T}(x[n]) = x[n]\sum_{k=0}^{\infty}\delta[n-k]$

i. The system is Stable. Suppose that $|x[n]| \leq B > 0$ for all n. Note that if $n \geq 0$, then $\sum_{k=0}^{\infty}\delta[n-k] = 1$, and if $n < 0$ then $\sum_{k=0}^{\infty}\delta[n-k] = 0$. Hence we have

$$|y[n]| = \begin{cases} |x[n]| \leq B \text{ if } n > 0 \\ 0 \leq B \text{ if } n > 0 \end{cases}$$

ii. The system is Causal. Suppose that $x_1[n] = x_2[n]$ for all $n < n_0$, and suppose that $\mathcal{T}(x_1[n]) = y_1[n]$ and $\mathcal{T}(x_2[n]) = y_2[n]$. if $n < 0$, we have $y_1[n] = y_2[n] = 0$, and if $0 \leq n \leq n_0$, we have $y_1[n] = x_1[n] = x_2[n] = y_2[n]$. Hence $y_1[n] = y_2[n]$ for all $n \leq n_0$. Hence the system is causal.

iii. The system is Linear.

$$\begin{aligned} \mathcal{T}(ax_1[n] + bx_2[n]) &= \begin{cases} 0 & \text{if } n < 0 \\ ax_1[n] + bx_2[n] & \text{if } n \geq 0 \end{cases} \\ &= a\mathcal{T}(x_1[n]) + b\mathcal{T}(x_2[n]) \end{aligned}$$

iv. The system is not Time Invariant. Let $x_1[n] = \delta[n]$ and $x_2[n] = \delta[n+1]$. Then $\mathcal{T}(x_1[n]) = \delta[n]$ and $\mathcal{T}(x_2[n]) = 0$.

(d) $\mathcal{T}(x[\quad) \quad \sum_{k=n-1}^{\infty} x[k]$

i. The system is not stable. If $x[n] = u[n]$, then $y[n \ = \infty$ for all n.

ii. The system is not causal. If $x_1[n] = \delta[n]$, and $x_2[n] = 0$, then clearly $x_1[n] = x_2[n]$ for all $n < 0$. However, we have $y_1[-1] = 1$ and $y_2[-1] = 0$. Hence the system is not causal.

iii. The system is linear.

$$\begin{aligned}
\mathcal{T}(ax_1[n] + bx_2[n]) &= \sum_{k=n-1}^{\infty} ax_1[n] + bx_2[n] \\
&= \sum_{k=n-1}^{\infty} ax_1[n] + \sum_{k=n-1}^{\infty} bx_2[n] \\
&= a \sum_{k=n-1}^{\infty} x_1[n] + b \sum_{k=n-1}^{\infty} x_2[n] \\
&= a\mathcal{T}(x_1[n]) + b\mathcal{T}(x_2[n])
\end{aligned}$$

iv. The system is Time Invariant. If $x_1[n - n_d] = x_2[n]$, then we have

$$\begin{aligned}
y_2[n & \quad \mathcal{T}(x_2[n]) \\
& \quad \sum_{k=n-1}^{\infty} x_2[k] \\
& \quad \sum_{k=n-1}^{\infty} x_1[k - n_d] \\
& \quad \sum_{j=n-n_d-1}^{\infty} x_1[j] \\
&= y_1[n - n_d]
\end{aligned}$$

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6. **[Oppenheim/Schafer/Buck Problem #2.35]**

(a) We know that the system is time-invariant. If the system is both time-invariant and linear, then since $x_2[n] = 2x_3[n + 3]$, we should have $y_2[n] = 2y_3[n + 3]$. But this is clearly not the case, and so the system cannot be LTI. Since it is time-invariant, it cannot be linear.

Alternatively note that $x_3[n + 4] + x_2[n] = x_1[n]$. If the system were LTI, then we would have $y_3[n + 4] + y_2[n] = y_1[n]$. Again, this is not the case, and so the system cannot be LTI. Since it is time-invariant, it cannot be linear.

(b) Note that $\delta[n] = x_3[n + 4]$. Since we know that the system is time-invariant, the impulse response must be $y_3[n + 4]$.

c) The on y in ormation that we have about the system (besides the three esponses shown) is the fact that the system is time-invariant. Therefore, the only sequences for which we can predict the output of the system is various time shifts of the $x_1[n]$, $x_2[n]$ and $x_3[n]$. Therefre, we can determine the response of the system to a sequence $x[n]$ if and only if it is of the form $x[n] = x_1[n - n_d]$ or $x[n] = x_2[n - n_d]$ or $x[n] = x_3[n - n_d]$ for some $n_d \in \mathbb{Z}$.

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7. **[Oppenheim/Schafer/Buck Problem #2.36]**

(a) Using the result from part (b) below, we have the impulse response is $h[n] = 2\delta[n + 2] + \delta[n + 1] - 2\delta[n] + 3\delta[n - 1] + 2\delta[n - 2] + \delta[n - 3]$. Now, we have $x_3[n] = \delta[n] + \delta[n - 1]$. If the system were in fact LTI, we would have $y_3[n] = h[n] + h[n - 1]$. However while $y_3[0] = -3$, we have $h[0] + h[-1] = -2 + 3 = 1$. Therefore, the system cannot be LTI, and it is linear, it cannot be time-invariant.

(b) Note that $\delta[n] = x_3[n] + 0.5(x_1[n] - x_2[n])$. Since the system is linear, the impulse response should be

$$y_3[n] - 0.5(y_1[n] - y_2[n] = 2\delta[n+2] + \delta[n+1] - 2\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$$

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8. **[McClellan et al. #1.1]** We have $y[n] = -0.9y[n - 2] + 0.3x[n] + 0.6x[n - 1] + 0.3x[n - 2]$, assuming initial rest, the response to an impulse is given by

$$\begin{aligned}
y[0] &= 0.3x[0] = 0.3 \\
y[1] &= 0.6x[0] = 0.6 \\
y[2] &= -0.9y[0] + 0.3x[0] = 0.3 - 0.27 = 0.03 \\
y[3] &= -0.9y[1] = -0.54 \\
y[4] &= -0.9y[2] = -0.027 \\
y[2n] &= -0.9y[2n - 2] = (-0.9)^{n-2} * (-0.027) \\
y[2n + 1] &= -0.9y[2n - 1] = (-0.9)^{n-1} * (-0.54)
\end{aligned}$$

The stem plot of $y[n]$ is shown in Figure 3, and the code to produce this plot is shown in Figure 1.

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9. **[McClellan et al. #1.2]** We have $y[n] = 1.63y[n-1] - 0.81y[n-2] + x[n] + 0.5x[n-1]$, assuming initial rest, the first few (non-negative time) response to an impulse is given by

$$\begin{aligned}
y[0] &= x[0] = 1 \\
y[1] &= 0.5x[0] + 1.63y[n - 1] = 2.1305 \\
y[2] &= 1.63y[1] - 0.81y[0] = 2.6636 \\
y[3] &= 1.63y[2] - 0.81y[1] = 2.6173x
\end{aligned}$$

◊ مسئله‌های کامپیوتری

```
a=[1 0 0.9]';
b=[0.  0.6 0.3]';
x=zeros(1,128);
x(1)=1;
y=filter(b,a,x);
t=0:12 ;
stem(t,y);
```

Figure 1: Matlab co e for [MC] 1.1

```
a=[1 -1.8*cos(7/16) 0.81]';
b=[1 0.5 0]';
x=zeros(1,111);
x(11)=1;
y=filter(b,a,x);
t=-10:100;
stem(t,y);
```

Figure 2: Matlab co e for [MC] 1.2

The stem plot is shown in Figure 4, and the code to produce the pl t in in Figure 2.

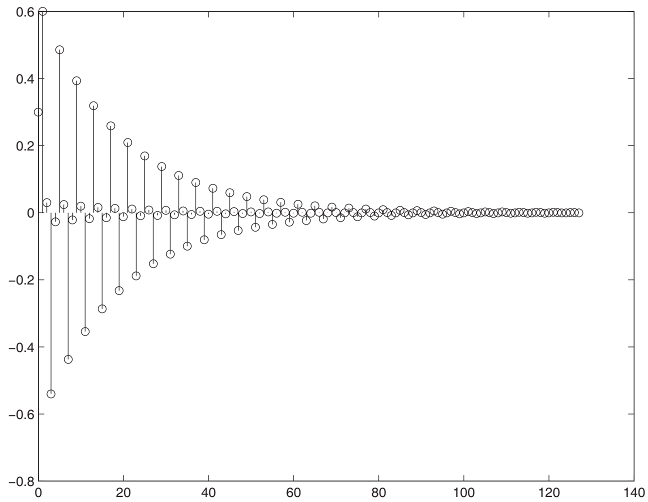

Figure 3: Stem plot for [MC] 1.1

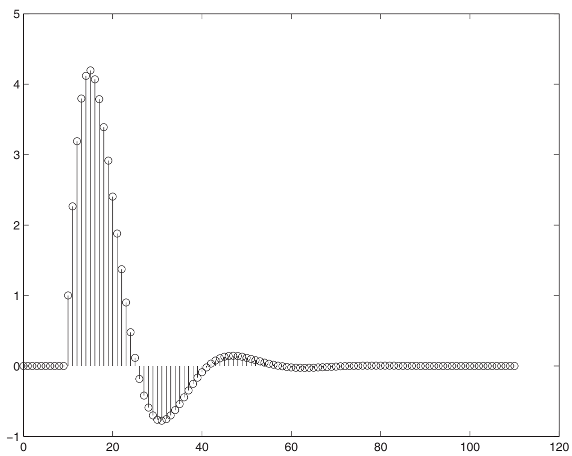

Figure 4: Stem plot for [MC] 1.2